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ON NONDISTRIBUTIVE STEINER QUASIGROUPS

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A well known result of R. Dedekind states that a lattice is nonmodular if and only if it has a sublattice isomorphic to N_5. Similarly, a lattice is nondistributive if and only if it has a sublattice isomorphic to N_5 or M_3 (see [11]). Recently, a few results in this spirit were obtained involving the number of polynomials of an algebra (see e.g. [1], [3], [5], [6]). In this paper we prove that a nondistributive Steiner quasigroup $(G,\cdot)$ has at least 21 essentially ternary polynomials (which improves the recent result obtained in [7]) and this bound is achieved if and only if $(G,\cdot)$ satisfies the identity $(xz\cdot yz)\cdot(xy)z=(xz)y\cdot x$. Moreover, we prove that a Steiner quasigroup $(G,\cdot)$ with 21 essentially ternary polynomials contains isomorphically a certain Steiner quasigroup $(M,\cdot)$, which we describe in Section 1.

1. Introduction. A *Steiner quasigroup* is an idempotent commutative groupoid $(G,\cdot)$ satisfying the condition $(xy)y=x$. Recall that Steiner quasigroups are in one-to-one correspondence with Steiner triple systems and, as has been shown by M. Reiss in 1859, an n-element Steiner quasigroup exists if and only if $n\equiv 1$ or $3 \pmod 6$ (see e.g. [2]). The least nontrivial (with more than one element) Steiner quasigroup is $\mathfrak{S}_3=(\{0,1,2\},\cdot)$, where the binary operation "$\cdot$" can be described as $x\cdot y=2x+2y \pmod 3$. Note that $\mathfrak{S}_3$ is *medial* (i.e. satisfies the identity $xy\cdot uv=xu\cdot yv$) and consequently, it is *distributive* (i.e. satisfies the conditions $(xy)z=xz\cdot yz$ and $z(xy)=zx\cdot zy$). Clearly, $\mathfrak{S}_3$ is the unique 3-element Steiner quasigroup and the following holds.

(1.i) *Every nontrivial Steiner quasigroup contains an isomorphic copy of $\mathfrak{S}_3$ as a subgroupoid.*

The least *nondistributive* Steiner quasigroup is $\mathfrak{S}_7=(\{0,1,\ldots,6\},\cdot)$, where the operation "$\cdot$" has a well known graphical representation given in Figure 1. This 7-element Steiner quasigroup is unique up to isomorphism.

In order to construct the quasigroup $(M,\cdot)$ mentioned at the beginning,

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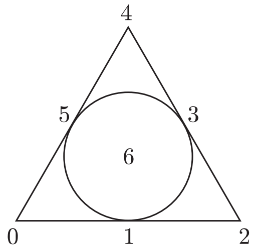

Fig. 1. The product of two elements lying on the circle, on a side or on a height of the triangle is the third element lying on that part of the triangle.

we first give a similar graphical representation for the 9-element Steiner quasigroup. Let $\mathfrak{G}_9 = (\{0, 1, \ldots, 8\}, \cdot)$ be a groupoid with the operation "·" given by the square in Figure 2. Note that $\mathfrak{G}_9$ is isomorphic to the product $\mathfrak{G}_3 \times \mathfrak{G}_3$. It is the unique (up to isomorphism) 9-element Steiner quasigroup. Obviously, it is medial and hence distributive (see also [8]).

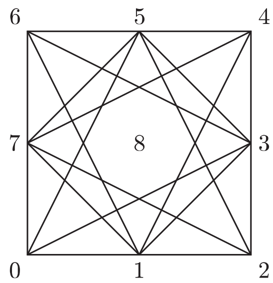

Fig. 2. The product of two elements—lying on a side, a symmetry line of the square or on a triangle with two vertices lying in the centers of adjacent sides and the third vertex at the common point of two remaining sides of the square (e.g. the triangle $[0, 3, 5]$)—is defined as the third element lying on the same part of the figure.

Now consider the set $M = \{e_0, e_1, e_2, d_0, d_1, d_2, f_0, f_1, f_2, g_0, g_1, g_2, h_0, h_1, h_2, p_0, p_1, p_2, c_0, c_1, c_2, a_0, a_1, a_2, q_0, q_1, q_2\}$ and consider the binary operation illustrated in Figure 3. Elements with the same indices form a 9-element Steiner subquasigroup with the binary operation given by Figure 2. The product of elements having two different indices is an element with the third index. The product of two elements lying in the same position in two different squares is defined as the element in the same position in the third square, e.g., $e_0 \cdot e_1 = d_2$. For every such product we define two more products by permutation of indices. In the above example we get $e_1 \cdot e_2 = d_0$ and $e_2 \cdot e_0 = d_1$. The remaining products of elements with different indices are given by the following Steiner triples:

$$[e_{\sigma(0)}, p_{\sigma(1)}, c_{\sigma(2)}], [e_{\sigma(0)}, a_{\sigma(1)}, q_{\sigma(2)}], [d_{\sigma(0)}, h_{\sigma(1)}, a_{\sigma(2)}],$$
$$[d_{\sigma(0)}, c_{\sigma(1)}, q_{\sigma(2)}], [g_{\sigma(0)}, h_{\sigma(1)}, c_{\sigma(2)}], [g_{\sigma(0)}, p_{\sigma(1)}, a_{\sigma(2)}]$$

for any permutation $\sigma \in S_3$. For example, using the first triple, we get $c_0 \cdot p_2 = e_1$, $e_2 \cdot c_1 = p_0$. It is not difficult (although tedious) to check that $(M, \cdot)$ is a Steiner quasigroup satisfying the identity

$$(xz \cdot yz) \cdot (xy)z = (xz)y \cdot x. \tag{1.ii}$$

We have done it using a computer. Since, e.g., $(e_0 \cdot e_1)e_2 = d_2 \cdot e_2 = f_2$ and $(e_0 \cdot e_2)(e_1 \cdot e_2) = d_1 \cdot d_0 = g_2$, $(M, \cdot)$ is nondistributive. Further, we prove that $p_3(M, \cdot) = 21$ (see (ii) of Theorem).

