COLLOQUIUM MATHEMATICUM

VOL. 73 1997 NO. 2

ON THE WITT RINGS OF FUNCTION FIELDS OF QUASIHOMOGENEOUS VARIETIES

BY

PIOTR JAWORSKI (WARSZAWA)

1. Introduction. Let V be a quasihomogeneous normal variety (i.e. a quasihomogeneous cone). The aim of this paper is to describe the Witt ring of the function field of V in terms of the second residue homomorphisms associated with the subvarieties and the resolution data of V.

Let K be a field of characteristic not equal to 2 and $\widehat{K}$ its algebraic closure. Let V be a normal irreducible affine variety contained in K^n. Then

$$V = \operatorname{Spec} K[x_1, \ldots, x_n]/I.$$

Let $a = (a_1, \ldots, a_n)$ be an n-tuple of relatively prime positive integers. We say that the affine variety V is *quasihomogeneous* of type a if its algebraic closure $\widehat{V} = V \otimes \widehat{K}$ is invariant under the weighted multiplication by elements of $\widehat{K}$:

$$t \cdot \alpha = (t^{a_1}\alpha_1, \ldots, t^{a_n}\alpha_n), \quad t \in \widehat{K},\ a \in \widehat{V}.$$

Analogously, we say that the polynomial $F(x_1, \ldots, x_n)$ is *quasihomogeneous* of type a if

$$F(t^{a_1}x_1, \ldots, t^{a_n}x_n) = t^d F(x_1, \ldots, x_n).$$

The integer d is called the *weight* of the polynomial F. This induces a natural grading of the polynomial algebra $K[x_1, \ldots, x_n]$.

Obviously a quasihomogeneous affine variety may be defined by a system of quasihomogeneous polynomials, i.e. homogeneous with respect to the graded structure. Therefore the factor algebra $K[x_1, \ldots, x_n]/I$ is graded, too.

Let $K(V)$ be the field of rational functions on V. A rational function is said to be *quasihomogeneous* if it is a quotient of quasihomogeneous polynomials; its weight is the difference of their weights. Let $\widetilde{K}$ be the subfield of $K(V)$ containing all quasihomogeneous rational functions of weight 0 and

1991 *Mathematics Subject Classification*: 11E81, 13A02, 13A18, 14B05.

Key words and phrases: Witt rings, residue homomorphisms, quasihomogeneous varieties, graded rings, selfdual vector bundles.

Supported by KBN(1994) 2 P301 010 06.

0 itself. We remark that trdeg $\widetilde{K} : K$ is one less than trdeg $K(V) : K$ and $\widetilde{K}$ is the function field of a one-less dimensional variety C over the ground field K; namely:

$$C = \operatorname{Proj} K[x_1, \ldots, x_n]/I.$$

Moreover, V is a quasihomogeneous cone over C.

It is well-known that many invariants of varieties with singularities can be described in terms of their resolutions. In the case of quasihomogeneous varieties the main resolution data are: the associated projective variety and the *irregular* orbits of the weighted multiplication (compare [12]). Our goal is to describe the Witt ring of the function field $K(V)$ in terms of the second residue homomorphisms associated to the prime divisors of the variety V *and* the Witt ring of the function field of the associated projective variety C. We show that the kernel and cokernel of the direct sum of all second residue homomorphisms associated to the prime divisors of the variety V can be described in terms of kernels and cokernels of two direct sums of residue homomorphisms associated to the prime divisors of the projective variety C. The choice of these direct sums is determined by the set of fixed points of the weighted multiplication by -1. We remark that similar results are also valid for certain algebroid surfaces (see [4]) and for Milnor K-theory groups (see [5]). We also investigate the link between the kernel of the above direct sum and the graded selfdual modules over the ring of regular functions on V.

Furthermore, we construct the Pardon type long exact sequence for the quasihomogeneous surface S with an isolated singularity (compare [10]) and describe the Witt ring of the surface $S \setminus \{0\}$. We deal in more detail with the affine plane and with the surfaces described by the equation

$$yz - x^n = 0, \qquad n \geq 2.$$

In this case, we find the selfdual line bundles which generate the Witt rings of the complements of the origin and show that, although the above surfaces are birationally equivalent, the kernel and the cokernel of the direct sum of all second residue homomorphisms associated with prime divisors may be different.

Acknowledgements. The author would like to express his gratitude to Prof. Winfried Scharlau for many valuable discussions during the preparation of the manuscript.

2. Notation

2.1. *Homomorphisms of Witt rings.* We consider the *Witt rings* over integral domains as defined by Knebusch (see [6], [11] or [8]), i.e. $W(R)$ is the Grothendieck ring of nondegenerate bilinear forms on projective modules over R modulo the metabolic (split) ones. We recall the basic facts.

