COLLOQUIUM MATHEMATICUM

VOL. 73 1997 NO. 2

ON COMPACT SYMPLECTIC AND KÄHLERIAN SOLVMANIFOLDS WHICH ARE NOT COMPLETELY SOLVABLE

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We are interested in the problem of describing compact solvmanifolds admitting symplectic and Kählerian structures. This was first considered in [3, 4] and [7]. These papers used the Hattori theorem concerning the cohomology of solvmanifolds, hence the results obtained covered only the *completely solvable case*. Our results do not use the assumption of complete solvability. We apply our methods to construct a new example of a compact symplectic non-Kählerian solvmanifold.

1. Introduction. Recently, there has been an interest in examples of compact symplectic manifolds with no Kähler structures ([1, 5, 7, 8, 10, 15, 16] and others). With the exception of [7, 16] and the surgery technique of [8], known examples are nilmanifolds coming from the following general theorem proved by C. Benson and C. Gordon [3]:

Let M be a compact $K(\Gamma,1)$-manifold where Γ is a discrete, finitely generated, torsion free, nilpotent group. If M admits a Kähler structure, then Γ is abelian and M is diffeomorphic to a torus.

This theorem implies that any *non-toral* symplectic $K(\Gamma,1)$-manifold with nilpotent Γ yields the desired example.

Motivated by this, the authors of [3, 4, 7] raised the same question for *aspherical manifolds* and, in particular, for *solvmanifolds*.

The case of solvmanifolds, however, differs essentially from that of nilmanifolds for several reasons. In general, the Nomizu theorem concerning the cohomology of a nilmanifold is not available, therefore, the minimal model of a solvmanifold cannot be used directly. Moreover, the known examples of symplectic and Kählerian solvmanifolds show that *all possible cohomology types, Kählerian and non-Kählerian, may occur* (see [4, 7]).

1991 *Mathematics Subject Classification*: 53C15, 55P62.

Key words and phrases: solvmanifold, Kähler structure, symplectic structure.

In [4] and [7], the authors investigated the case of a solvmanifold G/Γ of a *completely solvable* Lie group G and established the necessary conditions for the existence of Kählerian structures on such manifolds. The key to the proof of the cited results is the Nomizu–Hattori theorem [11] regarding the cohomology of G/Γ which allowed the authors to establish the *rational model* $(\overline{A}^*(G/\Gamma), d) = (\Lambda L(G)^*, \delta)$. For arbitrary Lie groups, *the Nomizu–Hattori theorem does not hold* (see [19] and [26]) and the technique of [4] is not available. Moreover, there are many types of solvmanifolds which are not completely solvable [26], e.g. solvmanifolds of (R)-type, (E)-type, mixed types etc.

We attack the problem in a different manner. The main results of this article are the following three theorems. Theorem 1 is the main algebraic tool. It shows that *for any compact solvmanifold* G/Γ (*completely solvable or not*) the existence of a Kählerian structure implies some strong algebraic conditions on the cohomology complex $(\Lambda L(G)^*, \delta)$ of the Lie algebra $L(G)$. The most interesting feature of this result is the fact that although $(\Lambda L(G)^*, \delta)$ is *not a rational model of* G/Γ in the non-completely solvable case, Theorem 1 is strong enough to obtain results analogous to [7] in the general case. In particular, Theorem 2 yields a new example of a compact symplectic non-Kählerian manifold M^8 (it is 8-dimensional). Theorem 3 describes algebraic properties of rational models of some aspherical Kählerian manifolds provided that these models admit a structure of a twisted tensor product. In particular, these results hold for any solvmanifold which has a free model represented as a twisted tensor product (in the spirit of [20]). Unfortunately, the author does not know which solvmanifolds admit free twisted tensor products as rational models except those considered in Corollary 1 to Theorem 3. It is worth mentioning that applications of rational homotopy theory to geometry are limited presently to the nilpotent space situation and are not directly applicable to the solvmanifold case. Therefore, Theorem 3 can be viewed as a step along the lines of the original twisted models of Sullivan.

THEOREM 1. *Let $M = G/\Gamma$ be a compact solvmanifold carrying a Kählerian structure. Then the cochain complex $(\Lambda L(G)^*, \delta)$ of the Lie algebra $L(G)$ satisfies the following property*: *all triple Massey products and higher order Massey products of the pair $(\Lambda L(G)^*, H^*(\Lambda L(G)^*))$ vanish as cohomology classes in $H^*(\Lambda L(G)^*)$* (*that is*, *there is a choice of cochains in $\Lambda L(G)^*$ representing all triple Massey products and higher order Massey products so that these cochains are exact*).

Remark. This condition is stronger than saying simply that all Massey products vanish, since, for instance, a triple Massey product $\langle [a], [b], [c] \rangle$ is defined with an indeterminacy lying in the ideal generated by $[a]$ and $[c]$.

Theorem 2. *Let $L(G)$ be a Lie algebra defined as follows*:

$$L(G) = \mathrm{Span}(A, B, X_1, X_2, X_3, X_4, X_5, X_6), \tag{1}$$

$$\begin{gathered}[A, X_1] = X_1, \quad [A, X_2] = -X_2, \quad [A, X_3] = X_3, \quad [A, X_4] = -X_4, \\ [A, X_5] = \alpha X_6, \quad [A, X_6] = -\alpha X_5, \quad \alpha \in \mathbb{R}, \\ [B, X_3] = X_1, \quad [B, X_4] = X_2 \end{gathered} \tag{2}$$

(*the other brackets are assumed to be zero*). *Let G be a simply connected Lie group corresponding to $L(G)$. Then*:

(i) *G is a solvable non-nilpotent and not completely solvable Lie group*;

(ii) *G contains a lattice Γ for some particular α*;

(iii) *the homogeneous space G/Γ is a compact symplectic solvmanifold with no Kähler structure.*

To formulate Theorem 3, we introduce the notion of the *twisted tensor product of graded commutative differential algebras*. Here and in the sequel we use traditional notations of rational homotopy theory and solvmanifolds. Nevertheless, some of them are explained in the next section.

Let $(\mathcal{R}, \overline{d})$ and $(\mathcal{S}, \delta)$ be graded commutative differential algebras. Introduce the tensor product $\mathcal{R} \otimes \mathcal{S}$ of $\mathcal{R}$ and $\mathcal{S}$ as graded algebras and define the "twisted" derivation d by the formula

$$\begin{aligned} &d(r \otimes 1) = \overline{d}(r) \otimes 1, \qquad r \in \mathcal{R}, \\ &d(1 \otimes s) = 1 \otimes \delta(s) + \sum_{i \geq 0} (-1)^{(i+1)\deg(s)} \sum_{\xi \geq 1} \phi_i^\xi(s) \otimes r_{i+1}^\xi, \qquad s \in \mathcal{S}; \end{aligned} \tag{3}$$

here ϕ_i^ξ is a derivation of $(\mathcal{S}, \delta)$ decreasing the degree by i, that is,

$$\phi_i^\xi(xy) = \phi_i^\xi(x)y + (-1)^{i\deg(x)} x \phi_i^\xi(y)$$

and $\{r_{i+1}^\xi, \xi = 1, 2, \ldots\}$ constitute a basis of $\mathcal{R}^{i+1}$. By definition, the graded differential algebra $(\mathcal{R} \otimes \mathcal{S}, d)$ is called the *twisted tensor product* of $(\mathcal{R}, \overline{d})$ and $(\mathcal{S}, \delta)$. In the sequel we denote the twisted tensor product by $(\mathcal{R}, \overline{d}) \otimes_\tau (\mathcal{S}, \delta)$.

Remark. Twisted tensor products could be considered in a more general framework [12, 20].