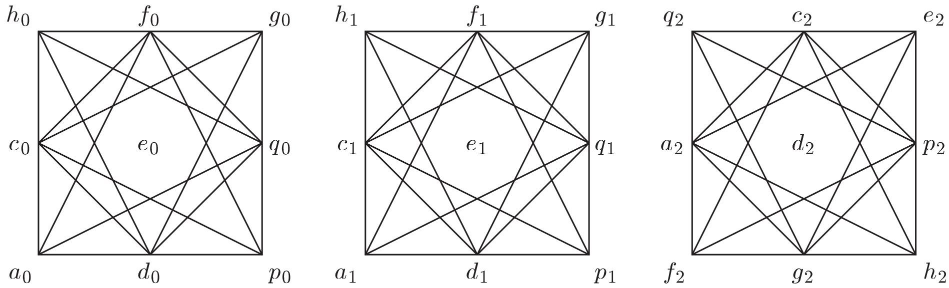

Fig. 3. The binary operation of the Steiner quasigroup $(M, \cdot)$

Let $\mathfrak{A} = (A, F)$ be an algebra. By $p_n(\mathfrak{A})$ we denote the number of essentially n-ary polynomials over $\mathfrak{A}$ (i.e., polynomials depending on all their variables) for $n > 0$ and let $p_0(\mathfrak{A})$ be the number of constant unary polynomials over $\mathfrak{A}$. We say that an algebra $\mathfrak{A}$ *represents* the sequence $(a_0, a_1, \ldots, a_n, \ldots)$ if $a_n = p_n(\mathfrak{A})$ for all n.

Let f be an n-ary polynomial of an algebra $\mathfrak{A} = (A, F)$. We say that the polynomial f *admits* a permutation σ or a permutation σ is *admissible* for f, if for every $a_1, \ldots, a_n \in A$ we have $f(a_{\sigma(1)}, \ldots, a_{\sigma(n)}) = f(a_1, \ldots, a_n)$. The set $G(f)$ of all permutations admissible for f is a subgroup of S_n called the *symmetry group* of f.

Recall that a nontrivial Steiner quasigroup $(G, \cdot)$ represents the sequence $(0, 1, 1)$. If $(G, \cdot)$ is distributive, then $p_3(G, \cdot) = 3$ (see Lemma 4.6 of [4]). Assume that $(G, \cdot)$ is nondistributive. Then J. Dudek and J. Gałuszka show in [7] that $p_3(G, \cdot) \geq 15$. We improve this result. Since we have checked that $p_3(\mathfrak{G}_7)$ is more than 51, the fact that there exists a Steiner quasigroup with only 21 essentially ternary polynomials was a surprise. One can also regard the result below as a small contribution to Problem 8 of G. Grätzer and A. Kisielewicz of [9]. In this paper we prove the following

THEOREM. *If* $(G, \cdot)$ *is a nondistributive Steiner quasigroup, then*

(i) $p_3(G, \cdot) \geq 21$,

(ii) $p_3(G, \cdot) = 21$ *if and only if* $(G, \cdot)$ *satisfies the identity*

$$(xz \cdot yz) \cdot (xy)z = (xz)y \cdot x,$$

(iii) *if* $p_3(G,\cdot) = 21$, *then* $(G,\cdot)$ *contains isomorphically the Steiner quasigroup* $(M,\cdot)$ *as a subgroupoid.*

2. Proof of Theorem. In the proof of the Theorem we consider some special ternary polynomials and we examine their symmetry groups using the well known fact that

(2.i) *If* f *is an* (*essentially*) n-*ary polynomial in an algebra* $\mathfrak{A}$, *then there exist* $n/\operatorname{card} G(f)$ *different* (*essentially*) n-*ary polynomials obtained from* f *by permuting its variables.*

Note that

(2.ii) *If* $(G,\cdot)$ *is a nontrivial Steiner quasigroup*, *then a ternary polynomial* $\varphi(x,y,z)$ *of* $(G,\cdot)$ *does not admit any* 3-*element cycle permutation of its variables.*

Indeed, if φ is not an essentially ternary polynomial in a nontrivial Steiner quasigroup $(G,\cdot)$, then the identity

(2.iii)
$$\varphi(x,y,z) = \varphi(y,z,x)$$

implies immediately that $(G,\cdot)$ is trivial, a contradiction. Suppose that φ is essentially ternary. Since the identity (2.iii) is not satisfied in $\mathfrak{G}_3$, the condition (1.i) implies that the identity (2.iii) is not satisfied in $(G,\cdot)$ either.

