COLLOQUIUM MATHEMATICUM

VOL. 73 1997 NO. 2

ON CERTAIN HARMONIC MEASURES ON THE UNIT DISK

BY

DIMITRIOS BETSAKOS (SAINT LOUIS, MISSOURI)

We will denote by $\operatorname{clos} E$ the closure of the set $E \subset \mathbb{C}$ and by $\omega(z, E, D)$ the harmonic measure at z of the set $\operatorname{clos} E \cap \operatorname{clos} D$ relative to the component of $D \setminus \operatorname{clos} E$ that contains z.

Beurling in his dissertation (see [1], pp. 58–62) proved the following theorem:

THEOREM 1 (Beurling's shove theorem). *Let K be the union of a finite number of intervals on the radius $(0,1)$ of the unit disk $\mathbb{D}$. Let l be the total logarithmic measure of K. Then*

$$\omega(0, K, \mathbb{D}) \ge \omega(0, K_0, \mathbb{D}) = \frac{2}{\pi} \arcsin \frac{e^l - 1}{e^l + 1}, \tag{1.1}$$

where K_0 stands for the interval $(e^{-l}, 1)$. Equality occurs only for the case $K = K_0$.

This is a natural counterpart to the Beurling–Nevanlinna projection theorem. Nevanlinna's book [7], pp. 108–110, contains a proof of both theorems. Nevanlinna also remarks that the proof of the shove theorem gives also the following result.

THEOREM 2. *Let K be as above and let m be the total length of K. Then*

$$\omega(0, K, \mathbb{D}) \ge \omega(0, K_1, \mathbb{D}), \tag{1.2}$$

where $K_1 = [1-m, 1]$.

In 1989 Essèn and Haliste [4] proved some generalizations of Beurling's shove theorem. In particular, they proved the following theorem.

THEOREM 3 (Essèn and Haliste). *Let K be the union of a finite number of closed intervals on the diameter $[-1,1]$ of $\mathbb{D}$ having total length $2m$. Assume that $-K = K$, i.e. K is symmetric with respect to the imaginary axis. Then*

$$\omega(0, K, \mathbb{D}) \ge \omega(0, K^*, \mathbb{D}), \tag{1.3}$$

where $K^* = [-1, -1+m] \cap [1-m, 1]$.

1991 *Mathematics Subject Classification*: 30C85, 31A15.

Segawa [10] made the following conjecture:

CONJECTURE 1 (Segawa). *Let* $K \subset [-1,1]$ *be the union of a finite number of intervals of total length* $2m$. *Then*

$$\omega(0,K,\mathbb{D}) \geq \omega(0,K^*,\mathbb{D}), \tag{1.4}$$

where $K^* = [-1,-1+m] \cap [1-m,1]$.

Thus Essèn and Haliste proved Segawa's conjecture in the special case where K is symmetric with respect to the imaginary axis. Segawa's conjecture may be viewed as a symmetrization result because the Steiner symmetrization of $\mathbb{D}\setminus K$ is congruent to $\mathbb{D}\setminus K^*$. In this note we prove some special cases of the conjecture as well as some results on extremal distances and harmonic measures associated with the geometric configuration in Segawa's conjecture.

First we prove the conjecture in the special case where K consists of two slits and $\mathbb{D}\setminus K$ is a simply connected domain. In this case the harmonic measures of K and K^* can be computed explicitly. The proof is based on direct calculations and the following lemma whose easy proof is omitted.

LEMMA 1. *Let* $\Phi,\Psi,\Pi_1,\Pi_2 \in \mathbb{R}$. *If* $0<\Pi_1\leq\Pi_2$ *and* $\Phi+\Psi\geq 0$, *then*

$$\frac{\Phi}{\Pi_1}+\frac{\Psi}{\Pi_2}\geq 0. \tag{1.5}$$

THEOREM 4. *For* $0<b<1$, $0<a<1$, *let* $K_1=[-1,-b]$, $K_2=[a,1]$, $K=K_1\cup K_2$, $K_1^*=[-1,-(a+b)/2]$, $K_2^*=[(a+b)/2,1]$, $K^*=K_1^*\cup K_2^*$. *Then*

$$\omega(0,K,\mathbb{D}) \geq \omega(0,K^*,\mathbb{D}). \tag{1.6}$$

Equality holds if and only if $K=K^*$ (*i.e.* $a=b$).