Theorem 3. *Let $F \to E \to T^k$ be a fiber bundle with the fiber F of homotopy type $K(\pi, 1)$ with nilpotent π over the homotopy torus T^k. Assume that the total space E of the bundle admits a model $(A^*(E), d)$ which can be represented as a twisted tensor product*

$$(A^*(E), d) = (\Lambda X, \overline{d} = 0) \otimes_\tau (\Lambda Y, \delta),$$

where X and Y are finite-dimensional vector spaces of elements of degree 1, and $(\Lambda Y, \delta)$ is a minimal differential algebra. Then if E admits a Kählerian structure, the model $(A^*(E), d)$ has the following properties:

(i) *$(A^*(E), d)$ is a Lefschetz algebra;*

(ii) *for the bigrading $\Lambda^{ij} = \Lambda^i X \otimes \Lambda^j Y$ the following equalities hold:*

$$H^1(A^*) = \Lambda^{1,0}, \quad H^{n-1}(A^*) = \Lambda^{k,l-1}, \quad k = \dim X,\ l = \dim Y;$$

(iii) *the Lefschetz element $\omega \in \Lambda^2(X \oplus Y)$ can be chosen in the form*

$$\omega = \omega^{2,0} + \omega^{0,2}$$

and $\omega^{2,0}$ and $\omega^{0,2}$ are non-degenerate 2-forms on X^* and Y^*, which are closed and non-exact with respect to d and δ.

(iv) $\dim X$ *and* $\dim Y$ *are even and X^* and Y^* are ω-orthogonal.*

Corollary 1. *Let $F \to E \to T^k$ be a fiber bundle with the base and the fiber as in Theorem 3. Assume that the $\pi_1(T^k)$-action on $H^*(F)$ is nilpotent. Then, if E is endowed with a Kählerian structure, there exists a free graded differential algebra $(A^*(E), d)$ which has the structure of the free tensor product given by Theorem 3 and possesses algebraic properties* (i)–(iv).

In particular, let $M = G/\Gamma$ be a compact solvmanifold carrying a Kählerian structure. Assume that the corresponding Mostow bundle satisfies the nilpotency condition for the $\pi_1(G/N\Gamma)$-action on $H^*(N\Gamma/\Gamma)$. There exists a free graded differential algebra $(A^*(G/\Gamma), d)$ which is a free rational model for G/Γ and has the properties (i)–(iv) *of Theorem* 3.

Remark. The definition of the Mostow bundle is given is Section 5. We included the solvmanifold case into the formulation of the corollary, although the author does not know whether there are solvmanifolds *non-diffeomorphic* to nilmanifolds whose Mostow bundle satisfies the nilpotency condition. So far, the last part of the corollary may be trivial. However, at the end of the paper we describe (very briefly) what the $\pi_1(T^k)$-action on $H^*(F)$ looks like in the solvmanifold case. This suggests that for *non-abelian* fibers the nilpotency condition may appear.

To illustrate the usefulness of Theorem 3 we also show that it generalizes the Benson–Gordon theorem [4].

Corollary 2. *If G is completely solvable and G/Γ is a solvmanifold that admits a Kähler structure, then*

(i) *there is an abelian complement $\mathcal{A}$ in $L(G)$ of the derived algebra $\mathcal{N} = [L(G), L(G)]$;*

(ii) *$\mathcal{A}$ and $\mathcal{N}$ are even-dimensional;*

(iii) *the center of $L(G)$ intersects $\mathcal{N}$ trivially;*

(iv) *the Kählerian form is cohomologous to a left-invariant symplectic form* $\omega = \omega_0 + \omega_1$, *where* $\mathcal{N} = \ker(\omega_0)$ *and* $\mathcal{A} = \ker(\omega_1)$;

(v) *both* ω_0 *and* ω_1 *are closed but non-exact in* $L(G)$ *and also in* $\mathcal{N}$ *and* $\mathcal{A}$;

(vi) *the adjoint action of* $\mathcal{A}$ *on* $\mathcal{N}$ *is by infinitesimal symplectomorphisms.*

Remark. (i) Corollary 2 follows from *the proof of Theorem* 3.

(ii) The properties (i)–(iv) stated in Theorem 3 are the algebraic properties established in [4] for $(\Lambda L(G)^*, \delta)$ in the completely solvable case.

2. Preliminaries. This work is situated, in fact, in the *framework of rational homotopy theory*. Since there are many books and research articles devoted to this topic, we assume that the reader is familiar with it and refer to [6, 9, 14, 23].

In the sequel we consider the category $\mathbb{R}$-DGA of graded differential algebras over the field of real numbers (although the algebraic results are valid for an arbitrary field of zero characteristic). By definition, a *model* of a manifold M is a *free graded differential algebra* $A^*(M)$ such that there exists either a homomorphism

$$\alpha : A^*(M) \to \Omega_{\mathrm{DR}}(M)$$

or a homomorphism

$$\beta : \Omega_{\mathrm{DR}}(M) \to A^*(M)$$

inducing isomorphism in cohomology. Of course, $A^*(M)$ need not be minimal and therefore is not unique. We use the notion of the *minimal model* in the usual sense. The cohomology functor for $\mathbb{R}$-DGA is denoted by H^*. By definition we call a free finitely-graded differential algebra $A = \bigoplus_{i=0}^{n} A^i$ *oriented* if $H^n(A) \neq 0$.

Definition. (i) A finitely-graded free differential algebra $(A, d) = (\bigoplus_{i=0}^{2n} A^i, d)$ is called *symplectic* if there exists an element $[\omega] \in H^2(A)$ such that $[\omega]^n \neq 0, \omega \in A^2$.

(ii) An algebra (A, d) satisfying (i) is called *Lefschetz*, or satisfies the *hard Lefschetz condition*, if all homomorphisms

$$L_{\omega^r} : H^{n-r} \to H^{n+r}, \quad L_{\omega^r}([h]) = [\omega^r][h], \quad r \geq 1,$$

are isomorphisms. The element $\omega \in A^2$ is called *Lefschetz*.

Observe that symplectic algebras satisfying Poincaré duality are models of compact symplectic manifolds and Lefschetz algebras are those of compact Kählerian manifolds. This fact will play the crucial role in Section 5.

One of the ingredients of proofs in this article is the Thomas theorem concerning the models of Serre fibrations [24] and [25], therefore, we reproduce its exact formulation (see [24] for the proof).

Let

$$(*) \qquad F \to E \to B$$

be a Serre fibration with B and F of finite cohomology type. As usual, the fundamental group $\pi_1(B)$ acts on $H^*(F)$ via a representation $\pi_1(B) \to \mathrm{Aut}(H^*(F))$ (see, e.g., [21]). We assume that E,F and B are path-connected. Recall that by definition an action of a group G on an abelian group C is called *nilpotent* if G acts on C by automorphisms and the inductively defined sequence

$$\Gamma_0 C = C \supset \Gamma_1 C \supset \ldots \supset \Gamma_{n+1} C = \{gc - c : g \in G, \quad c \in \Gamma_n C\} \supset \ldots$$

vanishes for some $n \geq 1$, ($\{\ldots\}$ denotes the $\mathbb{Z}G$-submodule in C generated by $gc - c$).

THOMAS THEOREM. *Let* $(*)$ *be a Serre fibration satisfying the above assumptions. Suppose that* $\pi_1(B)$ *acts nilpotently on* $H^*(F)$. *Then there exists a model* $(A^*(E), d)$ *which is a twisted tensor product*

$$(A^*(E), d) = (\mathcal{M}_B, d_B) \otimes_\tau (\mathcal{M}_F, d_F)$$

of the minimal models $(\mathcal{M}_B, d_B)$ *and* $(\mathcal{M}_F, d_F)$ *of* B *and* F *respectively. The twisting* τ *is of the form* (3).

In the sequel we denote the vector space dual to the vector space X by X^*.

Following [4], we consider as *solvmanifolds* only homogeneous spaces G/Γ, where G is a solvable simply connected Lie group and Γ is a lattice in G, that is, a discrete co-compact subgroup (see [2, 18] for the general theory of solvmanifolds).

We introduce *nilmanifolds* as homogeneous spaces N/Γ, where N is a simply connected nilpotent Lie group and Γ a lattice in N.

To compare our results with [4, 7], we recall the definition of a *completely solvable* Lie group: a Lie group G is completely solvable if all endomorphisms $\mathrm{ad}\, V : L(G) \to L(G)$, $V \in L(G)$, possess only *real* eigenvalues. Of course, there are many solvable Lie groups which are not completely solvable (see e.g. [2, 26]).

