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ON A CONJECTURE OF MĄKOWSKI AND SCHINZEL

BY

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1. Introduction. Let σ and ϕ denote the sum-of-divisors function and Euler's function, respectively. If otherwise unspecified, then, except for these, Roman and Greek letters will denote natural numbers, with p, q, r reserved for primes. We write $F_i = 2^{2^i} + 1$ for $i \ge 0$; these are Fermat numbers.

Mąkowski and Schinzel [5] conjectured in 1964 that

$$\frac{\sigma(\phi(n))}{n} \ge \frac{1}{2} \quad \text{for all } n. \tag{1}$$

They noted that Mrs. K. Kuhn had shown the inequality to be true for all n with at most six prime factors.

Pomerance [6] proved in 1989 that $\inf \sigma(\phi(n))/n > 0$ and Balakrishnan [1] recently verified (1) for squarefull values of n (satisfying $p^2 \mid n$ when $p \mid n$). Filaseta, Graham and Nicol [2] have shown that (1) is true when n is the product of the first k primes, for sufficiently large k. The Mąkowski–Schinzel conjecture is included in B42 of Guy [3].

We shall prove here that the conjecture is true in general if it is true for squarefree integers, and we shall verify the conjecture for various classes of numbers, such as:

(i) All numbers of the form $2^a m$, where the distinct prime factors of m are Fermat primes or primes congruent to 1 (mod 3), with at most eight of the latter.

(ii) Any product of any primes less than 1780.

(iii) All numbers of the form $2^a m$, where m is a product of primes $1+2^b r$, for any b and any prime r. Thus m is any product of any primes in the set $\{3, 5, 7, 11, 13, 17, 23, 29, 41, 47, 53, 59, 83, 89, 97, \ldots\}$.

Our proof will require the following three lemmas.

LEMMA 1. (a) *For any v, w with $v \mid w$, $\sigma(v)/v \le \sigma(w)/w$. There is equality if and only if $v = w$.*

(b) *For any u, v, $\sigma(uv) \ge u\sigma(v)$. There is equality if and only if $u = 1$.*

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Proof. (a) follows quickly from the observation that $\sigma(w)/w = \sum_{d \mid w} 1/d$. Setting $w = uv$ shows (b) to be equivalent to (a). ∎

LEMMA 2. *If* $0 \le d_1 < d_2 < \ldots < d_t$, *then*

$$2^{1+\sum_{i=1}^{t} 2^{d_i}} - 1 \ge \prod_{i=1}^{t} (1 + 2^{2^{d_i}}). \tag{2}$$

Equality holds if and only if $d_i = i - 1$ *for* $i = 1, \ldots, t$.

Proof. Multiply out the right-hand side of (2) and note that each summand is a divisor of $N = 2^{\sum_{i=1}^{t} 2^{d_i}}$. The sum of all such divisors is $\sigma(N)$, the left-hand side of (2). Since *all* divisors are obtained on the right if and only if $d_i = i - 1$ for each i, we obtain then the statement on equality. ∎

LEMMA 3. *For any real number* $x > 1$ *and any natural number* k,

$$\frac{x^{k+1} - 1}{x - 1} \ge (x + z_1)(x + z_2) \ldots (x + z_k),$$

where $z_1, z_2, \ldots, z_k$ *are any positive real numbers such that* $z_1 + z_2 + \ldots + z_k \le 1$.

Proof. This follows from the following inequality given in Hardy [4, p. 34]. If $x_1, x_2, \ldots, x_k$ are all positive real numbers, and $y_k = x_1 + x_2 + \ldots + x_k$, then

$$(1 + x_1)(1 + x_2) \ldots (1 + x_k) \le 1 + y_k + \frac{y_k^2}{2!} + \ldots + \frac{y_k^k}{k!}. \quad ∎$$

There is a well-known equivalent form of the conjecture in (1). It is an easier form for our purposes and is proved here for completeness.