Every ring homomorphism $i : R \to S$ induces a ring homomorphism of Witt rings (*the extension of scalars*)

$$i^* : W(R) \to W(S).$$

The so-called residue homomorphisms are other examples of widely used mappings of Witt rings (see [7, 8, 9, 11]). Let $v : K \to \Gamma \cup \{\infty\}$ be a discrete valuation on a field K. Then

$$V = \{a \in K : v(a) \geq 0\}$$

is the discrete valuation ring with the maximal ideal

$$m = \{a \in K : v(a) > 0\}.$$

The factor field V/m is called the *residue field* and is denoted by K_v. Any generator π of the ideal m is called a *uniformizer* of the valuation. Obviously, the value group Γ is generated by the weight of π:

$$\Gamma = \mathbb{Z} \cdot \gamma, \qquad \gamma = v(\pi).$$

Every element of the field K may be uniquely written as a product $\pi^k a$, where $k \in \mathbb{Z}$, and $a \in V \setminus m$ (obviously $v(\pi^k a) = k\gamma$). The *first* and the *second residue homomorphisms* are defined as follows:

$$\partial^i : W(K) \to W(K_v), \qquad i = 1, 2,$$
$$\partial^i \langle \pi^k \cdot a \rangle = \begin{cases} \langle \overline{a} \rangle & \text{if } k + i \text{ is odd,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\overline{a}$ is the image of a in the residue field $K_v = V/m$.

We remark that the residue homomorphisms are only group homomorphisms. But they define a ring homomorphism (see [11], Ch. 6, §2)

$$\partial = \partial^1 + \overline{\gamma} \cdot \partial^2 : W(K) \to W(K_v)[\Gamma/2\Gamma]$$

where $W(K_v)[\Gamma/2\Gamma]$ is the group ring and $\overline{\gamma}$ is the image of the generator γ:

$$\partial(\langle \pi^k \cdot a \rangle) = \langle \overline{a} \rangle \cdot \overline{\gamma}^{\bar{k}}, \qquad \overline{k} = k \pmod 2.$$

We consider $\Gamma/2\Gamma$ as the multiplicative group; $\overline{\gamma}^2 = 1$.

Moreover, we remark that the ring residue homomorphism and the second group residue homomorphism depend on the choice of the uniformizer π. Namely, if π and $\pi' = \omega\pi$ are two uniformizers then the residue homomorphisms ∂^2_π and $\partial^2_{\pi'}$, respectively ∂_π and $\partial_{\pi'}$, are related in the following way:

$$\partial^2_{\pi'}(\alpha) = \langle \overline{\omega} \rangle \partial^2_\pi(\alpha), \qquad \partial_{\pi'}(\alpha) = \partial^1_\pi(\alpha) + \langle \overline{\omega} \rangle \partial^2_\pi(\alpha)\overline{\gamma}.$$

Thus they differ only by an isomorphism of the target space.

2.2. *Prime divisors and valuations*. Let Z be an integral scheme. By a *prime divisor* we shall mean a closed integral (i.e. irreducible and reduced) subscheme of Z of codimension 1. If Z is an *affine scheme*, i.e. $Z = \operatorname{Spec} A$

for some ring A, then there is a one-to-one correspondence between prime divisors and prime ideals of height $= 1$.

Every prime divisor P defines a valuation on the function field $K(Z)$; for every $f \in K(Z)$ the weight $v(f)$ is equal to the order of 0 of f at P or to minus the order of the pole of f at P. The corresponding valuation ring V is the integral closure of the local ring $\mathcal{O}_{Z,p}$, where p is a generic point of P. We remark that when p is a regular point of the scheme Z then the local ring $\mathcal{O}_{Z,p}$ is integrally closed and the residue field K_v of the corresponding valuation is isomorphic to the function field $K(P) = K_p$ of the scheme P. Therefore the images of the induced residue homomorphisms are the Witt rings of the field K_p:

$$\partial_p^i : W(K(Z)) \to W(K_p), \qquad i = 1, 2.$$

Other examples of valuations are the quasihomogeneous ones. Let $a = (a_1, \ldots, a_n)$ be an n-tuple of relatively prime positive integers. A *quasihomogeneous valuation* of type a on the field of rational functions $K(x_1, \ldots, x_n)$ is defined by the following rules:

$$v : K(x_1, \ldots, x_n) \to \mathbb{Z} \cup \{\infty\},$$
$$v(1) = 0, \quad v(0) = \infty, \quad v(x_i) = a_i.$$