To prove the main results we need the following fact, which is well known and can be found in explicit form in [10]:

PROPOSITION. *The minimal model of any compact nilmanifold* N/Γ *is of the form*

$$(\mathcal{M}_{N/\Gamma}, d) \simeq (\Lambda L(N)^*, \delta)$$

where δ *is a standard derivation determining the cohomology of* $L(N)$:

$$(4) \quad \delta\alpha(X_0, \ldots, X_k) = \sum_{i<j} (-1)^{i+j-1} \alpha([X_i, X_j], X_0, \ldots, \widehat{X}_i, \widehat{X}_j, \ldots, X_k),$$

$X_0, \ldots, X_k \in L(N), \alpha \in \Lambda^k L(N)^*$.

DEFINITION ([6]). (i) Let (A, d) be a free graded commutative differential algebra. We call it *formal* if there exists a quasi-isomorphism

$$\varrho : (A, d) \to (H^*(A, d), 0),$$

where $(H^*(A, d), 0)$ denotes the cohomology algebra considered as a graded differential algebra with zero derivation.

(ii) We say that a graded commutative differential algebra *is a formal consequence* of its cohomology algebra if the minimal model $(\mathcal{M}_A, d)$ of (A, d_A) is formal.

Remark. We slightly generalized the notion of formality, extending it to free (not necessarily minimal) algebras. This terminology is a bit more convenient for us. On the other hand, it is important to *distinguish the notions of formality and that of being a formal consequence*. In general a graded differential algebra may have the formal minimal model, but nevertheless, there may be no quasiisomorphisms between the original algebra and its cohomology (compare e.g. the main theorem in [6] concerning the de Rham algebra of any Kählerian manifold).

The classical theorem of Deligne–Griffiths–Morgan–Sullivan [6] will be used in the proofs of Theorem 1 and Theorem 2. We reproduce it here.

DGMS-THEOREM. *For any compact Kählerian manifold its de Rham algebra is a formal consequence of its cohomology algebra.*

In Section 4 we use the notions of the *Massey product* and the *higher order Massey product* referring to [13] for the definition.

A free graded algebra over a graded vector space X is denoted by ΛX, the degree of an element $x \in X$ is denoted by $\deg(x)$.

3. Proof of Theorem 1. Recall that for introducing Massey products one needs *a pair* $(A, H^*(A))$ (a differential graded algebra A together with its cohomology algebra). It is important for us to stress this fact, since in the sequel we will *change algebras without changing cohomologies* and, therefore, Massey products will also vary for different pairs. For this reason, we will use a slightly different terminology considering *Massey products of the pair* $(A, H^*(A))$.

LEMMA 1. *Let*

$$\psi : (A, d_A) \to (B, d_B) \tag{6}$$

be a morphism in $\mathbb{R}$-DGA, *inducing a monomorphism in cohomology. Then the following implication holds*: *if* (B, d_B) *is a formal consequence of its cohomology algebra*, *then all triple Massey products and higher order Massey products of the pair* $(A, H^*(A))$ *vanish as cohomology classes in* $H^*(A)$ (*that* *is, there is a choice of cochains in A representing all triple Massey products and higher order Massey products so that these cochains are exact).*

Proof. Consider, first, triple Massey products (of course, we could proceed with Massey products of an arbitrary order, but we prefer to accomplish the proof separately for triple and then for *quadruple Massey products* presenting all essential moments of the proof but avoiding clumsy notation).

Note that for each morphism $\varphi : (A, d_A) \to (B, d_B)$,

$$\varphi^*\langle [a], [b], [c]\rangle = \langle \varphi^*[a], \varphi^*[b], \varphi^*[c]\rangle,$$

where the right-hand side of this identity is considered as a cohomology class in $H^*(B, d_B)$ and the right-hand product $\langle\ ,\ ,\ \rangle$ is taken with respect to the pair $(B, H^*(B))$. Indeed, the cohomology class $[y]$ representing the triple Massey product is determined by the following procedure: if

$$d_A x^{12} = \overline{a}b, \quad d_A x^{23} = \overline{b}c, \quad x^{12}, x^{23} \in A,$$

then

$$y = \overline{a}x^{23} + \overline{x}^{12}c$$

(here and in the sequel $\overline{x}$ denotes $(-1)^p x$ for $x \in A^p$). We take *an arbitrary cocycle y* obtained by this procedure and fix it. We do not care about the indeterminacy lying in $([a], [c])$. Then

$$\varphi(y) = \overline{\varphi(a)}\varphi(x^{23}) + \overline{\varphi(x^{12})}\varphi(c)$$

and

$$\varphi(d_A x^{12}) = d_B \varphi(x^{12}) = \overline{\varphi(a)}\varphi(b),$$
$$\varphi(d_A x^{23}) = d_B \varphi(x^{23}) = \overline{\varphi(b)}\varphi(c),$$

so that $\varphi(y)$ represents, by definition, a triple Massey product

$$\langle [\varphi(a)], [\varphi(b)], [\varphi(c)]\rangle$$

of the pair $(B, H^*(B))$. Finally,

$$\varphi^*[y] = \langle [\varphi^*[a], \varphi^*[b], \varphi^*[c]\rangle$$

as expected.

Now, let us prove the lemma for triple Massey products. Let $(\mathcal{M}_B, d)$ be the minimal model of (B, d_B) and ϱ be the corresponding quasi-isomorphism

$$\varrho : (\mathcal{M}_B, d) \to (B, d_B).$$

Assume that there exists a non-zero cohomology class $[y] \in H^*(A, d_A)$ represented as a triple Massey product $[y] = \langle [a], [b], [c]\rangle$. We have already shown that $\psi^*[y] = \langle \psi^*[a], \psi^*[b], \psi^*[c]\rangle$. Since ψ^* is injective, $\psi^*[y]$ represents a non-zero cohomology class in $H^*(B)$ which is represented as a triple Massey product of the pair $(B, H^*(B))$. Since ϱ^* is an isomorphism, there exist cocycles $a_M, b_M, c_M \in \mathcal{M}_B$ such that

$$\varrho^*[a_M] = \psi^*[a], \quad \varrho^*[b_M] = \psi^*[b], \quad \varrho^*[c_M] = \psi^*[c].$$

Recall that

$$\psi^*[a]\psi^*[b] = 0, \quad \psi^*[b]\psi^*[c] = 0,$$

which implies $\varrho^*[a_M]\varrho^*[b_M] = 0, \varrho^*[b_M]\varrho^*[c_M] = 0$ and

$$[a_M][b_M] = 0, \quad [b_M][c_M] = 0.$$

Hence,

$$dz^{12} = \overline{a}_M b_M, \quad dz^{23} = \overline{b}_M c_M$$

for cochains $z^{12}, z^{23} \in \mathcal{M}_B$. Take

$$y_M = \overline{a}_M z^{23} + \overline{z}^{12} c_M,$$

which is a cocycle representing the triple Massey product $\langle [a_M], [b_M], [c_M] \rangle$ of the pair $(\mathcal{M}_B, H^*(\mathcal{M}_B) \simeq H^*(B))$. Thus,

$$\begin{aligned} \varrho^*[y_M] &= \varrho^*\langle [a_M], [b_M], [c_M] \rangle = \langle \varrho^*[a_M], \varrho^*[b_M], \varrho^*[c_M] \rangle \\ &= \langle \psi^*[a], \psi^*[b], \psi^*[c] \rangle = \psi^*[y] \neq 0. \end{aligned}$$

But, since $(\mathcal{M}_B, d)$ is formal, $[y_M]$ must vanish as a cohomology class in $H^*(\mathcal{M}_B)$ because of the following general fact proved in [6] (p. 262): *if $(\mathcal{M}_B, d)$ is formal, one can make uniform choices so that the cochains representing all Massey products and higher order Massey products are exact*. This contradiction implies $\langle [a], [b], [c] \rangle = 0$ in $H^*(A)$.