LEMMA 4. *We have* $\sigma(\phi(n)) \ge n/2$ *for all* n *if and only if*

$$\sigma(\phi(m)) \ge m$$

for all odd m. *We have* $\sigma(\phi(n)) = n/2$ *if and only if* $n = 2m$, m *is odd*, *and* $\sigma(\phi(m)) = m$.

Proof. Suppose $\sigma(\phi(n)) \ge n/2$ for all n, and let m be any odd number. Then

$$\frac{\sigma(\phi(m))}{m} = 2\,\frac{\sigma(\phi(2m))}{2m} \ge 2 \cdot \frac{1}{2} = 1.$$

This shows also that if m is odd and $\sigma(\phi(m)) = m$, then $\sigma(\phi(n)) = n/2$, where $n = 2m$.

Suppose next that $\sigma(\phi(m)) \ge m$ for all odd m. Let n be any even number, so $n = 2^a m$, say, where m is odd. Then $\phi(n) = 2^{a-1}\phi(m)$, so, by Lemma 1(b), $\sigma(\phi(n)) \ge 2^{a-1}\sigma(\phi(m))$. Hence

$$\frac{\sigma(\phi(n))}{n} \ge \frac{2^{a-1}}{2^a} \cdot \frac{\sigma(\phi(m))}{m} \ge \frac{1}{2}.$$

This shows also that if $\sigma(\phi(n)) = n/2$ then $a = 1$ and $\sigma(\phi(m)) = m$. ∎

2. Theorems

THEOREM 1. *We have $\sigma(\phi(m)) \geq m$ whenever m is a product of Fermat primes. For such a product, there is equality if and only if $m = F_0F_1 \dots F_k$ for $0 \leq k \leq 4$.*

Proof. Write $m = \prod_{i=1}^{t} p_i^{a_i}$, where $a_i \geq 1$ and $p_i = 2^{2^{d_i}} + 1$ is prime, for $i = 1, \dots, t$. Assume $0 \leq d_1 < d_2 < \dots < d_t$, and set $T = \sum_{i=1}^{t} 2^{d_i}$. We have

$$\phi(m) = \prod_{i=1}^{t} (p_i - 1)p_i^{a_i - 1} = 2^T \prod_{i=1}^{t} p_i^{a_i - 1},$$

and

$$\sigma(\phi(m)) = (2^{T+1} - 1) \prod_{i=1}^{t} \frac{p_i^{a_i} - 1}{p_i - 1}.$$

Then

$$\frac{\sigma(\phi(m))}{m} = (2^{T+1} - 1) \prod_{i=1}^{t} \frac{1}{p_i} \cdot \prod_{i=1}^{t} \frac{p_i^{a_i} - 1}{p_i^{a_i - 1}(p_i - 1)} \geq 1,$$

by virtue of Lemma 2.

Furthermore, it follows from this and Lemma 2 that $\sigma(\phi(m)) = m$ if and only if $a_i = 1$ and $d_i = i - 1$ for each i. The latter requires $p_i = 2^{2^{i-1}} + 1 = F_{i-1}$ and since $F_0, \dots, F_4$ are prime and F_5 is not, we have $\sigma(\phi(m)) = m$ if and only if m is one of the stated products. (These solutions were known to Mąkowski and Schinzel [5].) ∎

THEOREM 2. *The Mąkowski–Schinzel conjecture is true in general if it is true for squarefree integers.*

Proof. Write $m = \prod_{i=1}^{t} q_i^{b_i}$, for distinct odd primes $q_1, \dots, q_t$, and $m' = \prod_{i=1}^{t} q_i$. For each $i = 1, \dots, t$, write

$$q_i - 1 = \prod_{j=1}^{t} q_j^{\beta_{ij}} \cdot w_i,$$

where $\beta_{ij} \geq 0$ and $q_j \nmid w_i$ for any i, j. Write $W = \prod_{i=1}^{t} w_i$, and $B_j = \sum_{i=1}^{t} \beta_{ij}$, for each $j = 1, \dots, t$. Then

$$\phi(m') = \prod_{i=1}^{t} (q_i - 1) = \prod_{j=1}^{t} q_j^{B_j} \cdot W,$$

and $q_j \nmid W$ for any $j = 1, \dots, t$, so that

$$\sigma(\phi(m')) = \prod_{j=1}^{t} \frac{q_j^{B_j + 1} - 1}{q_j - 1} \cdot \sigma(W).$$