Proof of (i). Let $(G,\cdot)$ be a nondistributive Steiner quasigroup. Using Lemmas 3.2 and 4.1 of [4] we infer that the polynomials

$$f(x,y,z) = (xy)z, \quad g(x,y,z) = xz\cdot yz, \quad h(x,y,z) = (xz)y\cdot x$$

are essentially ternary and pairwise distinct.

Note that both f and g are symmetric in x and y. It follows, in view of (2.ii), that they admit no other transposition of variables. Consequently, there are precisely 6 pairwise distinct polynomials obtained from f and g by permuting variables. Now consider two cases according to whether $G(h)$ is trivial or not.

First, assume that $G(h)$ is nontrivial. Then, in view of Lemma 4.5 of [4], $(G,\cdot)$ satisfies the identity

(2.iv)
$$(xz)y\cdot x = (yz)x\cdot y,$$

and consequently, $p_3(G,\cdot) \geq 9$.

To improve this, consider the polynomial

$$p(x,y,z) = (xz)y\cdot zy.$$

Observe that

(2.v) *If* $(G,\cdot)$ *is a Steiner quasigroup*, *then the identities* (2.iv) *and* $(xz)y\cdot zy = (yz)x\cdot zx$ *are equivalent.*

In fact, by $(xy)y = x$ we get $(yz)x = x(zy) = z(zx) \cdot zy = ((zy)y \cdot zx) \cdot zy$. Applying (2.iv) we have $(yz)x = ((zx)y \cdot zy) \cdot zx$ and hence

$$(xz)y \cdot zy = (yz)x \cdot zx.$$

Conversely, putting xz instead of z in the last identity we get

$$zy \cdot (xz)y = ((xz)y)x \cdot z.$$

The polynomial $((xz)y)x \cdot z$ admits the same transpositions as $p(x, y, z)$, thus we obtain

$$((yz)x)y \cdot z = ((xz)y)x \cdot z.$$

Finally, by the identity $(xy)y = x$ we have condition (2.iv), which completes the proof of (2.v).

Since $p(x, y, z) = p(y, x, z)$, the polynomial p does not admit any other transposition of its variables, because of (2.ii). Observe that the polynomial p depends on x and y, because the assumption $p(x, y, z) = p(z, z, z) = z$ implies a contradiction $(xz)y = y$. The polynomial p depends on z, since $p(x, y, z) = p(x, y, x) = xy$ gives $(xz)y = xy \cdot zy$, a contradiction. Thus p is essentially ternary. Note that the assumption $p(x, y, z) = \varphi(x, y, z)$ implies a contradiction $xy = p(x, y, x) = \varphi(x, y, x) = y$ for every polynomial $\varphi \in \{f, g, h\}$. Thus p is a new polynomial. Using (2.i) we obtain $p_3(G, \cdot) \geq 12$.

Now consider the polynomials

$$a(x, y, z) = (xz \cdot yz)z, \qquad c(x, y, z) = (zx)y \cdot (zy)x,$$

introduced in Sections 5 and 7 of [4]. Using Proposition 2 and 4 of [4] we infer that the polynomials are essentially ternary. Obviously, both of them admit the transposition (x, y). Using (2.ii) we infer that $G(a) = G(c) = \{\mathrm{id}, (x, y)\}$. The polynomials a and c are not equal to any of the polynomials considered above. Indeed, for every $\varphi \in \{f, g, h\}$ the identity $c(x, x, z) = \varphi(x, x, z)$ is $z = xz$, a contradiction, and the assumption $c(x, y, z) = p(x, y, z)$ gives $(zy)x = zy$, also a contradiction. Similarly, the assumption $a(x, x, z) = \psi(x, x, z)$ for $\psi \in \{f, g, h, c\}$ implies $x = z$, a contradiction, and if we suppose that $a(xz, y, z) = p(xz, y, z)$, then we also get a contradiction: $h(z, x, y) = g(z, x, y)$. This proves that $p_3(G, \cdot) \geq 18$.

Finally, consider the polynomial

$$q(x, y, z) = (xy \cdot xz) \cdot (xz)y.$$

Note that q admits the transposition (x, y). Indeed, by $(xy)y = x$ we get

$$x \cdot (xz)y = y \cdot (zy)x,$$
$$((xz)y \cdot xz)(xy) \cdot (xz)y = ((zy)x \cdot zy)(xy) \cdot (zy)x,$$
$$((xy \cdot xz) \cdot (xz)y)(xy) = ((xy \cdot zy) \cdot (zy)x)(xy),$$
$$(xy \cdot xz) \cdot (xz)y = (xy \cdot zy) \cdot (zy)x.$$

Using (2.ii) we have $G(q)=\{\mathrm{id},(x,y)\}$. The assumption that q does not depend on x implies that $q(x,y,z)=q(z,y,z)=z(zy)\cdot zy=z$ and hence we obtain a contradiction $xy\cdot xz=(xz)y\cdot z$. Since $q(x,y,z)=q(y,x,z)$, the polynomial q depends also on y. Similarly, we show that q depends on z and consequently it is essentially ternary. The polynomial q differs from every polynomial considered earlier. Indeed, the identities $q(x,y,z)=f(x,y,z)$, $q(x,y,z)=p(x,y,z)$ and $q(x,y,z)=c(x,y,z)$ give contradictions immediately. The assumption $q\in\{g,h\}$ implies that $x=q(x,y,x)=\varphi(x,y,x)=y$ for $\varphi\in\{g,h\}$. If $q=a$, then we have $x=q(x,y,x)=a(x,y,x)=xy$, a contradiction. Applying (2.i) we get $p_3(G,\cdot)\ge 21$, as required.