Proof. A long but simple calculation (using various conformal mappings) gives

$$\omega(0,K,\mathbb{D}) = \frac{1}{\pi}\cos^{-1}\left[\frac{4-(a+1/a)+(b+1/b)}{a+1/a+b+1/b}\right] + 1 - \frac{1}{\pi}\cos^{-1}\left[\frac{-4-(a+1/a)+(b+1/b)}{a+1/a+b+1/b}\right] \tag{1.7}$$

(see [12], exercise 273). Based on (1.7) we fix a real number $l\in(0,2)$ and we define the function

$$f(x) = 1-\frac{1}{\pi}\cos^{-1}\left[\frac{-4-(x+1/x)+l-x+1/(l-x)}{x+1/x+l-x+1/(l-x)}\right] + \frac{1}{\pi}\cos^{-1}\left[\frac{4-(x+1/x)+l-x+1/(l-x)}{x+1/x+l-x+1/(l-x)}\right]$$

for $x\in(0,l)$.

Thus to prove (1.6) it suffices to prove that f attains its minimum for $x = l/2$. It is easy to see that $f(x) = f(l-x)$. So we only need to prove that

$$f'(x) < 0 \quad \text{for all } x \in (0, l/2), \quad \text{and} \quad f'(l/2) = 0. \tag{1.8}$$

For this purpose we define, for $x \in (0, l/2]$,

$$A = A(x) = x + 1/x, \qquad B = B(x) = l - x + 1/(l - x),$$
$$g(x) = \cos^{-1}\left[\frac{-4 - A + B}{A + B}\right] - \cos^{-1}\left[\frac{4 - A + B}{A + B}\right].$$

It suffices to prove that $g'(x) > 0$ for all $x \in (0, l/2)$, and $g'(l/2) = 0$. So we differentiate and after some easy calculations we find that $g'(x) > 0$ if and only if

$$\frac{AB' - A'B - 2A' - 2B'}{(AB + 2A - 2B - 4)^{1/2}} + \frac{-AB' + A'B - 2A' - 2B'}{(AB + 2B - 2A - 4)^{1/2}} > 0. \tag{1.9}$$

Now we will apply Lemma 1. Put

$$\Phi = AB' - A'B - 2A' - 2B', \qquad \Pi_1 = (AB + 2A - 2B - 4)^{1/2},$$
$$\Psi = -AB' + A'B - 2A' - 2B', \qquad \Pi_2 = (AB + 2B - 2A - 4)^{1/2}.$$

It is easy to check that the assumptions of the lemma are satisfied. So by the lemma we conclude that $g'(x) > 0$ for all $x \in (0, l/2)$. We can check directly that $g'(l/2) = 0$.

We will prove Segawa's conjecture in the "totally non-symmetric case" $K \cap (-K) = \emptyset$. We need the following lemma.

LEMMA 2. *Let* $0 < a < 1$, $K = [-1, -a] \cup [a, 1]$, $K' = [-1, -(1 + a)/2] \cup [(1 + a)/2, 1]$. *Then*

$$\tfrac{1}{2}\omega(0, K, \mathbb{D}) > \omega(0, K', \mathbb{D}). \tag{1.10}$$

Proof. The proof is again a direct calculation. By [12], exercise 273, we have

$$\pi\omega(0, K, \mathbb{D}) = 2\cos^{-1}\left[\frac{2}{a + 1/a}\right], \tag{1.11}$$

$$\pi\omega(0, K', \mathbb{D}) = 2\cos^{-1}\left[\frac{2}{(1 + a)/2 + 2/(1 + a)}\right]. \tag{1.12}$$

Let

$$f(x) = 2\cos^{-1}\left[\frac{2}{x + 1/x}\right] - 4\cos^{-1}\left[\frac{2}{(1 + x)/2 + 2/(1 + x)}\right], \qquad x \in (0, 1).$$

It suffices to prove that $f(x) > 0$ for all $x \in (0, 1)$. This can be easily proven by a little calculus.