Consider now *quadruple Massey products*. Recall the procedure of constructing the corresponding cohomology classes. Again, we assume that *all cohomology classes representing triple Massey products of the pair $(A, H^*(A))$ vanish* (this is the condition under which quadruple Massey products are well-defined, sometimes this condition is called "simultaneous vanishing"). Assume that $[w] \in H^*(A)$ is a quadruple Massey product:

$$[w] = \langle [a], [b], [c], [u] \rangle.$$

Again, we begin with the general observation that $\varphi^*[w] = \langle \varphi^*[a], \varphi^*[b], \varphi^*[c], \varphi^*[u] \rangle$ for each $\mathbb{R}$-DGA-morphism φ. Indeed, the construction of the quadruple Massey product is carried out as follows: one takes cochains $x^{12}, x^{23}, x^{34} \in A$ such that

$$d_A x^{12} = \overline{a} b, \quad d_A x^{23} = \overline{b} c, \quad d_A x^{34} = \overline{c} u$$

and cochains $x^{13}, x^{24} \in A$ such that

$$d_A x^{13} = \overline{a} x^{23} + \overline{x}^{12} c, \quad d_A x^{24} = \overline{b} x^{34} + \overline{x}^{23} u$$

and forms a cocycle

$$w = \overline{a} x^{24} + \overline{x}^{12} x^{34} + \overline{x}^{13} u.$$

Again, as in the previous case,

$$\varphi d_A x^{12} = d_B\varphi(x^{12}) = \overline{\varphi(a)}\varphi(b), \qquad \varphi d_A x^{23} = d_B\varphi(x^{23}) = \overline{\varphi(b)}\varphi(c),$$
$$\varphi d_A x^{34} = d_B\varphi(x^{34}) = \overline{\varphi(c)}\varphi(u),$$
$$\varphi d_A x^{13} = d_B\varphi(x^{13}) = \overline{\varphi(a)}\varphi(x^{23}) + \overline{\varphi(x^{12})}\varphi(c),$$
$$\varphi d_A x^{24} = d_B\varphi(x^{24}) = \overline{\varphi(b)}\varphi(x^{34}) + \overline{\varphi(x^{23})}\varphi(u),$$

which means that

$$\varphi(w) = \overline{\varphi(a)}\varphi(x^{24}) + \overline{\varphi(x^{12})}\varphi(x^{34}) + \overline{\varphi(x^{13})}\varphi(u)$$

represents the corresponding quadruple Massey product of the pair $(B, H^*(B))$. Finally,

$$\varphi^*[w] = \langle[\varphi^*[a], \varphi^*[b], \varphi^*[c], \varphi^*[u]\rangle.$$

Again, take the minimal model $(\mathcal{M}_B, d)$ and cocycles $[a_M], [b_M], [c_M]$ and $[u_M]$ in $\mathcal{M}_B$ such that $\varrho^*[a_M] = \psi^*[a]$, $\varrho^*[b_M] = \psi^*[b]$, $\varrho^*[c_M] = \psi^*[c]$, $\varrho^*[u_M] = \psi^*[u]$. Since ϱ^* is an isomorphism, the identities involving cohomology classes in $H^*(A)$ are transferred to $H^*(\mathcal{M}_B)$:

$$[a_M][b_M] = 0, \quad [b_M][c_M] = 0, \quad [c_M][u_M] = 0$$

(for example, $[a][b] = 0 \Rightarrow \psi^*[a]\psi^*[b] = \varrho^*[a_M]\varrho^*[b_M] = 0$). Hence

$$dz^{12} = \overline{a}_M b, \quad dz^{23} = \overline{b}_M c_M, \quad dz^{34} = \overline{c}_M u_M, \quad z^{12}, z^{23}, z^{34} \in \mathcal{M}_B.$$

Since $\langle[a], [b], [c]\rangle = 0$ and $\langle[b], [c], [u]\rangle = 0$ as *cohomology classes in* $H^*(A)$ ("vanish simultaneously"), the same is valid for their images under ψ^* and ϱ^* (see the corresponding identity for arbitrary φ^*). Therefore,

$$\langle[a_M], [b_M], [c_M]\rangle = 0, \quad \langle[b_M], [c_M], [u_M]\rangle = 0$$

as cohomology classes in $H^*(\mathcal{M}_B)$ and there exist cochains $z^{13}, z^{24} \in \mathcal{M}_B$ such that

$$dz^{13} = \overline{a}_M z^{32} + \overline{z}^{12} c_M, \quad dz^{24} = \overline{b}_M z^{34} + \overline{z}^{23} u_M.$$

Hence, the quadruple Massey product is well-defined with respect to the pair $(\mathcal{M}_B, H^*(\mathcal{M}_B))$:

$$[w_M] = \langle[a_M], [b_M], [c_M], [u_M]\rangle$$

and

$$\begin{aligned}\varrho^*\langle[a_M], [b_M], [c_M], [u_M]\rangle &= \langle\varrho^*[a_M], \varrho^*[b_M], \varrho^*[c_M], \varrho^*[u_M]\rangle \\ &= \langle\psi^*[a], \psi^*[b], \psi^*[c], \psi^*[u]\rangle \\ &= \psi^*\langle[a], [b], [c], [u]\rangle \neq 0\end{aligned}$$

if we assume that $\langle[a], [b], [c], [u]\rangle \neq 0$, since ψ^* is an injection. However, $(\mathcal{M}_B, d)$ is formal and $\langle[a_M], [b_M], [c_M], [u_M]\rangle$ must vanish as a cohomology class in $H^*(\mathcal{M}_B)$ (again, appeal to the remark on p. 262 in [6]). This contradiction completes the proof. The same argument is obviously valid for Massey products of arbitrary orders.

Proof of Theorem 1. Let $\Omega_{\mathrm{DR}}(G/\Gamma)$ be the de Rham algebra of G/Γ. Consider $(\Lambda L(G)^*, \delta)$ as the algebra of all left-invariant differential forms on G:

$$(\Lambda L(G)^*, \delta) \simeq (\Omega_{\mathrm{DR}}^{\mathrm{inv}}(G), d).$$

Since Γ is a lattice in G and G is simply connected, the natural projection $G \to G/\Gamma$ is a covering with a discrete fiber Γ. Therefore, this covering determines the *injection*

$$(\Omega_{\mathrm{DR}}^{\mathrm{inv}}(G), d) \to \Omega_{\mathrm{DR}}(G/\Gamma).$$

Thus, finally

$$(\Lambda L(G)^*, \delta) \to \Omega_{\mathrm{DR}}(G/\Gamma)$$

is injective. It is known that this morphism induces a *monomorphism* on cohomology (see [19]), and the proof follows from the DGMS-theorem and Lemma 1.

4. The solvmanifold M^8. In this section we prove Theorem 2. The proof will consist of the three lemmas below.

LEMMA 2. *The Lie group G determined by* (1)–(2) *is the semidirect product*

$$G = \mathbb{R}^2 \times_\varphi \mathbb{R}^6,$$

where $\varphi : \mathbb{R}^2 \to \mathrm{GL}(\mathbb{R}^6)$ is the homomorphism defined by the formula

$$\varphi(t,x) = \begin{pmatrix} e^t & 0 & xe^t & 0 & 0 & 0 \\ 0 & e^{-t} & 0 & xe^{-t} & 0 & 0 \\ 0 & 0 & e^t & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-t} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\alpha t & \sin\alpha t \\ 0 & 0 & 0 & 0 & -\sin\alpha t & \cos\alpha t \end{pmatrix}$$

for $(t,x) \in \mathbb{R}^2$.

Proof. Observe that $L(G) = \mathcal{A} \times_{\varphi_*} \mathcal{N}$, where $\mathcal{A} = \mathrm{Span}(A, B), \mathcal{N} = \mathrm{Span}(X_1, \ldots, X_6)$ and $\varphi_* : \mathcal{A} \to \mathrm{End}(\mathcal{N})$ is defined by the rule

$$\varphi_*(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, \quad \varphi_*(B) = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Indeed, since $\mathcal{A}$ and $\mathcal{N}$ are abelian, $\mathrm{End}(\mathcal{N}) = \mathrm{Der}(\mathcal{N})$ and since $[\varphi_*(A), \varphi_*(B)] = 0$, the linear mapping determined by the above matrices is a homomorphism of the Lie algebras $\mathcal{A}$ and $\mathrm{Der}(\mathcal{N})$, which means that $L(G) = \mathcal{A} \times_{\varphi_*} \mathcal{N}$.

From (2), $L(G)$ is solvable but *non-nilpotent* ($[A, X_1] = X_1$ contradicts the nilpotency condition, while the solvability follows from the inclusion $[L(G), L(G)] \subset \mathcal{N}$ and the commutativity of $\mathcal{N}$).

Now, computing the exponential mapping from the standard formula

$$\varphi(\exp tA) = e^{\varphi_*(tA)}, \qquad \varphi(\exp tB) = e^{\varphi_*(tB)}$$

one obtains the expression for $\varphi(t, x)$. The lemma is proved.