It remains to consider the case where the identity (2.iv) does not hold and $\operatorname{card} G(h)=1$. Using (2.i) we obtain 6 different essentially ternary polynomials by permuting the variables of the polynomial h. They are not equal to any of the polynomials obtained from $(xy)z$ and $xy\cdot xz$ because of the difference of the symmetry groups. Thus $p_3(G,\cdot)\ge 12$. Consider the essentially ternary polynomial p. By means of (2.v) we get $\operatorname{card} G(p)=1$. Using (2.i) we obtain 6 different polynomials from p. It is easy to check that none of them is equal to any of the polynomials obtained from h. It follows that $p_3(G,\cdot)\ge 18$. When considering the polynomials a and c we do not use the identity (2.iv). Thus we get immediately $p_3(G,\cdot)\ge 24$, which completes the proof of (i).

Proof of (ii). Let $(G,\cdot)$ be a Steiner quasigroup with $p_3(G,\cdot)=21$. Take

$$r(x,y,z)=(xz\cdot yz)\cdot(xy)z.$$

Since $r(x,y,z)=r(y,x,z)$ and $r(y,y,z)=yz$ we infer that r is essentially ternary. By means of (2.ii) we get $G(r)=\{\mathrm{id},(x,y)\}$. It is easy to see that $r\notin\{f,g,p,c,a,q\}$. According to (2.i) the assumption $p_3(G,\cdot)=21$ implies that in the groupoid $(G,\cdot)$ the polynomials r and h are equal, hence $(G,\cdot)$ satisfies the condition (1.ii). Conversely, assume that a nondistributive Steiner quasigroup $(G,\cdot)$ satisfies the identity (1.ii). Using Marczewski's description of the set $\mathbf{A}^{(3)}(G,\cdot)$ (see [10]) and the identities of the groupoid $(G,\cdot)$ we prove that $\mathbf{A}_4^{(3)}(G,\cdot)=\mathbf{A}_3^{(3)}(G,\cdot)$, where

$$\begin{aligned}\mathbf{A}_3^{(3)}(G,\cdot)=\{&x,y,z,\ xy,\ yz,\ zx,\ (xy)z,\ (yz)x,\ (zx)y,\ xz\cdot yz,\ yx\cdot zx,\\ &zy\cdot xy,\ (xz)y\cdot x,\ (yx)z\cdot y,\ (zy)x\cdot z,(xz)y\cdot zy,\ (yx)z\cdot xz,\\ &(zy)x\cdot yx,\ \ (zx)y\cdot(zy)x,\ (xy)z\cdot(xz)y,(yz)x\cdot(yx)z,\\ &(xz\cdot yz)z,\ (yx\cdot zx)x,\ (zy\cdot xy)y,\ (xy\cdot xz)\cdot(xz)y,\\ &(yz\cdot yx)\cdot(yx)z,\ (zx\cdot zy)\cdot(zy)x\,\}\end{aligned}$$

and hence $p_3(G,\cdot)=21$ (see the proof of (i)). The calculations here are rather routine. We illustrate them only with two most elaborate examples, obtaining as a by-product some identities that will be used later. First, we show that the polynomial

$$b(x,y,z) = (xz \cdot yz) \cdot xy$$

considered in Section 6 of [4] and belonging to the set $\mathbf{A}_3^{(3)}(G,\cdot)$ is equal to the polynomial $c(x,y,z) \in \mathbf{A}_3^{(3)}(G,\cdot)$, i.e.

$$(xz \cdot yz) \cdot xy = (zx)y \cdot (zy)x. \tag{2.vi}$$

Indeed, putting zx instead of x in the identity (2.iv) we get $zy \cdot xz = (yz \cdot xz)y$ and hence

$$xz \cdot yz = (xy \cdot xz)y. \tag{2.vii}$$

Now, using (2.iv) and (2.vii) we obtain

$$\begin{aligned}(xz \cdot yz) \cdot xy &= (xy \cdot xz)y \cdot xy = (y(xz) \cdot xy)y = (yx \cdot (zx)y)y \\ &= ((zx)y \cdot x)y \cdot (zx)y = ((zy)x \cdot y)y \cdot (zx)y \\ &= (zy)x \cdot (zx)y.\end{aligned}$$

In the second example we consider the polynomial $(zy \cdot xy) \cdot ((yx)z)y \in \mathbf{A}_4^{(3)}(G,\cdot)$, a product of two different elements of the set $\mathbf{A}_3^{(3)}(G,\cdot)$, and we prove that it belongs to the set $\mathbf{A}_3^{(3)}(G,\cdot)$ and

$$(zy \cdot xy) \cdot ((yx)z)y = (zx)y \cdot (zy)x. \tag{2.viii}$$

Indeed, the identity (2.iv) implies that

$$\begin{aligned}(xz)y &= ((yz)x)y \cdot x = ((yz)x \cdot (yx)x)x \\ &= (x(yx) \cdot x(zy))x = (x(zy) \cdot yx)x \cdot x(zy)\end{aligned}$$

and hence $x(zy) \cdot (xz)y = (x(zy) \cdot yx)x$. Putting xy instead of x in the last identity we obtain $(xy \cdot zy) \cdot ((xy)z)y = (xy \cdot zy)x \cdot xy$ and then, using (2.vii), we have $(xy \cdot zy) \cdot ((xy)z)y = (xz \cdot yz) \cdot xy$. Now, by the identity (2.vi) we get the statement (2.viii).