THEOREM 5. *Let K be the union of a finite number of intervals on the diameter $[-1, 1]$ of $\mathbb{D}$. Let m be the total length of these intervals. Assume* *that* $(-K)\cap K=\emptyset$. *Then*

$$\omega(0,K,\mathbb{D})>\omega(0,K^*,\mathbb{D}), \tag{1.13}$$

where $K^*=[-1,-1+m/2]\cup[1-m/2,1]$.

Proof. By applying Øksendal's reflection lemma [9] (see also Baernstein's lemma in [5]) we see that

$$\omega(0,K,\mathbb{D})>\omega(0,K_1,\mathbb{D}), \tag{1.14}$$

where $K_1=-(K\cap\mathbb{H})\cup(-\mathbb{H}\cap K)$. Here $\mathbb{H}$ is the right half-plane. Note that the sets $-(K\cap\mathbb{H})$ and $(-\mathbb{H}\cap K)$ are disjoint and lie on the radius $[-1,0)$ of the unit disk.

Now Beurling's shove theorem yields $\omega(0,K_1,\mathbb{D})\ge\omega(0,K_2,\mathbb{D})$, where $K_2=[-1,-1+m]$.

Let $u(z)=\omega(z,K_2,\mathbb{D})$ and $v(z)=(u(z)+u(\overline{z}))/2$. Then by the maximum principle

$$u(0)=v(0)>\tfrac{1}{2}\omega(0,K_3,\mathbb{D}), \tag{1.15}$$

where $K_3=[-1,-1+m]\cup[1-m,1]$. Now Lemma 2 finishes the proof of the theorem.

We will now prove some results for extremal length. If E and F are two compact sets lying in $\operatorname{clos}D$, where D is a domain in the plane, then we denote by $\lambda(E,F,D)$ the extremal distance between E and F relative to $D\setminus F\setminus E$. See [1], p. 361, or [8] for the definition and main properties of extremal length.

THEOREM 6. *Let* $0<b<a<1$, $K_1=[-1,-b]$, $K_2=[a,1]$, $K_1^*=[-1,-(a+b)/2]$, $K_2^*=[(a+b)/2,1]$, $\lambda=\lambda(K_1,K_2,\mathbb{D})$, $\lambda^*=\lambda(K_1^*,K_2^*,\mathbb{D})$. *Then* $\lambda>\lambda^*$.

Proof. The proof is based on an explicit calculation of λ and λ^*. We reflect K_1,K_2,K_1^*,K_2^* in the unit circle and let E_1 (resp. E_2,E_1^*,E_2^*) be the union of K_1 (resp. K_2,K_1^*,K_2^*) with its reflection. Then because of symmetry (see [8], §2.12) we have

$$\lambda/2=\lambda(E_1,E_2,\mathbb{C})=:\lambda_1 \quad\text{and}\quad \lambda^*/2=\lambda(E_1^*,E_2^*,\mathbb{C})=:\lambda_1^*. \tag{1.16}$$

Now λ_1 and λ_1^* can be computed in terms of the Grötzsch-ring function ν (see [8], §2.16). By using the fact that ν is a decreasing function we find easily that

$$\lambda_1^*<\lambda_1 \Leftrightarrow \frac{(1/b-b)(1/a-a)}{(a+1/b)(b+1/a)}<\frac{(2/(a+b)-(a+b)/2)^2}{((a+b)/2+2/(a+b))^2}. \tag{1.17}$$

Some additional easy calculations show that the right hand inequality holds. So $\lambda_1^*<\lambda_1$ and (1.16) implies $\lambda^*<\lambda$.