Remark. We have also proven that G is solvable, non-nilpotent and *not completely solvable*, since $\operatorname{ad} A$ possesses an imaginary eigenvalue (see the expression of $\operatorname{ad} A = \varphi_*(A)$).

Lemma 3. *Let $S \in \mathrm{SL}(2, \mathbb{Z})$ be a matrix with two distinct real eigenvalues, say, λ and λ^{-1}. Let $P \in \mathrm{GL}(2, \mathbb{R})$ be a matrix such that*

$$PSP^{-1} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}.$$

Put $a_0 = \ln \lambda$. If one takes $\alpha = 2\pi/a_0$ in (2), *then the set*

$$\Gamma = ((a_0\mathbb{Z}) \times \mathbb{Z}) \times_\varphi L, \qquad L = (P(m_1, m_2), P(n_1, n_2), (q_1, q_2)),$$

is a lattice in an appropriate Lie group G.

Proof. It is enough to prove that L is invariant under $\varphi((a_0\mathbb{Z}) \times \mathbb{Z})$. From Lemma 2 one obtains by straightforward calculation

$$\varphi(a_0p_1, p_2) = \begin{pmatrix} PS^{p_1}P^{-1} & p_2PS^{p_1}P^{-1} & 0 \\ 0 & PS^{p_1}P^{-1} & 0 \\ 0 & 0 & E_2 \end{pmatrix}.$$

Applying the latter linear transformation to L, one obtains $\varphi(a_0p_1, p_2)(L) \subset L$ (the calculation is straightforward). The lemma is proved.

Lemma 4. *Under the assumptions of Theorem* 2, *the pair*

$$((\Lambda L(G)^*, \delta), H^*(\Lambda L(G)^*, \delta))$$

has a non-vanishing quadruple Massey product. The graded differential algebra $(\Lambda L(G)^*, \delta)$ is cohomologically symplectic.

Proof. Consider the dual base in $L(G)^*$, which we denote by

$$\{a, b, x_1, x_2, x_3, x_4, x_5, x_6\}.$$

From (4) (which is valid for any $(\Lambda L(G)^*, \delta)$) the following formulas hold:

$$\delta a = \delta b = 0;$$

$$\delta x_1 = -ax_1 - bx_3, \quad \delta x_2 = ax_2 - bx_4, \quad \delta x_3 = ax_3, \quad \delta x_4 = -ax_4,$$
$$\delta x_5 = ax_6, \quad \delta x_6 = -ax_5.$$

The element

$$\omega = ab + x_3x_4 + x_1x_3 + x_2x_4 + x_5x_6$$

shows that $(\Lambda L(G)^*, \delta)$ is symplectic, since $\omega^4 \neq 0$ (it is proportional to $abx_1x_2x_3x_4x_5x_6$). The latter product *cannot be exact*, since a straightforward computation shows that

$$\delta\Big(\mu a x_1x_2x_3x_4x_5x_6 + \nu b x_1x_2x_3x_4x_5x_6 + \sum_{ij} \gamma_{ij} abx_1 \ldots \widehat{x}_i \ldots \widehat{x}_j \ldots x_6\Big) = 0$$

(the vanishing of δ on the first two terms follows from the fact that all δx_j contain either a or b, and the vanishing of δ on the sum of the other terms can be verified directly).

A calculation of the same kind shows that $\delta\omega = 0$ and thus $\Lambda L(G)^*$ is symplectic.

The pair $((\Lambda L(G)^*, \delta), H^*(\Lambda L(G)^*))$ has a non-vanishing quadruple Massey product. To prove this, use a straightforward generalization of the argument in [7]. We reproduce it here for the convenience of the reader. To simplify notations, we denote the Chevalley–Eilenberg complex $(\Lambda L(G)^*, \delta)$ by the symbol (A, δ). If all the triple Massey products in (A, δ) vanish, one can form a *quadruple* Massey product as follows. Let $[\lambda_1] \in H^p(A), [\lambda_2] \in H^q(A), [\lambda_3] \in H^r(A), [\lambda_4] \in H^s(A)$ and assume that

$$\langle [\lambda_1], [\lambda_2], [\lambda_3] \rangle = 0, \qquad \langle [\lambda_2], [\lambda_3], [\lambda_4] \rangle = 0.$$

Then one can define the quadruple Massey product $\langle [\lambda_1], [\lambda_2], [\lambda_3], [\lambda_4] \rangle$, which vanishes if and only if there exist $f_1 \in A^{p+q-1}, f_2 \in A^{q+r-1}, f_3 \in A^{r+s-1}, \mu_1 \in A^{p+q+r-2}, \mu_2 \in A^{q+r+s-2}$ such that

$$\lambda_1 \wedge \lambda_2 = \delta f_1, \tag{5}$$
$$\lambda_2 \wedge \lambda_3 = \delta f_2, \tag{6}$$
$$\lambda_3 \wedge \lambda_4 = \delta f_3, \tag{7}$$
$$\lambda_1 \wedge f_2 + (-1)^{p+1} f_1 \wedge \lambda_3 = \delta \mu_1, \tag{8}$$
$$\lambda_2 \wedge f_3 + (-1)^{q+1} f_2 \wedge \lambda_4 = \delta \mu_2, \tag{9}$$
$$[(-1)^{p+1} \lambda \wedge \mu_2 + (-1)^{q+1} \mu_1 \wedge \lambda_4 + f_1 \wedge f_3] = 0. \tag{10}$$

Take $\lambda_1 = x_3 \wedge x_4, \lambda_2 = \lambda_3 = \lambda_4 = b$. One can check that the triple Massey products $\langle [x_3 \wedge x_4], [b], [b] \rangle$ and $\langle [b], [b], [b] \rangle$ vanish. Therefore, one can define the following quadruple Massey product:

$$\langle [x_3] \wedge [x_4], [b], [b], [b] \rangle.$$

Assume that it vanishes. Then one can rewrite (5)–(10) as follows:

$$x_3 \wedge x_4 \wedge b = \delta f_1, \tag{5'}$$

$$0 = \delta f_2, \tag{6'}$$

$$0 = \delta f_3, \tag{7'}$$

$$x_3 \wedge x_4 \wedge f_2 - f_1 \wedge b = \delta \mu_1, \tag{8'}$$

$$b \wedge f_3 + f_2 \wedge b = \delta \mu_2, \tag{9'}$$

$$[-x_3 \wedge x_4 \wedge \mu_2 + \mu_1 \wedge b + f_1 \wedge f_3] = 0. \tag{10'}$$

Observe that

$$\delta(x_1 \wedge x_4) = (-a \wedge x_1 - b \wedge x_3) \wedge x_4 - x_1 \wedge a \wedge x_4 = -b \wedge x_3 \wedge x_4$$

and from $(5')$ one obtains

$$\delta f_1 = -\delta(x_1 \wedge x_4) \Rightarrow f_1 = -x_1 \wedge x_4 + f_1', \delta f_1' = 0. \tag{11}$$

Substituting (11) to $(4')$ we have

$$x_3 \wedge x_4 \wedge f_2 + x_1 \wedge x_4 \wedge b + f_1' \wedge b = \delta \mu_1, \tag{12}$$

which implies

$$[x_3 \wedge x_4] \cdot [f_2] \in [b] \cdot H^2(A).$$

A straightforward calculation shows that

$$H^2(A) = \{[a \wedge b], [x_3 \wedge x_4], [x_1 \wedge x_4 + x_2 \wedge x_3], [x_5 \wedge x_6]\},$$

which implies

$$f_2 = t \cdot b, \qquad t \in \mathbb{R}. \tag{13}$$

From $(1')$,

$$[x_3 \wedge x_4] \cdot [b] = 0 \Rightarrow [x_3 \wedge x_4] \cdot [f_2] = 0$$

and therefore, from (12),

$$[b \wedge x_1 \wedge x_4] = [b] \cdot [f_1'].$$

The latter equality implies

$$f_1' = \tfrac{1}{2}(x_1 \wedge x_4 + x_2 \wedge x_3) + p \cdot a \wedge b + q \cdot x_3 \wedge x_4. \tag{14}$$