If

$$g(x_1, \ldots, x_n) = \sum g_{i_1,\ldots,i_n} x_1^{i_1} \ldots x_n^{i_n}$$

then we put

$$v(g) = \min\{i_1 a_1 + \ldots + i_n a_n : g_{i_1,\ldots,i_n} \neq 0\}.$$

Moreover, the weight of a fraction is the difference of weights:

$$v\left(\frac{f}{g}\right) = v(f) - v(g).$$

We remark that

$$v(f(x_1, \ldots, x_n)) = \mathrm{ord}_t\, f(t^{a_1} x_1, \ldots, t^{a_n} x_n).$$

Now let V be an irreducible quasihomogeneous variety (of type a):

$$V = \operatorname{Spec} K[x_1, \ldots, x_n]/I,$$

where I is a prime ideal generated by quasihomogeneous polynomials. The above defined quasihomogeneous valuation v on the field $K(x_1, \ldots, x_n)$ induces a quasihomogeneous valuation v_* on the function field $K(V)$. Namely, if g is a polynomial then

$$g = \sum_{i=0}^{\infty} g_i,$$

where g_i are either 0 or quasihomogeneous polynomials of weight i, and we put

$$v_*(g) = \min\{i : g_i \notin I\}.$$

Moreover,

$$v_*\left(\frac{f}{g}\right) = v_*(f) - v_*(g).$$

We remark that the residue field of the induced quasihomogeneous valuation v_* is isomorphic to the subfield $\widetilde{K}$ of the function field $K(V)$ consisting of all quasihomogeneous elements of weight 0 and 0 itself. The value group Γ_I of v_* is a subgroup of the value group of the quasihomogeneous valuation on the field of rational functions $K(x_1, \ldots, x_n)$ (i.e. of $\mathbb{Z}$). We observe that if V is not contained in any linear subscheme $\{x_i = 0\}$ then $\Gamma_I = \mathbb{Z}$. Moreover, $\Gamma_I \subset 2\mathbb{Z}$ if and only if every point of V is invariant under the weighted multiplication by -1:

$$(x_1, \ldots, x_n) \to ((-1)^{a_1} x_1, \ldots, (-1)^{a_n} x_n).$$

2.3. *Quasihomogeneous and projective varieties.* The quasihomogeneous valuation induces a graded structure on the polynomial ring $K[x_1, \ldots, x_n]$:

$$K[x_1, \ldots, x_n] = \bigoplus_{i=0}^{\infty} S_i,$$

where $S_0 = K$ and S_i, $i > 0$, are K-linear spaces generated by monomials of weight i.

Let I be a prime quasihomogeneous ideal. There are two schemes associated with I (compare [3], II.2):

(i) the quasihomogeneous affine variety

$$V = \operatorname{Spec} K[x_1, \ldots, x_n]/I;$$

(ii) the projective variety

$$C = \operatorname{Proj} V = \operatorname{Proj} K[x_1, \ldots, x_n]/I.$$

Obviously, the function field of Proj V is the subfield $\widetilde{K}$ of the function field $K(V)$ consisting of all quasihomogeneous elements of weight 0 and 0 itself. We remark that if V is normal then so is $\operatorname{Proj} V$. Therefore if V is a normal surface then $\operatorname{Proj} V$ is a smooth curve. Moreover, in general, the complement of the set of normal points of a quasihomogeneous variety is contained in a proper quasihomogeneous subvariety (see [1], V.1.8 and IV.3.1).

We remark that quasihomogeneous means homogeneous with respect to the grading induced by the quasihomogeneous weight.

2.4. *Graded rings.* Let S be a graded ring with unity:

$$S = \bigoplus_{i=0}^{\infty} S_i,$$

where S_i, $i \geq 0$, are abelian groups and $S_i \cdot S_j \subset S_{i+j}$. We assume that S is an integral domain and $1/2$ belongs to S_0.

We denote by $K(S)$ the field of quotients of S and by T the multiplicative set of nonzero homogeneous elements of the ring S:

$$T = \bigcup_{i=0}^{\infty} S_i^*.$$

The graded structure of the ring S induces a valuation on the field $K(S)$,

$$v : K(S) \to \mathbb{Z} \cup \{\infty\},$$

defined by the rule:

- if $a \in S_i^*$ then $v(a) = i$,
- if $a, b \in S^*$ then $v(a/b) = v(j(a)) - v(j(b))$,

where j is the multiplicative map which associates with each nonzero element of the graded ring S its leading form:

$$j : S^* \to T,$$

if $a = \sum_{i \geq i_0} a_i$, where $a_i \in S_i$ and $a_{i_0} \neq 0$, then $j(a) = a_{i_0}$.