THEOREM 7. *Let $K_1 \subset [-1,0)$ be a finite union of closed intervals of total length m_1. Let $K_2 \subset (0,1]$ be a finite union of closed intervals of total length m_2. Then*

$$\lambda(K_1, K_2, \mathbb{D}) \le \lambda(K_1', K_2', \mathbb{D}), \tag{1.18}$$

where $K_1' = [-1, -1+m_1]$ and $K_2' = [1-m_2, 1]$.

If in the above theorem we replace $\mathbb{D}$ by $\mathbb{C}$ then we obtain a special case of a theorem proven by Tamrazov [11]. The proof of Theorem 7 is similar to Dubinin's proof of Tamrazov's theorem (see [3]).

Proof of Theorem 7. For any $A \subset \mathbb{C}$, let $\widehat{A}$ denote the reflection of A in the unit circle. Because of symmetry we have

$$\lambda(K_1 \cup \widehat{k}_1, K_2 \cup \widehat{k}_2, \mathbb{C}) = \tfrac{1}{2}\lambda(K_1, K_2, \mathbb{D}). \tag{1.19}$$

Now by making successive polarizations with respect to suitable vertical axes we get (see [3])

$$\lambda(K_1 \cup \widehat{k}_1, K_2 \cup \widehat{k}_2, \mathbb{C}) \le \lambda(K_1' \cup \widehat{k}_1, K_2' \cup \widehat{k}_2, \mathbb{C}). \tag{1.20}$$

We reflect $K_1' \cup \widehat{k}_1$ and $K_2' \cup \widehat{k}_2$ in $\partial\mathbb{D}$ and by symmetry we have

$$\lambda(K_1' \cup \widehat{k}_1, K_2' \cup \widehat{k}_2, \mathbb{C}) = \lambda(\widehat{k}_1' \cup K_1, \widehat{k}_2' \cup K_2, \mathbb{C}). \tag{1.21}$$

Again successive polarizations with respect to the same axes as previously give

$$\lambda(\widehat{k}_1' \cup K_1, \widehat{k}_2' \cup K_2, \mathbb{C}) \le \lambda(K_1' \cup \widehat{k}_1', K_2' \cup \widehat{k}_2', \mathbb{C}). \tag{1.22}$$

Finally, by symmetry,

$$\lambda(K_1' \cup \widehat{k}_1', K_2' \cup \widehat{k}_2', \mathbb{C}) = \tfrac{1}{2}\lambda(K_1', K_2', \mathbb{D}). \tag{1.23}$$

Now (1.19)–(1.23) give (1.18).

COROLLARY 1. *Let K_1, K_2 be as in Theorem 7. Then*

$$\lambda(K_1, K_2, \mathbb{D}) \le \lambda(K_1^*, K_2^*, \mathbb{D}), \tag{1.24}$$

where $K_1^* = [-1, -1+(m_1+m_2)/2]$ and $K_2^* = [1-(m_1+m_2)/2, 1]$.

Proof. By Theorems 7 and 6 we have

$$\lambda(K_1, K_2, \mathbb{D}) \le \lambda(K_1', K_2', \mathbb{D}) \le \lambda(K_1^*, K_2^*, \mathbb{D}). \tag{1.25}$$

Theorems 7 and 6 suggest the following conjecture.

CONJECTURE 2. *Let K_1, K_2, K_1', K_2' be as in Theorem 7. Then*

$$\omega(0, K_1 \cup K_2, \mathbb{D}) \ge \omega(0, K_1' \cup K_2', \mathbb{D}). \tag{1.26}$$

This conjecture implies Segawa's conjecture because of Theorem 4.

We return to results on harmonic measure.

THEOREM 8. *Let $a \in (0,1)$ and K be the union of a finite number of intervals lying in $(a,1)$ and having total length m. Let $D = \mathbb{D}\setminus[-1,a]\setminus[a,1]$. Then*

$$\omega(0,-K^*\cup K^*,D) \le \omega(0,-K\cup K,D) \le \omega(0,-K_*\cup K_*,D), \tag{1.27}$$

where $K^* = [1-m,1]$ and $K_* = [a,a+m]$.