P r o o f o f (iii). Let $(G,\cdot)$ be a Steiner quasigroup with $p_3(G,\cdot) = 21$. Denote by $(\widehat{G},\cdot)$ the subgroupoid of $(G,\cdot)$ generated by three elements $\mathfrak{a},\mathfrak{b},\mathfrak{c} \in G$ such that $(\widehat{G},\cdot)$ is nondistributive. By means of (ii) the Steiner quasigroup $(\widehat{G},\cdot)$ satisfies the identity (1.ii) and hence

$$\begin{aligned}\widehat{G} = \{&\mathfrak{a},\mathfrak{b},\mathfrak{c},\ \mathfrak{ab},\ \mathfrak{bc},\ \mathfrak{ca},\ (\mathfrak{ab})\mathfrak{c},(\mathfrak{bc})\mathfrak{a},(\mathfrak{ca})\mathfrak{b},\ \mathfrak{ac}\cdot\mathfrak{bc},\ \mathfrak{ba}\cdot\mathfrak{ca},\ \mathfrak{cb}\cdot\mathfrak{ab},\ (\mathfrak{ac})\mathfrak{b}\cdot\mathfrak{a},\\ &(\mathfrak{ba})\mathfrak{c}\cdot\mathfrak{b},\ (\mathfrak{cb})\mathfrak{a}\cdot\mathfrak{c},\ (\mathfrak{ac})\mathfrak{b}\cdot\mathfrak{cb},\ (\mathfrak{ba})\mathfrak{c}\cdot\mathfrak{ac},\ (\mathfrak{cb})\mathfrak{a}\cdot\mathfrak{ba},\ (\mathfrak{ca})\mathfrak{b}\cdot(\mathfrak{cb})\mathfrak{a},\\ &(\mathfrak{ab})\mathfrak{c}\cdot(\mathfrak{ac})\mathfrak{b},(\mathfrak{bc})\mathfrak{a}\cdot(\mathfrak{ba})\mathfrak{c},\ (\mathfrak{ac}\cdot\mathfrak{bc})\mathfrak{c},(\mathfrak{ba}\cdot\mathfrak{ca})\mathfrak{a},\ (\mathfrak{cb}\cdot\mathfrak{ab})\mathfrak{b},\\ &(\mathfrak{ab}\cdot\mathfrak{ac})\cdot(\mathfrak{ac})\mathfrak{b},\ (\mathfrak{bc}\cdot\mathfrak{ba})\cdot(\mathfrak{ba})\mathfrak{c},\ (\mathfrak{ca}\cdot\mathfrak{cb})\cdot(\mathfrak{cb})\mathfrak{a}\}.\end{aligned}$$

We check that any equality of arbitrary two elements of the set $\widehat{G}$ implies that $(\widehat{G},\cdot)$ is either a 3- or a 9-element Steiner quasigroup and hence it is distributive. In the proof of this fact we apply the following remarks.

(2.ix) *If* $(\check{G}, \cdot)$ *is a Steiner quasigroup generated by two different elements, then* $\mathrm{card}(\check{G}, \cdot) = 3$.

(2.x) *If a Steiner quasigroup* $(\check{G}, \cdot)$ *with* $p_3(\check{G}, \cdot) = 21$ *is generated by three different elements* a, b, c *and* $[a, b, c]$ *forms a distributive triple* (*i.e.*, $(ab)c = ac \cdot bc$), *then* $\mathrm{card}(\check{G}, \cdot) = 9$.

The proof of the first remark is obvious. In the proof of (2.x) we need an observation that (2.iv) implies the identity

$$((xy)z)x \cdot xy = (xz \cdot xy)x. \tag{2.xi}$$

Now, we prove that the condition $(ab)c = ac \cdot bc$ implies that $(bc)a = ba \cdot ca$ and $(ca)b = cb \cdot ab$. Indeed, the assumption that $[a, b, c]$ forms a distributive triple implies that $(ab)c \cdot a = (ac \cdot bc)a$. By (2.vii) we get $(ab)c \cdot a = ab \cdot cb$ and consequently $[(ab)c]a \cdot ab = cb$. Now, using (2.ix) we get $(ac \cdot ab)a = cb$ and hence $ac \cdot ab = a(cb)$. Thus $[b, c, a]$ forms a distributive triple. Similarly, multiplying both sides of the equality $(ab)c = ac \cdot bc$ by the element b we prove that $ab \cdot cb = (ac)b$. Applying the identities $(xy)y = x$ and (2.vii), we get immediately that $\check{G} = \{a, b, c, ab, ac, bc, (ab)c, (ac)b, (bc)a\}$ and the binary operation of the Steiner quasigroup $(\check{G}, \cdot)$ is given by the following Figure 4.

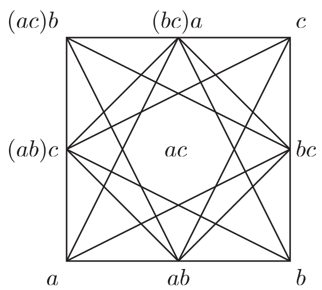

Fig. 4. The graphical representation of the Steiner quasigroup $(\check{G}, \cdot)$.