To prove (14) observe that

$$\begin{aligned}\delta(x_1 \wedge x_2) &= (-a \wedge x_1 - b \wedge x_3) \wedge x_2 - x_1 \wedge (a \wedge x_2 - b \wedge x_4)\\ &= b \wedge x_2 \wedge x_3 - b \wedge x_1 \wedge x_4\end{aligned}$$

and thus

$$[b \wedge x_1 \wedge x_4] = [b \wedge x_2 \wedge x_3].$$

From (14) and (12),

$$\begin{aligned}f_1 = &- x_1 \wedge x_4 + \tfrac{1}{2}(x_1 \wedge x_4 + x_2 \wedge x_3) + p \cdot a \wedge b + q \cdot x_3 \wedge x_4\\ = &\tfrac{1}{2}(x_2 \wedge x_3 - x_1 \wedge x_4) + p \cdot a \wedge b + q \cdot x_3 \wedge x_4, \qquad p, q \in \mathbb{R}.\end{aligned} \tag{15}$$

From (13), (7′) and (9′) one obtains

$$f_2 - f_3 = s \cdot b, \qquad s \in \mathbb{R}. \tag{16}$$

To prove this equality, one should notice that $[f_2], [f_3] \in H^1(A) = \mathrm{Span}(a, b)$ and $[f_2 - f_3)] \wedge [b] = 0$ (from (9′)). Thus, (16) is the only possibility for f_2 and f_3. Since $f_3 = (t - s) \cdot b$, one obtains

$$\delta\mu_2 = 0. \tag{17}$$

A straightforward calculation shows that

$$f_1 \wedge f_3 = \delta\big((t - s) \cdot x_2 \wedge \big(q \cdot x_3 - \tfrac{1}{2} \cdot x_1\big)\big). \tag{18}$$

Thus, one can rewrite (8′) and (10′) as follows

$$(t + q) \cdot b \wedge x_3 \wedge x_4 - \tfrac{1}{2} b \wedge (x_2 \wedge x_3 - x_1 \wedge x_4) = \delta\mu_1, \tag{8′′}$$
$$-[x_3 \wedge x_4] \wedge [\mu_2] + [\mu_1 \wedge b] = 0. \tag{10′′}$$

One can check that

$$b \wedge x_3 \wedge x_4 = \delta(-x_1 \wedge x_4), \qquad b \wedge (x_2 \wedge x_3 - x_1 \wedge x_1) = \delta(x_1 \wedge x_2).$$

These equations together with (8′′) imply that there exists a closed element μ_1' such that

$$\mu_1 = -(t + q) x_1 \wedge x_4 - \tfrac{1}{2} x_1 \wedge x_2 + \mu_1'.$$

The latter equation implies

$$[\mu_1 \wedge b] = -(t + q) \cdot [x_1 \wedge x_4 \wedge b] - \tfrac{1}{2}[x_1 \wedge x_2 \wedge b] + [\mu_1'] \wedge [b].$$

Therefore, one can rewrite (10′′) as follows:

$$[-x_3 \wedge x_4] \wedge [\mu_2] - (t + q) \cdot [x_1 \wedge x_4 \wedge b] - \tfrac{1}{2}[x_1 \wedge x_2 \wedge b] + [\mu_1'] \wedge [b] = 0.$$

So, the cohomology class $[x_1 \wedge x_2 \wedge b]$ satisfies the condition

$$[x_1 \wedge x_2 \wedge b] \in [b] \cdot H^2(A) + [x_3 \wedge x_4] \cdot H^1(A),$$

which is generated by $[b \wedge x_1 \wedge x_4], [a \wedge x_3 \wedge x_4], [b \wedge x_5 \wedge x_6]$. This is impossible because the cohomology classes $[x_1 \wedge x_2 \wedge b], [b \wedge x_1 \wedge x_4], [b \wedge x_5 \wedge x_6], [a \wedge x_3 \wedge x_4]$ are linearly independent (this may be verified directly). The proof is complete.

P r o o f o f T h e o r e m 2. Lemma 2 implies (i); (ii) follows from Lemma 3 and the remark after Lemma 2. To prove (iii), we use again the isomorphism between the graded differential algebra $(\Lambda L(G)^*, \delta)$ and the algebra of left-invariant differential forms on G. The inequality $\omega^4 \neq 0$ in case of *left-invariant* differential forms implies the *non-degeneracy* of the appropriate matrix of the 2-form ω. Since $G \to G/\Gamma$ is a covering, the matrix of the differential form induced by ω on G/Γ remains the same and, therefore, remains non-degenerate. Thus, ω determines a symplectic structure on G/Γ. Since Lemma 4 implies that the pair $((\Lambda L(G)^*,\delta), H^*(\Lambda L(G)^*))$ has a non-zero quadruple Massey product, the proof follows from Theorem 1 and the DGMS-theorem (see Section 2).

5. Proof of Theorem 3 and corollaries. Let $(\Lambda X \otimes \Lambda Y, d)$ satisfy the conditions of Theorem 3. In the sequel we assume X and Y to be finite-dimensional and fix the bases $x_1,\ldots,x_k$, $y_1,\ldots,y_l$ of X and Y respectively. We begin with

LEMMA 5. *Let $(\Lambda X\otimes\Lambda Y, d)$ satisfy the conditions of Theorem 3. Suppose that $(\Lambda X \otimes \Lambda Y, d)$ is oriented. Then, necessarily, the derivation d is of the form*

$$dy_j = x^{(j)} + y_{s_j t_j} + \sum_{s\neq j} \widetilde{x}_s y_s + \overline{x}_j y_j, \qquad j = 1,\ldots,l,$$

where

$$\sum_{j=1}^{l} \overline{x}_j = 0, \tag{13}$$

$$x^{(j)} \in \Lambda^2 X, \quad y_{s_j t_j} \in \Lambda^2 Y, \quad \deg(\widetilde{x}_s) = 1, \quad \deg(y_s) = 1, \quad \deg(\overline{x}_j) = 1.$$

Proof. Since $H^n(\Lambda X \otimes \Lambda Y, d) \neq 0$ and since $(\Lambda X \otimes \Lambda Y)$ is the exterior algebra,

$$\dim H^n(\Lambda X \otimes \Lambda Y) = \dim Z^n(\Lambda X \otimes \Lambda Y) = \Lambda^n(X \oplus Y) = 1.$$

Suppose that $\Lambda^{n-1}(X \oplus Y) \neq Z^{n-1}(\Lambda(X \oplus Y))$, so that there exists $v \in \Lambda^{n-1}(X \oplus Y)$ with $dv \neq 0, dv \in \Lambda^n(X \oplus Y)$. By the previous remark $dv = \alpha u$, where u is a generator in $\Lambda^n(X\oplus Y)$ corresponding to the non-zero cohomology class, which is a contradiction.

Thus

$$\Lambda^{n-1}(X \oplus Y) = \Lambda^{k-1,l} + \Lambda^{k,l-1} = Z^{n-1}(X \oplus Y). \tag{14}$$

Therefore

$$u_i = x_1 \ldots \widehat{x}_i \ldots x_k y_1 \ldots y_l \in \Lambda^{k-1,l}, \qquad i = 1,\ldots,k,$$

are cocycles (here and in the sequel $\widehat{x}_i$ denotes the absence of x_i). Calculating du_i explicitly one obtains

$$
\begin{aligned}
du_i = x_1 \dots \widehat{x}_i \dots x_k \Big((-1)^{j-1} \sum_{j=1}^{l} y_1 \dots y_{j-1} dy_j y_{j+1} \dots y_l\Big) \\
&= x_1 \dots \widehat{x}_i \dots x_k \Big\{(-1)^{j-1} \sum_{j=1}^{l} y_1 \dots y_{j-1} \Big(x^{(j)} + y_{s_j t_j} \\
&\quad + \sum_{s \neq j} \widetilde{x}_s y_s + \overline{x}_j y_j\Big) y_{j+1} \dots y_l\Big\} \\
&= \sum_{j=1}^{l} (-1)^{j-1} \Big\{x_1 \dots \widehat{x}_i \dots x_k y_1 \dots y_{j-1} x^{(j)} y_{j+1} \dots y_l \\
&\quad + x_1 \dots \widehat{x}_i \dots x_k y_1 \dots y_{j-1} y_{s_j t_j} y_{j+1} \dots y_l \\
&\quad + x_1 \dots \widehat{x}_i \dots x_k y_1 \dots y_{j-1} \Big(\sum_{s \neq j} \widetilde{x}_s y_s\Big) y_{j+1} \dots y_l \\
&\quad + x_1 \dots \widehat{x}_i \dots x_k y_1 \dots y_{j-1} \overline{x}_j y_j y_{j+1} \dots y_l\Big\} = 0.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
x_1 \dots \widehat{x}_i \dots x_k y_1 \dots y_{j-1} x^{(j)} y_{j+1} \dots y_l \in \Lambda^{k+1,l} &= \{0\}, \\
x_1 \dots x_i \dots x_k y_1 \dots y_{j-1} y_{s_j t_j} y_{j+1} \dots y_l \in \Lambda^{k-1,l+1} &= \{0\}, \\
x_1 \dots \widehat{x}_i \dots x_k y_1 \dots y_{j-1} \Big(\sum_{s \neq j} \widetilde{x}_s y_s\Big) y_{j+1} \dots y_l &= 0.
\end{aligned}
$$