Proof. The proof is a direct calculation. Let F be the conformal mapping of D onto $\mathbb{D}$ with $F(0)=0$ and $F'(0)>0$. The mapping $w=F(z)$ is given by the formula

$$w + \frac{1}{w} = 2\,\frac{z+1/z}{a+1/a}, \qquad z\in D,\ w\in\mathbb{D}. \tag{1.28}$$

The extension of F on the boundary maps $t\in[a,1]$ to $e^{i\theta}$, $\theta\in[0,\pi]$ (t is actually the prime end "approached from above"). By (1.28) we easily find

$$\theta = \theta(t) = \cos^{-1}\left[\frac{t+1/t}{a+1/a}\right]. \tag{1.29}$$

Now we can easily prove that $\theta''(t) < 0$ for all $t\in[a,1]$, and this implies both inequalities in (1.27).

Now we will use Theorem 8 to prove the following result that involves rearrangements of functions. See [6] for the basic facts about rearrangements.

THEOREM 9. *Let $a\in(0,1)$ and $D = \mathbb{D}\setminus[a,1]\setminus[-1,-a]$. Let f be a positive, bounded, Borel function on $[a,1]$ and let H_f be the harmonic function in D with boundary values $H_f = 0$ on $\partial\mathbb{D}$, $H_f = f$ on $[a,1]$ and $H_f(t) = f(-t)$ for $t\in[-1,-a]$. Let $\widetilde{f}$ be the increasing rearrangement of f and let $H_{\tilde f}$ be the harmonic function in D with boundary values $H_{\tilde f} = 0$ on $\partial\mathbb{D}$, $H_{\tilde f} = \widetilde{f}$ on $[a,1]$ and $H_{\tilde f}(t) = \widetilde{f}(-t)$ for $t\in[-1,-a]$. Then $H_{\tilde f}(0) \le H_f(0)$.*

Proof. First note that Theorem 8 is a special case of Theorem 9. Theorem 8 says that Theorem 9 holds if f is the characteristic function of the union of a finite number of intervals on $[a,1]$. Now suppose that f is a simple function s. Let $\{c_1, c_1+c_2, c_1+c_2+c_3,\ldots,c_1+c_2+\ldots+c_n\}$ be the set of values of s ($c_j\ge 0$, $j=1,\ldots,n$). Thus

$$s(x) = c_1\chi_{E_1}(x) + \ldots + c_n\chi_{E_n}(x), \tag{1.30}$$

where $E_n\subset E_{n-1}\subset\ldots\subset E_1 := [a,1]$. ($E_j$ may be assumed to be a finite union of intervals, $j=1,\ldots,n$.) We also write

$$s_i(x) = c_1\chi_{E_1}(x) + \ldots + c_i\chi_{E_i}(x), \qquad i=1,\ldots,n. \tag{1.31}$$

So $s_n(x) = s(x)$.

Let u_i, $i = 1, \ldots, n$, be the harmonic function in D with boundary values $u_i = 0$ on $\partial\mathbb{D}$, $u_i(t) = s_i(t)$, $t \in [a, 1]$, and $u_i(t) = s_i(-t)$, $t \in [-1, -a]$. So $u_n = H_s$. The increasing rearrangement of s is given by

$$\widetilde{s}(x) = c_1\chi_{E_1^*}(x) + \ldots + c_n\chi_{E_n^*}(x), \tag{1.32}$$

where $E_i^* = [1 - |E_i|, 1]$, $i = 1, \ldots, n$.

Similarly, the increasing rearrangement of s_i is given by

$$\widetilde{s}_i(x) = c_1\chi_{E_1^*}(x) + \ldots + c_i\chi_{E_i^*}(x), \qquad i = 1, \ldots, n. \tag{1.33}$$