We now have

$$\phi(m) = \prod_{i=1}^{t} q_i^{b_i - 1} \prod_{i=1}^{t} (q_i - 1) = \prod_{j=1}^{t} q_j^{b_j + B_j - 1} \cdot W,$$

and then

$$\sigma(\phi(m)) = \prod_{j=1}^{t} \frac{q_j^{b_j + B_j} - 1}{q_j - 1} \cdot \sigma(W) = \prod_{j=1}^{t} \frac{q_j^{b_j + B_j} - 1}{q_j^{B_j + 1} - 1} \cdot \sigma(\phi(m')).$$

Hence

$$\frac{\sigma(\phi(m))}{m} = \prod_{j=1}^{t} \frac{q_j^{b_j + B_j} - 1}{q_j^{b_j - 1}(q_j^{B_j + 1} - 1)} \cdot \frac{\sigma(\phi(m'))}{\prod_{i=1}^{t} q_i} \geq \frac{\sigma(\phi(m'))}{m'}. \tag{3}$$

Now let n be any even number and put $n = 2^a m$, with m and m' as before. If the Mąkowski–Schinzel conjecture is true for squarefree integers, we then have

$$\frac{\sigma(\phi(n))}{n} = \frac{\sigma(\phi(2^a m))}{2^a m} \geq \frac{\sigma(\phi(m))}{2m} \geq \frac{\sigma(\phi(m'))}{2m'} = \frac{\sigma(\phi(2m'))}{2m'} \geq \frac{1}{2}.$$

That is, the conjecture is true for all even integers, and clearly also for all odd integers. Note, in particular, that there is strict inequality above if m is not squarefree (that is, if $b_j \geq 2$ for at least one $j = 1, \ldots, t$). ■

We now adopt an alternative notation and write

$$m = \prod_{p_i \in P} p_i^{a_i} \prod_{q_i \in Q} q_i^{b_i},$$

where P is the set of Fermat primes that divide m and $Q \neq \emptyset$ is the set of remaining distinct prime factors of m. For each $p_i \in P$, set $p_i = 2^{2^{d_i}} + 1$, and put $T_1 = \sum_{p_i \in P} 2^{d_i}$. For each $q_i \in Q$, suppose $2^{\delta_i} \parallel (q_i - 1)$ (so that $2^{\delta_i} \mid (q_i - 1)$, but $2^{\delta_i + 1} \nmid (q_i - 1)$) and set $T_2 = \sum_{q_i \in Q} \delta_i$.

From (3),

$$\begin{aligned} \frac{\sigma(\phi(m))}{m} &\geq \frac{\sigma(2^{T_1} \prod_{q_i \in Q}(q_i - 1))}{\prod_{p_i \in P} p_i \prod_{q_i \in Q} q_i} \\ &= \frac{2^{T_1 + T_2 + 1} - 1}{2^{T_2 + 1} - 1} \cdot \frac{\sigma(\prod_{q_i \in Q}(q_i - 1))}{\prod_{p_i \in P} p_i \prod_{q_i \in Q} q_i} \\ &\geq \frac{2^{T_1 + T_2 + 1} - 1}{(2^{T_1 + 1} - 1)(2^{T_2 + 1} - 1)} \cdot \frac{\sigma(\prod_{q_i \in Q}(q_i - 1))}{\prod_{q_i \in Q} q_i}, \end{aligned}$$

by Lemma 2, and, since $T_2 > 0$,

$$\frac{\sigma(\phi(m))}{m} > \frac{2^{T_2}}{2^{T_2 + 1} - 1} \cdot \frac{\sigma(\prod_{q_i \in Q}(q_i - 1))}{\prod_{q_i \in Q} q_i}. \tag{4}$$