The proof of remark (2.x) is complete.

Now we can demonstrate that $\mathrm{card}(\widehat{G}, \cdot) = 27$. This fact is proved in several steps. We start from

(2.xii) *If* $\mathfrak{a} = (\mathfrak{ab})\mathfrak{c} \cdot (\mathfrak{ac})\mathfrak{b}$ *or* $\mathfrak{a} = (\mathfrak{ab} \cdot \mathfrak{cb}) \cdot (\mathfrak{ab})\mathfrak{c}$, *then* $\mathrm{card}(\widehat{G}, \cdot) \le 9$, *and if* $\mathfrak{a}$ *is equal to another element of the Steiner quasigroup* $(\widehat{G}, \cdot)$, *then* $\mathrm{card}(\widehat{G}, \cdot) \le 3$.

If $\mathfrak{a} = (\mathfrak{ab})\mathfrak{c} \cdot (\mathfrak{ac})\mathfrak{b}$, then by (2.vi), $\mathfrak{a} = \mathfrak{cb} \cdot (\mathfrak{ba} \cdot \mathfrak{ca})$ and hence $(\mathfrak{bc})\mathfrak{a} = \mathfrak{ba} \cdot \mathfrak{ca}$. Applying (2.x) we infer that $\mathrm{card}(\widehat{G}, \cdot) \le 9$. Similarly the assumption $\mathfrak{a} = (\mathfrak{ab} \cdot \mathfrak{cb}) \cdot (\mathfrak{ab})\mathfrak{c}$ implies that $\mathfrak{ab} \cdot \mathfrak{cb} = (\mathfrak{ab})\mathfrak{c} \cdot \mathfrak{a}$, $(\mathfrak{ab} \cdot \mathfrak{cb})\mathfrak{a} = (\mathfrak{ab})\mathfrak{c}$ and hence by (2.vii), $\mathfrak{ac}\cdot\mathfrak{bc}=(\mathfrak{ab})\mathfrak{c}$. Thus $\mathrm{card}(\widehat{G},\cdot)\leq 9$. If $\mathfrak{a}$ is equal to one of the remaining elements, we prove that $(\widehat{G},\cdot)$ is generated by at most two different elements and in consequence using (2.ix) we obtain $\mathrm{card}(\widehat{G},\cdot)\leq 3$. By means of (2.xii), we infer that $\mathfrak{a}$ is not equal to any of the remaining elements of $(\widehat{G},\cdot)$. The same statement is true for $\mathfrak{b}$ and $\mathfrak{c}$.

Now let us take the element $\mathfrak{ab}\in\widehat{G}\setminus\{\mathfrak{a},\mathfrak{b},\mathfrak{c}\}$ and note that

(2.xiii) *If* $\mathfrak{ab}=(\mathfrak{ac})\mathfrak{b}\cdot\mathfrak{cb}$ *or* $\mathfrak{ab}=(\mathfrak{ac}\cdot\mathfrak{bc})\mathfrak{c}$, *then* $\mathrm{card}(\widehat{G},\cdot)\leq 9$, *and if* $\mathfrak{a}$ *is equal to another element of the Steiner quasigroup* $(\widehat{G},\cdot)$, *then* $\mathrm{card}(\widehat{G},\cdot)\leq 3$.

The assumption $\mathfrak{ab}=(\mathfrak{ac})\mathfrak{b}\cdot\mathfrak{cb}$ implies that $\mathfrak{ab}\cdot\mathfrak{cb}=(\mathfrak{ac})\mathfrak{b}$. Hence by (2.x), $\mathrm{card}(\widehat{G},\cdot)\leq 9$. Similarly, $\mathfrak{ab}=(\mathfrak{ac}\cdot\mathfrak{bc})\mathfrak{c}$ implies the same statement. We prove that if $\mathfrak{ab}$ is equal to one of the remaining elements of $(\widehat{G},\cdot)$, then the groupoid $(\widehat{G},\cdot)$ is generated by at most two different elements and $\mathrm{card}(\widehat{G},\cdot)\leq 3$. By (2.xiii), $\mathfrak{ab}$ is not equal to any of the remaining elements of $(\widehat{G},\cdot)$ and similarly $\mathfrak{ac}$ and $\mathfrak{bc}$ have the same property.

Now we deal with the element $(\mathfrak{ab})\mathfrak{c}\in\widehat{G}\setminus\{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{ab},\mathfrak{ac},\mathfrak{bc}\}$ and we observe that

(2.xiv) *If* $(\mathfrak{ab})\mathfrak{c}=\mathfrak{ac}\cdot\mathfrak{bc}$ *or* $(\mathfrak{ab})\mathfrak{c}=(\mathfrak{ac})\mathfrak{b}\cdot\mathfrak{a}$, *then* $\mathrm{card}(\widehat{G},\cdot)\leq 9$, *and if* $\mathfrak{a}$ *is equal to another element of the Steiner quasigroup* $(\widehat{G},\cdot)$, *then* $\mathrm{card}(\widehat{G},\cdot)\leq 3$.