Let u_i^*, $i = 1, \ldots, n$, be the harmonic function in D with boundary values $u_i^* = 0$ on $\partial\mathbb{D}$, $u_i^*(t) = s_i^*(t)$, $t \in [a, 1]$, and $u_i^*(t) = s_i^*(-t)$, $t \in [-1, -a]$. It is easy to see that for $z \in D$ we have

$$u_n(z) - u_{n-1}(z) = c_n\omega(z, -E_n \cup E_n, D), \tag{1.34}$$

$$u_n^*(z) - u_{n-1}^*(z) = c_n\omega(z, -E_n^* \cup E_n^*, D). \tag{1.35}$$

So by Theorem 8 we get

$$u_n(0) - u_{n-1}(0) \geq u_n^*(0) - u_{n-1}^*(0). \tag{1.36}$$

Similarly we prove that

$$u_{n-1}(0) - u_{n-2}(0) \geq u_{n-1}^*(0) - u_{n-2}^*(0), \tag{1.37}$$

$$u_{n-2}(0) - u_{n-3}(0) \geq u_{n-2}^*(0) - u_{n-3}^*(0), \tag{1.38}$$

$$\vdots$$

$$u_2(0) - u_1(0) \geq u_2^*(0) - u_1^*(0), \tag{1.39}$$

$$u_1(0) \geq u_1^*(0). \tag{1.40}$$

We add the above inequalities and get

$$H_s(0) = u_n(0) \geq u_n^*(0) = H_s^*(0). \tag{1.41}$$

So we proved the theorem in case f is a simple function. The general case can be easily proved by using standard approximating theorems: Simple functions approximate a Borel function; the rearrangements of these simple functions approximate the rearrangement of the Borel function; harmonic extensions approximate the corresponding harmonic extension.

Remark. We can prove an analogous result for decreasing rearrangements. In this case we use the last of the inequalities in (1.27). Actually, the proof of Theorem 8 shows that a "continuous symmetrization" theorem holds. Roughly speaking, this theorem would say: "If we move K continuously to the right, the harmonic measure of $-K \cup K$ decreases". Then using this result we can prove a continuous version of Theorem 9. The increasing rearrangement should be replaced by the appropriate continuous rearrangement of f (see [2]).

I would like to thank Professor Albert Baernstein for helpful suggestions.

REFERENCES

[1] A. Beurling, *The Collected Works of Arne Beurling. Vol.* 1, *Complex Analysis*, L. Carleson *et al.* (eds.), Birkhäuser, 1989.
[2] F. Brock, *Continuous Steiner-symmetrization*, Math. Nachr. 172 (1995), 25–48.
[3] V. N. Dubinin, *Capacities and geometric transformations of subsets in n-space*, Geom. Funct. Anal. 3 (1993), 342–369.
[4] M. Essèn and K. Haliste, *On Beurling's theorem for harmonic measure and the rings of Saturn*, Complex Variables Theory Appl. 12 (1989), 137–152.
[5] J. L. Fernandez, J. Heinonen and O. Martio, *Quasilines and conformal mappings*, J. Anal. Math. 52 (1989), 117–132.
[6] B. Kawohl, *Rearrangements and Convexity of Level Sets in PDE*, Lecture Notes in Math. 1150, Springer, 1985.
[7] R. Nevanlinna, *Analytic Functions*, Springer, 1970 (first ed. 1936).
[8] M. Ohtsuka, *Dirichlet Problem*, *Extremal Length and Prime Ends*, Van Nostrand, 1970.
[9] B. Øksendal, *Projection estimates for harmonic measure*, Ark. Mat. 21 (1983), 191–203.
[10] S. Segawa, *Martin boundaries of Denjoy domains*, Proc. Amer. Math Soc. 103 (1988), 177–183.
[11] P. M. Tamrazov, *Condenser capacities. A method of intermixing loads*, Mat. Sb. 115 (1981), 40–73 (in Russian); English transl.: Math. USSR-Sb. 43 (1982), 33–62.
[12] L. I. Volkovyskiĭ, G. L. Lunts and I. G. Aramanovich, *A Collection of Problems on Complex Analysis*, Dover, 1991.

Department of Mathematics
Washington University
Saint Louis, Missouri 63130
U.S.A.
E-mail: db@math.wustl.edu

Current address:
Anatolikis Romilias 7
Kozani 50100, Greece

Received 26 March 1996;
revised 13 September 1996