THEOREM 3. *For each* $q_i \in Q$, *write* $q_i - 1 = 2^{\delta_i} r_i l_i$, *where* r_i *is an odd prime and* l_i *is odd. Suppose, renumbering the* $q_i \in Q$ *if necessary, that*

$$\prod_{q_i \in Q} r_i = \prod_{j=1}^{k} r_j^{c_j}, \tag{5}$$

where $r_1, \ldots, r_k$ *are distinct. Use these primes to partition* Q, *so that, renaming the* $q_i \in Q$, $q_{ij} = 1 + 2^{\delta_{ij}} r_j l_{ij}$, *where* l_{ij} *is odd, for* $j = 1, \ldots, k$ *and* $i = 1, \ldots, c_j$. *Then* $\sigma(\phi(m)) > m$ *if*

$$\frac{\sigma(r_j^{c_j})}{r_j^{c_j}} \prod_{i=1}^{c_j} \frac{q_{ij} - 1}{q_{ij}} \geq 1 \tag{6}$$

for each $j = 1, \ldots, k$.

Proof. With T_2 as above, we have

$$\prod_{q_i \in Q} (q_i - 1) = 2^{T_2} \prod_{q_i \in Q} r_i \prod_{q_i \in Q} l_i,$$

so that, using Lemma 1(a),

$$\frac{\sigma(\prod_{q_i \in Q} (q_i - 1))}{\prod_{q_i \in Q} (q_i - 1)} \geq \frac{2^{T_2+1} - 1}{2^{T_2}} \cdot \frac{\sigma(\prod_{q_i \in Q} r_i)}{\prod_{q_i \in Q} r_i}.$$

Then, continuing from (4) and using (5),

$$\begin{aligned}
\frac{\sigma(\phi(m))}{m} &> \frac{2^{T_2}}{2^{T_2+1} - 1} \cdot \frac{\sigma(\prod_{q_i \in Q} (q_i - 1))}{\prod_{q_i \in Q} (q_i - 1)} \prod_{q_i \in Q} \frac{q_i - 1}{q_i} \\
&\geq \frac{\sigma(\prod_{j=1}^{k} r_j^{c_j})}{\prod_{j=1}^{k} r_j^{c_j}} \prod_{q_i \in Q} \frac{q_i - 1}{q_i} = \prod_{j=1}^{k} \frac{\sigma(r_j^{c_j})}{r_j^{c_j}} \prod_{i=1}^{c_j} \frac{q_{ij} - 1}{q_{ij}}.
\end{aligned}$$

The result follows. ∎

We remark that it is easy to see that (6) is always true when $c_j = 1$ or 2. Also, the left-hand side of (6) is a decreasing function of c_j provided $q_{c_j j} \leq \sigma(r_j^{c_j})$, and this would appear to be generally the case.

COROLLARY. *With the notation of Theorem* 3, $\sigma(\phi(m)) > m$ *if*

$$r_j \sum_{i=1}^{c_j} \frac{1}{q_{ij} - 1} \leq 1$$

for each $j = 1, \ldots, k$.

Proof. For each $j = 1, \ldots, k$ and $i = 1, \ldots, c_j$, set $\xi_{ij} = \delta_{ij} + \log_2 l_{ij}$, so that $q_{ij} = 1 + 2^{\xi_{ij}} r_j$. Then, from the proof of Theorem 3,