The assumption $(\mathfrak{ab})\mathfrak{c}=\mathfrak{ac}\cdot\mathfrak{bc}$ implies immediately that $\mathrm{card}(\widehat{G},\cdot)\leq 9$. The equation $(\mathfrak{ab})\mathfrak{c}=(\mathfrak{ac})\mathfrak{b}\cdot\mathfrak{a}$ gives $(\mathfrak{ab})\mathfrak{c}=(\mathfrak{ac}\cdot\mathfrak{bc})\cdot(\mathfrak{ab})\mathfrak{c}$ and hence $\mathrm{card}(\widehat{G},\cdot)\leq 9$. Since $(\widehat{G},\cdot)$ is nondistributive, $(\mathfrak{ab})\mathfrak{c}$, $(\mathfrak{bc})\mathfrak{a}$ and $(\mathfrak{ca})\mathfrak{b}$ are not equal to any of the remaining elements of $(\widehat{G},\cdot)$.

Consider the element $\mathfrak{ac}\cdot\mathfrak{bc}\in\widehat{G}\setminus\{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{ab},\mathfrak{ac},\mathfrak{bc},(\mathfrak{ab})\mathfrak{c},(\mathfrak{bc})\mathfrak{a},(\mathfrak{ca})\mathfrak{b}\}$. We have

(2.xv) *If* $\mathfrak{ac}\cdot\mathfrak{bc}=(\mathfrak{ac})\mathfrak{b}\cdot\mathfrak{a}$, *then* $\mathrm{card}(\widehat{G},\cdot)\leq 9$, *and if* $\mathfrak{a}$ *is equal to another element of the Steiner quasigroup* $(\widehat{G},\cdot)$, *then* $\mathrm{card}(\widehat{G},\cdot)\leq 3$.

If $\mathfrak{ac}\cdot\mathfrak{bc}=(\mathfrak{ac})\mathfrak{b}\cdot\mathfrak{a}$, then $\mathfrak{ab}\cdot\mathfrak{cb}=(\mathfrak{ac}\cdot\mathfrak{bc})\mathfrak{a}=(\mathfrak{ac})\mathfrak{b}$ and by (2.x) we obtain $\mathrm{card}(\widehat{G},\cdot)\leq 9$. For the other equalities using (2.ix) we prove that $\mathrm{card}(\widehat{G},\cdot)\leq 3$. Similarly, $\mathfrak{ab}\cdot\mathfrak{ac}$ and $\mathfrak{ab}\cdot\mathfrak{cb}$ differ from the remaining elements of $(\widehat{G},\cdot)$.

Now we take $(\mathfrak{ac})\mathfrak{b}\cdot\mathfrak{a}\in\widehat{G}$ and an element $\varphi(\mathfrak{a},\mathfrak{b},\mathfrak{c})\in\widehat{G}\setminus\{\mathfrak{a},\mathfrak{b},\mathfrak{c},\mathfrak{ab},\mathfrak{ac},\mathfrak{bc},(\mathfrak{ab})\mathfrak{c},(\mathfrak{bc})\mathfrak{a},(\mathfrak{ca})\mathfrak{b},\mathfrak{ac}\cdot\mathfrak{bc},\mathfrak{ba}\cdot\mathfrak{ca},\mathfrak{cb}\cdot\mathfrak{ab}\}$. We prove that

(2.xvi) *The equality* $(\mathfrak{ac})\mathfrak{b}\cdot\mathfrak{a}=\varphi(\mathfrak{a},\mathfrak{b},\mathfrak{c})$ *implies that* $\mathrm{card}(\widehat{G},\cdot)\leq 3$.

The above remark implies that in the nondistributive Steiner quasigroup

$(\widehat{G}, \cdot)$ the element $(\mathfrak{ac})\mathfrak{b} \cdot \mathfrak{a}$ and also $(\mathfrak{ab})\mathfrak{c} \cdot \mathfrak{a}$ and $(\mathfrak{ab})\mathfrak{c} \cdot \mathfrak{b}$ are not equal to any of the remaining elements.

Now suppose that the element $(\mathfrak{ac})\mathfrak{b} \cdot \mathfrak{cb} \in \{(\mathfrak{ba})\mathfrak{c} \cdot \mathfrak{ac}, (\mathfrak{cb})\mathfrak{a} \cdot \mathfrak{ba}, (\mathfrak{ca})\mathfrak{b} \cdot (\mathfrak{cb})\mathfrak{a}, (\mathfrak{ab})\mathfrak{c} \cdot (\mathfrak{ac})\mathfrak{b}, (\mathfrak{bc})\mathfrak{a} \cdot (\mathfrak{ba})\mathfrak{c}, (\mathfrak{ac} \cdot \mathfrak{bc})\mathfrak{c}, (\mathfrak{ba} \cdot \mathfrak{ca})\mathfrak{a}, (\mathfrak{cb} \cdot \mathfrak{ab})\mathfrak{b}, (\mathfrak{ab} \cdot \mathfrak{ac}) \cdot (\mathfrak{ac})\mathfrak{b}, (\mathfrak{bc} \cdot \mathfrak{ba}) \cdot (\mathfrak{ba})\mathfrak{c}, (\mathfrak{ca} \cdot \mathfrak{cb}) \cdot (\mathfrak{cb})\mathfrak{a}\}$. In this case we obtain

(2.xvii) *If* $(\mathfrak{ac})\mathfrak{b} \cdot \mathfrak{cb} = (\mathfrak{ac} \cdot \mathfrak{bc})\mathfrak{c}$, *then* $\operatorname{card}(\widehat{G}, \cdot) \leq 9$, *and if* $\mathfrak{a}$ *is equal to another element of the Steiner quasigroup* $(\widehat{G}, \cdot)$, *then* $\operatorname{card}(\widehat{G}, \cdot) \leq 3$.