The first two equalities are obvious, the third follows, since $s \in \{1, \dots, j-1, j+1, \dots, l\}$. Finally,

$$
x_1 \dots \widehat{x}_i \dots x_k \Big(\sum_{j=1}^{l} \overline{x}_j\Big) y_1 \dots y_{j-1} y_j y_{j+1} \dots y_l = 0,
$$

which means that

$$
\sum_{j=1}^{l} \overline{x}_j \in \mathrm{Ann}_{\Lambda^{1,0}}(x_1 \dots \widehat{x}_i \dots x_k).
$$

Since the above argument is valid for all i,

$$
\sum_{j=1}^{l} \overline{x}_j \in \bigcap_{i=1}^{k} \mathrm{Ann}_{\Lambda^{1,0}}(x_1 \dots \widehat{x}_i \dots x_k).
$$

Since the sum of $\overline{x}_j$ is of degree 1, (13) follows. Lemma 5 is proved.

LEMMA 6. *Under the conditions of Lemma 5 we have the inclusion*

$$d(\Lambda^{k-2,l}) \subset \Lambda^{k,l-1}. \tag{15}$$

Proof. Take the element $y_1 \ldots y_l$. Then

$$\begin{aligned} d(y_1 \ldots y_l) &= \sum_{i=1}^{l} (-1)^{i-1} y_1 \ldots dy_i \ldots y_l \\ &= \sum_{i=1}^{l} (-1)^{i-1} y_1 \ldots y_{i-1} \Big(x^{(i)} + y_{s_i t_i} + \sum_{s \neq i} \widetilde{x}_s y_s + \overline{x}_i y_i \Big) y_{i+1} \ldots y_l \\ &= \sum_{i=1}^{l} (-1)^{i-1} y_1 \ldots y_{i-1} \overline{x}^{(i)} y_{i+1} \ldots y_l \\ &\quad + \sum_{i=1}^{l} (-1)^{i-1} y_1 \ldots y_{i-1} \overline{x}_i y_i y_{i+1} \ldots y_l \\ &= \sum_{i=1}^{l} (-1)^{i-1} y_1 \ldots y_{i-1} x^{(i)} y_{i+1} \ldots y_l \in \Lambda^{2,l}, \end{aligned}$$

because

$$y_1 \ldots y_{i-1} y_{s_i t_i} y_{i+1} \ldots y_l \in \Lambda^{0,l+1} = \{0\},$$
$$y_1 \ldots y_{i-1} \Big(\sum_{s \neq i} \widetilde{x}_s y_s \Big) y_{i+1} \ldots y_l = 0$$

and

$$\sum_{i=1}^{l} (-1)^{2(i-1)} \overline{x}_i y_1 \ldots y_{i-1} y_i y_{i+1} \ldots y_l = \Big(\sum_{i=1}^{l} \overline{x}_i \Big) y_1 \ldots y_l = 0$$

from Lemma 5. Lemma 6 is proved.

LEMMA 7. *Let $(\Lambda X \otimes \Lambda Y, d)$ satisfy the conditions of Lemma 5. Assume in addition that the Poincaré duality holds. If*

$$H^1(\Lambda X \otimes \Lambda Y, d) = \Lambda^{1,0}, \tag{16}$$

then

$$H^{n-1}(\Lambda X \otimes \Lambda Y, d) = \Lambda^{k-1,l}. \tag{17}$$

Proof. From the Poincaré duality

$$\dim H^{n-1}(\Lambda X \otimes \Lambda Y) = k.$$

From (14),

$$Z^{n-1}(\Lambda X \otimes \Lambda Y) = \Lambda^{k-1,l} + \Lambda^{k,l-1}$$

and by comparing the degrees (the calculation is straightforward, although it is necessary to use the *minimality* of $(\Lambda Y, \delta)$ which guarantees that $y_{s_j t_j}$

do not contain expressions with y_j):

$$d(\Lambda^{k-1,l-1}) \subset \Lambda^{k,l-1}, \quad d(\Lambda^{k,l-2}) \subset \Lambda^{k,l-1}.$$

From Lemma 6 (formula (15)), $d(\Lambda^{k-2,l}) \subset \Lambda^{k,l-1}$ and therefore, the subspace B^{n-1} of coboundaries is contained in $\Lambda^{k,l-1}$. Thus,

$$\dim H^{n-1}(\Lambda X \otimes \Lambda Y) = \dim(\Lambda^{k-1,l} + \Lambda^{k,l-1}) - \dim B^{n-1} = k.$$

Therefore, since $\dim \Lambda^{k,l-1} = l, \dim \Lambda^{k-1,l} = k$, the assumption $B^{n-1} \neq \Lambda^{k,l-1}$ would imply

$$\dim H^{n-1} = \dim Z^{n-1} - \dim B^{n-1} = k + l - \dim B^{n-1} > k.$$

Thus

$$B^{n-1} = \Lambda^{k,l-1} \tag{18}$$

and Lemma 7 is proved.

Proof of Theorem 3. Since there exists a model $(A^*(E), d)$ which admits a structure of a twisted tensor product, it remains to prove that this algebra possesses the algebraic properties (i)–(iv) stated in Theorem 3. The first property follows from the existence of a Kählerian structure on E.

Suppose that $(\Lambda X \otimes \Lambda Y, d)$ satisfies the conditions of Lemma 7 and dy_j are linearly independent. Then (16) holds, since $d|_X = 0$ and $d|_Y$ is one-to-one. If some dy_j are linearly dependent, by the obvious base change one can assume that for the new basis vectors $\overline{y}_j$, $d\overline{y}_j = 0$ and thus $\overline{y}_j \in X$. Therefore one can assume the linear independence of dy_j without loss of generality and (17) follows. Now, the condition (ii) follows from Lemma 7.

To prove (iii)–(iv), take $y^* \in Y^*$ and observe that

$$(i(y^*)\omega)\omega^{m-1} = \frac{1}{m} i(y^*)\omega^m \in \Lambda^{k,l-1},$$

since $\omega^m \in \Lambda^n = \Lambda^{k+l}$. From (18), $i(y^*)\omega^m$ is a coboundary. Therefore, $i(y^*)\omega \notin \Lambda^{1,0}$ because otherwise the hard Lefschetz condition and Lemma 7 would give a contradiction. Thus there exists $\overline{y}^* \in Y^*$ such that $\omega(y^*, \overline{y}^*) \neq 0$ and therefore $\omega|_{Y^* \times Y^*}$ is non-degenerate. Observe that using non-degeneracy, one can choose X^* to be ω-orthogonal to Y^*. Using the appropriate identifications of exterior algebra elements and alternating forms one obtains the form of the Lefschetz element, that is, (iii) and, as a consequence, (iv). Theorem 3 is proved.

Proof of Corollary 1. Observe that the assumptions of Corollary 1 imply the possibility of applying the Thomas theorem to the fibration $F \to E \to T^k$. Since, obviously,

$$\mathcal{M}_{T^k} = (\Lambda X, 0), \quad X = \mathrm{Span}(x_1, \ldots, x_k), \quad \deg(x_i) = 1, \quad i = 1, \ldots, k,$$

and

$$\mathcal{M}_F = (\Lambda Y, \delta), \quad Y = \mathrm{Span}(y_1, \ldots, y_r), \quad \deg(y_j) = 1, \quad j = 1, \ldots, r,$$

the rational model $(A^*(E), d)$ is a twisted tensor product

$$(A^*(E), d) = (\Lambda X \otimes_\tau \Lambda Y, d)$$

where d satisfies the assumptions of Theorem 3.