$$\frac{\sigma(\phi(m))}{m} > \prod_{j=1}^{k} \frac{r_j^{c_j+1} - 1}{r_j^{c_j}(r_j - 1)} \prod_{i=1}^{c_j} \frac{q_{ij} - 1}{q_{ij}} = \prod_{j=1}^{k} \frac{r_j^{c_j+1} - 1}{r_j^{c_j}(r_j - 1)} \prod_{i=1}^{c_j} \frac{2^{\xi_{ij}} r_j}{1 + 2^{\xi_{ij}} r_j}$$
$$= \prod_{j=1}^{k} \frac{r_j^{c_j+1} - 1}{r_j - 1} \prod_{i=1}^{c_j} \frac{1}{r_j + 2^{-\xi_{ij}}} \ge 1,$$

by Lemma 3, provided $\sum_{i=1}^{c_j} 2^{-\xi_{ij}} \le 1$ for each $j = 1, \ldots, k$. The result follows, since $2^{-\xi_{ij}} = r_j/(q_{ij} - 1)$. ∎

3. Applications. For the first example given in Section 1, all primes $q \in Q$ satisfy $q \equiv 1 \pmod 3$ so that we may take $k = 1$ and $r_1 = 3$. Write c for c_1 and q_i for q_{i1}, $i = 1, \ldots, c$. Assume $q_1 < \ldots < q_c$. If $c \ge 9$, then $q_1 \ge 7$, $q_2 \ge 13$, $q_3 \ge 19$, $q_4 \ge 31$, $q_5 \ge 37$, $q_6 \ge 43$, $q_7 \ge 61$, $q_8 \ge 67$, $q_9 \ge 73, \ldots$ Denote the primes on the right by $q'_1, q'_2, \ldots$ Then a simple computer run gives us

$$\frac{\sigma(3^c)}{3^c} \prod_{i=1}^{c} \frac{q_i - 1}{q_i} \ge \frac{\sigma(3^c)}{3^c} \prod_{i=1}^{c} \frac{q'_i - 1}{q'_i} > 1$$

for $c = 1, \ldots, 8$ (but the latter product is less than 1 for $c = 9$). That completes the verification of example (i), using Theorem 3.

For example (ii), all primes $q \in Q$ satisfy $q < 1780$. Partition these primes according to the largest prime factor r of $q - 1$. If $r = 3$, 5, 7 or 11 then, respectively,

$$q \in \{7, 13, 19, 37, 73, 97, 109, \ldots, 1297, 1459\} \quad \text{(with 16 elements)};$$
$$q \in \{11, 31, 41, 61, 101, \ldots, 1601, 1621\};$$
$$q \in \{29, 43, 71, 113, \ldots, 1373, 1471\};$$
$$q \in \{23, 67, 89, 199, 331, 353, 397, 463, 617, 661, 727, 881, 991, 1321, 1409, 1453\};$$

and so on, for all $r < 890$. Then apply Theorem 3, as follows. Say $r_1 = 3$. Then $c_1 \le 16$ and (assuming $q_{11} < q_{21} < \ldots$) $q_{11} \ge 7$, $q_{21} \ge 13, \ldots$ Denote the primes on the right by $q'_{11}, q'_{21}, \ldots$ A computer run gives us

$$\frac{\sigma(3^{c_1})}{3^{c_1}} \prod_{i=1}^{c_1} \frac{q_{i1} - 1}{q_{i1}} \ge \frac{\sigma(3^{c_1})}{3^{c_1}} \prod_{i=1}^{c_1} \frac{q'_{i1} - 1}{q'_{i1}} > 1$$

for $c_1 = 1, \ldots, 16$. The corresponding computation for each possible value of r gives similar answers. The result is determined by the case $r = 11$: if all 16 primes shown $(23, 67, \ldots, 1453)$ occur in Q then the corresponding product indeed exceeds 1, but if the next possibility (1783) is also included then that product is less than 1.