Observe that if $(\mathfrak{ac})\mathfrak{b} \cdot \mathfrak{cb} = (\mathfrak{ac} \cdot \mathfrak{bc})\mathfrak{c} = [\mathfrak{a}(\mathfrak{bc})]\mathfrak{c} \cdot \mathfrak{bc}$, then $(\mathfrak{ac})\mathfrak{b} = (\mathfrak{bc})\mathfrak{a} \cdot \mathfrak{c}$ and $\operatorname{card}(\widehat{G}, \cdot) \leq 9$ as above. We show that any other equality implies $\operatorname{card}(\widehat{G}, \cdot) \leq 3$. Thus $(\mathfrak{ac})\mathfrak{b} \cdot \mathfrak{cb}$, $(\mathfrak{ba})\mathfrak{c} \cdot \mathfrak{ac}$ and $(\mathfrak{cb})\mathfrak{a} \cdot \mathfrak{ba}$ differ from the remaining elements of the considered set.

The consideration of the element $(\mathfrak{ca})\mathfrak{b} \cdot (\mathfrak{cb})\mathfrak{a}$ proves that for every $\psi(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}) \in \{(\mathfrak{ab})\mathfrak{c} \cdot (\mathfrak{ac})\mathfrak{b}, (\mathfrak{bc})\mathfrak{a} \cdot (\mathfrak{ba})\mathfrak{c}, (\mathfrak{ac} \cdot \mathfrak{bc})\mathfrak{c}, (\mathfrak{ba} \cdot \mathfrak{ca})\mathfrak{a}, (\mathfrak{cb} \cdot \mathfrak{ab})\mathfrak{b}, (\mathfrak{ab} \cdot \mathfrak{ac}) \cdot (\mathfrak{ac})\mathfrak{b}, (\mathfrak{bc} \cdot \mathfrak{ba}) \cdot (\mathfrak{ba})\mathfrak{c}, (\mathfrak{ca} \cdot \mathfrak{cb}) \cdot (\mathfrak{cb})\mathfrak{a}\}$ we have

(2.xviii) *The equality* $(\mathfrak{ca})\mathfrak{b} \cdot (\mathfrak{cb})\mathfrak{a} = \psi(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$ *implies that* $\operatorname{card}(\widehat{G}, \cdot) \leq 3$.

Since $(\widehat{G}, \cdot)$ is nondistributive, $(\mathfrak{ca})\mathfrak{b} \cdot (\mathfrak{cb})\mathfrak{a}$ and also $(\mathfrak{ab})\mathfrak{c} \cdot (\mathfrak{ac})\mathfrak{b}$ and $(\mathfrak{ab})\mathfrak{c} \cdot (\mathfrak{bc})\mathfrak{a}$ do not belong to the above set.

Similarly, considering the elements $(\mathfrak{ac} \cdot \mathfrak{bc})\mathfrak{c}$ and $(\mathfrak{ab} \cdot \mathfrak{ac}) \cdot (\mathfrak{ac})\mathfrak{b}$, we prove that

(2.xix) *If* $(\mathfrak{ac} \cdot \mathfrak{bc})\mathfrak{c} \in \{(\mathfrak{ba} \cdot \mathfrak{ca})\mathfrak{a}, (\mathfrak{cb} \cdot \mathfrak{ab})\mathfrak{b}, (\mathfrak{ab} \cdot \mathfrak{ac}) \cdot (\mathfrak{ac})\mathfrak{b}, (\mathfrak{bc} \cdot \mathfrak{ba}) \cdot (\mathfrak{ba})\mathfrak{c}, (\mathfrak{ca} \cdot \mathfrak{cb}) \cdot (\mathfrak{cb})\mathfrak{a}\}$, *then* $\operatorname{card}(\widehat{G}, \cdot) \leq 3$.

(2.xx) *If* $(\mathfrak{ab} \cdot \mathfrak{ac}) \cdot (\mathfrak{ac})\mathfrak{b} \in \{(\mathfrak{bc} \cdot \mathfrak{ba}) \cdot (\mathfrak{ba})\mathfrak{c}, (\mathfrak{ca} \cdot \mathfrak{cb}) \cdot (\mathfrak{cb})\mathfrak{a}\}$, *then* $\operatorname{card}(\widehat{G}, \cdot) \leq 3$.

Hence also $(\mathfrak{ab} \cdot \mathfrak{ac})\mathfrak{a}$ and $(\mathfrak{ab} \cdot \mathfrak{cb})\mathfrak{b}$ differ from the remaining elements and $(\mathfrak{ab} \cdot \mathfrak{ac}) \cdot (\mathfrak{ab})\mathfrak{c} \neq (\mathfrak{ac} \cdot \mathfrak{bc}) \cdot (\mathfrak{ac})\mathfrak{b}$.

Now, it is easy to see that $(\widehat{G}, \cdot)$ is isomorphic to the Steiner quasigroup $(M, \cdot)$ described in Section 1. Thus any Steiner quasigroup $(G, \cdot)$ with $p_3(G, \cdot) = 21$ contains an isomorphic copy of $(M, \cdot)$ as a subgroupoid. This completes the proof of the Theorem.

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