Coming back to the solvmanifold case, let N be the nilradical of G. From the Mostow theorem [16], $N \cap \Gamma$ is a lattice in N. The latter fact is equivalent to the closedness of $N\Gamma$ in G. Therefore, one can construct the *Mostow bundle*

$$\Gamma N/\Gamma = N/N \cap \Gamma \to G/\Gamma \to G/\Gamma N.$$

It is well known that $G/\Gamma N$ is a *torus* and since $\Gamma \cap N$ is a lattice, the fiber of the considered fibration is a *compact nilmanifold* and thus has the homotopy type of $K(\pi, 1)$ with *nilpotent* π. Since we assumed the nilpotency of the $\pi_1(G/\Gamma N)$-action on the cohomology of the fiber, the conditions of Theorem 3 are satisfied and there exists a free graded differential algebra, which is a rational model of G/Γ and satisfies the algebraic properties (i)–(iv) proved in Theorem 3. This model of course can be represented in the form $\mathcal{M}_{G/\Gamma N} \otimes_\tau \mathcal{M}_{N/N\cap\Gamma}$.

Proof of Corollary 2. Analyzing the proof of Theorem 3 one can notice that conditions (ii)–(iv) are valid for any twisted tensor product of the form given in Theorem 3 and satisfying the Lefschetz condition. Following [4] we assign to G/Γ a model $\overline{A}^*(G/\Gamma)$ which possesses properties (i)–(iv) of $A^*(G/\Gamma)$, although this new model in general differs from the latter. We observe that for completely solvable Lie groups,

$$H^*(G/\Gamma) \simeq H^*(L(G)) = H^*(\Lambda L(G)^*, \delta)$$

by Hattori's theorem [11]. Since

$$(\Lambda L(G)^*, \delta) \simeq (\Omega^{\mathrm{inv}}_{\mathrm{DR}}(G), d),$$

the cohomology class of a Kählerian form has a representative $\omega \in (\Lambda^2 L(G)^*, \delta)$. Since $\omega^m \neq 0$ and ω is an alternating 2-form on $L(G)$, it is non-degenerate. Therefore, $(\Lambda L(G)^*, \delta)$ *can be taken as a model of* G/Γ.

Take $\mathcal{N} = [L(G), L(G)]$ and decompose $L(G)^*$ as a vector space sum $L(G)^* = \mathcal{A}^* \oplus \mathcal{N}^*$, where

$$\mathcal{A}^* = \{\theta \in L(G)^* : \theta(\mathcal{N}) = 0\}, \quad \mathcal{N}^* = \{\alpha \in L(G)^* : \alpha(\mathcal{A}) = 0\}$$

(here $\mathcal{A}$ is an arbitrary complement to $\mathcal{N}$). Now, taking into account the equalities (the first follows from (4))

$$\delta\beta(U, V) = -\beta([X, Y]), \quad \mathcal{N} = [L(G), L(G)]$$

we have $\delta(\Lambda\mathcal{A}^*) = 0$. Since $\mathcal{N}$ is contained in the nilradical of $L(G)$, the ideal $\mathcal{N}$ is nilpotent and therefore $(\Lambda\mathcal{N}^*, \delta|_{\Lambda\mathcal{N}^*})$ is a *minimal graded differential algebra*. Therefore,

$$(\bar{A}^*(G/\Gamma) = (\Lambda L(G)^*, \delta) = (\Lambda\mathcal{A}^*, \delta = 0) \otimes_\tau (\Lambda\mathcal{N}^*, \delta|_{\Lambda\mathcal{N}^*})$$

(that is, the model for G/Γ can be represented as a twisted tensor product satisfying the condition (ii) of Theorem 3, the Lefschetz condition and the conditions of Lemmas 5–7 (the latter is guaranteed by the minimality of the second term in the twisted tensor product). Finally, $(\bar{A}^*(G/\Gamma), d)$ has the properties (i)–(iv) of Theorem 3 and as a consequence, the properties (ii), (iv) and (v) of Corollary 2.

Remark. These properties were proved in [4] directly for the particular case of $(\Lambda L(G)^*, \delta)$ with completely solvable G.

To finish the proof it is enough to repeat the argument in [4] (parts (i), (iii) and (vi). Here, of course, the proof is based on the particular choice of $(\bar{A}^*(G/\Gamma), d)$. For example, the commutativity of $\mathcal{A}$ follows from (v) of Theorem 2 and the formula

$$\begin{aligned} 0 &= \delta\omega(A,B,V) \\ &= -\omega([A,B],V) + \omega([A,V],B) - \omega([B,V],A), \qquad A,B \in \mathcal{A},\ V \in \mathcal{N} \end{aligned}$$

(see the proof in [4] for details). The corollary is proved.

Remark. There are examples of solvmanifolds for which the associated Mostow bundle satisfies the nilpotency condition for $\pi_1(G/N\Gamma)$. Consider the semidirect product $G = \mathbb{R}^2 \times_\varphi \mathbb{R}^4$ determined by the homomorphism $\varphi : \mathbb{R}^2 \to \mathrm{GL}(\mathbb{R}^4)$ of the form

$$\varphi(x,t) = \begin{pmatrix} \cos\pi 2t & \sin\pi 2t & x\cos\pi t & x\sin\pi t \\ -\sin\pi 2t & \cos\pi 2t & -x\sin\pi t & x\cos\pi t \\ 0 & 0 & \cos\pi 2t & \sin\pi 2t \\ 0 & 0 & -\sin\pi 2t & \cos\pi 2t \end{pmatrix}.$$

One can show by exactly the same method as in the proof of Theorem 2 that G is a solvable non-nilpotent Lie group possessing a lattice. The corresponding Mostow bundle is of the form $T^4 \to G/\Gamma \to T^2$. The action of $\pi_1(T^2)$ on $H^*(T^4)$ is nilpotent. To prove this, it is enough to combine several observations. First, since the generators of the fundamental group of the base can be represented as *submanifolds* in T^2, one can reduce the study of their action to the action of the same elements on the cohomology of the fiber in the corresponding *portion* of the Mostow bundle over S^1. Since the Mostow bundle is a smooth *Steenrod fiber bundle* [22] with structure group $N\Gamma/H$ (H is a maximal normal subgroup contained in Γ), its portions over S^1 can be reduced to bundles with *discrete* structure group (say, introducing a connection in the corresponding principal fiber bundles and using the holonomy theorem). But for Steenrod fiber bundles with discrete groups the fundamental group of the base acts on the cohomology of the fiber via the representation in the structure group of the bundle acting on the fiber [22]. Calculating now this action for our particular Mostow bundle, one can notice that a non-trivial action may come only from the elements of the lattice Γ generating non-trivial classes in $N\Gamma/H$. But $\varphi(m,n)$ is a *unipotent* matrix for all $(m,n)\in\mathbb{Z}^2$, which implies the nilpotency of the action on $H^*(N/N\cap\Gamma)$. Thus, we have constructed a solvmanifold for which the corresponding Mostow bundle satisfies the nilpotency condition. However, this example is not very interesting, since the total space of this bundle is diffeomorphic to a nilmanifold. The author failed to construct an example of a "pure" solvmanifold with the required property. The above argument might work for non-abelian fibers, since the torsion in $\Gamma\cap N/[\Gamma\cap N,\Gamma\cap N]=(\Gamma\cap N)_{\mathrm{ab}}$ might "kill" some generators in cohomology, e.g. in $H^1(N/N\cap\Gamma)\simeq(\Gamma\cap N)_{\mathrm{ab}}\otimes\mathbb{Q}$.

On the other hand, an example in [17] shows that there are solvmanifolds whose Mostow bundles do not satisfy the nilpotency condition.

Acknowledgements. This work was supported by the Polish KBN. The author is grateful to Martin Saralegi for the opportunity of reading the preprint [7] before its publication and to the referee for valuable advice and improvements. The final version of this paper was finished during the author's stay in Mathematisches Forschungsinstitut (Oberwolfach). The Volkswagen-Stiftung research grant "Research in Pairs" is greatly appreciated.

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Received 7 June 1995;
revised 2 October 1996