For the third application, suppose in fact that $q_{ij} = 1 + 2^{\delta_{ij}} r_j$ for each $j = 1, \ldots, k$ and $i = 1, \ldots, c_j$. We may assume that $q_{1j} < \ldots < q_{c_j j}$ for each j, so, looking to the proof of the Corollary, $\xi_{ij} = \delta_{ij} \geq i$ for $i = 1, \ldots, c_j$. Then

$$r_j \sum_{i=1}^{c_j} \frac{1}{q_{ij} - 1} = \sum_{i=1}^{c_j} \frac{1}{2^{\xi_{ij}}} \leq \sum_{i=1}^{c_j} \frac{1}{2^i} = \frac{2^{c_j} - 1}{2^{c_j}} < 1,$$

for each $j = 1, \ldots, k$, so that $\sigma(\phi(m)) > m$. This is example (iii) in Section 1: taking Lemma 4 and the Fermat primes into account, the Mąkowski–Schinzel conjecture holds for any product of any primes in the set

$$X = \{2, 3, 5, 7, 11, 13, 17, 23, 29, 41, 47, 53, 59, 83, 89, 97, \ldots\}.$$

We can append further primes to the (presumably infinite) set X by filling holes caused by non-prime values of $1 + 2^i r_j$. For example, $79 \notin X$ but, since $79 = 1 + 2 \cdot 3 \cdot 13$ and $27 = 1 + 2 \cdot 13$ is not prime and $79 > 27$, if $79 \mid m$ we may take 79 as the first prime in the subset of primes in Q determined by $r_j = 13$. We may similarly let 43, 67 and 71 stand in place of the composites 15, 45 and 57, respectively. The prime 19 can be appended to X only at the expense of either 7 or 13. Finally, 31 could replace the absent 21 or 25, 37 could replace 25, 61 could replace 21, 25 or 49, and 73 could replace 25 or 49.

The preceding paragraph may be summarised by saying that the Mąkowski–Schinzel conjecture in (1) holds for any n which is a product of any primes in the set X augmented by all other primes less than 100, except that at most two of 7, 13 and 19, and at most three of 31, 37, 61 and 73 may be included as prime factors of n.

4. Notes. In Guy [3], 24 solutions are listed of the equation $\phi(\sigma(k)) = k$. Then of course there are 24 solutions of the equation $\sigma(\phi(m)) = m$, given by $m = \sigma(k)$. The latter include $m = 1$ and the five solutions given in Theorem 1; the others are all even. Ten further solutions of $\phi(\sigma(k)) = k$ were found by Terry Raines: $2^8 3^6 7^2 13$, $2^9 3^4 5^2 11^2 31$, $2^{13} 3^3 5^4 7^3$, $2^{13} 3^8 5 \cdot 7^3$, $2^{13} 3^6 5 \cdot 7^3 13$, $2^{13} 3^3 5^4 7^4$, $2^{13} 3^8 5^2 7^3$, $2^{13} 3^6 5^2 7^3 13$, $2^{21} 3^5 5 \cdot 11^3 31$, and $2^{21} 3^5 5^2 11^3 31$. We have found eight more: $2^{13} 3^5 5^4 7^4$, $2^{17} 3^{10} 5^5 7^3 11^2 19$, $2^{17} 3^{10} 5^5 7^4 11^2 19$, $2^{24} 3^9 5^7 11 \cdot 13$, $2^{25} 3^{11} 5^6 7^3 13^2 31$, $2^{25} 3^{11} 5^6 7^4 13^2 31$, $2^{26} 3^{10} 5^5 7^3 11^2 19$, and $2^{26} 3^{10} 5^5 7^4 11^2 19$. We have confirmed that all solutions of $\phi(\sigma(k)) = k$ up to 10^9 are known.

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After submitting the paper, I was informed that Ram Gupta found the following simple proof of Theorem 2. He uses the fact that $\phi(p^a) = p\phi(p^{a-1})$ for $a \geq 2$. Then, using also Lemma 1(b), if $n > 1$ is any integer and $p \mid n$, we have

$$\frac{\sigma(\phi(pn))}{pn} = \frac{\sigma(p\phi(n))}{pn} > \frac{p\sigma(\phi(n))}{pn} = \frac{\sigma(\phi(n))}{n},$$

and the result follows.

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