The residue field of the induced valuation v is a subfield $\widetilde{K}$ of $K(S)$ generated by homogeneous fractions of degree 0:

$$\widetilde{K} = \left\{ \frac{a}{b} : a, b \in S_i,\ b \neq 0 \right\}.$$

Let $\mathcal{W}$ be the subring of the Witt ring $W(K(S))$ generated by homogeneous elements ($\mathcal{W} = W(T)$). We remark that the multiplicative mapping j induces a ring homomorphism

$$j^* : W(K(S)) \to \mathcal{W}, \qquad j^*(\langle a \rangle) = \langle j(a) \rangle.$$

We shall use the following notation:

Spec_i (respectively Speh_i, Spnh_i) denotes the subset of Spec of the graded ring consisting of prime ideals (resp. prime homogeneous, prime nonhomogeneous ideals) of height i.

Proj_i denotes the subset of Proj consisting of homogeneous ideals of height i and

$$\psi : \mathrm{Speh}_i \to \mathrm{Proj}_i$$

is the canonical bijection. Moreover, by $V(p)$ we denote the subset of Spec consisting of all prime ideals containing the ideal p.

2.5. *Graded modules.* An S-module M is called *graded* if it is a direct sum of abelian groups

$$M = \bigoplus M_i,$$

where $S_j \cdot M_i \subset M_{i+j}$.

Analogously an S-bilinear form $b(\cdot,\cdot)$ on M is *graded* if

$$b(M_i, M_j) \subset S_{i+j}.$$

We recall that a bilinear form is called *nondegenerate* (*regular*) if the induced mapping

$$b^* : M \to M^*, \quad (b^*(\alpha))(\beta) = b(\alpha, \beta),$$

is an isomorphism. In such a case M is selfdual and reflexive.

2.6. *Direct sums.* We shall use the following convention:

(i) If $\alpha : A \to B$ and $\beta : A \to C$ are homomorphisms (of groups or modules) then by $\alpha \oplus \beta$ we shall denote their direct sum

$$\alpha \oplus \beta : A \to B \oplus C, \qquad (\alpha \oplus \beta)(a) = (\alpha(a), \beta(a)).$$

(ii) We shall denote by the same symbol a homomorphism and its trivial extension to the direct sum: If $\alpha : A \to B$, is a homomorphism, then $\alpha : A \oplus C \to B$, denotes the mapping $(a, c) \to \alpha(a)$.

3. Main results. Let an integral domain S be a graded ring with unity such that $1/2$ belongs to S_0, and let v be the valuation on the field $K(S)$ induced by the graded structure.

Theorem 1. *The mapping*

$$\varDelta_S = \partial_v^1 \oplus \partial_v^2 \oplus \bigoplus_{p \in \mathrm{Spnh}_1} \partial_p^2 : W(K(S)) \to W(\widetilde{K}) \oplus W(\widetilde{K}) \oplus \bigoplus_{p \in \mathrm{Spnh}_1} W(K_p)$$

is an isomorphism of $W(\widetilde{K})$-modules (for any choice of uniformizers).

Furthermore, as the direct sum $W(\widetilde{K}) \oplus W(\widetilde{K})$ is isomorphic to the $W(\widetilde{K})$-module $\mathcal{W}$, we may replace $\partial_v^1 \oplus \partial_v^2$ by the mapping j^* induced by taking the leading forms.

Corollary 1. *The mapping*

$$\varDelta = j^* \oplus \bigoplus_{p \in \mathrm{Spnh}_1} \partial_p^2 : W(K(S)) \to \mathcal{W} \oplus \bigoplus_{p \in \mathrm{Spnh}_1} W(K_p),$$

is an isomorphism of $W(\widetilde{K})$-modules.

We can describe the kernel of $\bigoplus \partial_p^2$, $p \in \mathrm{Spnh}_1$, in more detail:

Corollary 2. *Let $b(\cdot,\cdot)$ be a nondegenerate bilinear form over a $K(S)$-vector space L. If the equivalence class of $(L, b(\cdot,\cdot))$ in the Witt ring*

$W(K(S))$ belongs to the kernel of ∂_p^2 for every $p \in \mathrm{Spnh}_1$, then there exists an orthogonal base $e_1, \ldots, e_n$ of L such that $b(e_i, e_i)$ is homogeneous with respect to the gradating of S for every i.

Moreover, it follows from Theorem 1 that the so-called "weak Hasse principle" is valid for the field $K(S)$, namely:

COROLLARY 3. *If a quadratic form with coefficients in $K(S)$ is hyperbolic over every completion of $K(S)$ with respect to a discrete $\widetilde{K}$-valuation then it is hyperbolic over $K(S)$.*

Let K be a field of characteristic not equal to 2. Let V be a normal quasihomogeneous affine variety contained in K^n,

$$V = \operatorname{Spec} K[x_1, \ldots, x_n]/I.$$

We assume that V is not contained in any linear subspace of $\operatorname{Spec} K[x_1, \ldots, x_n]$, i.e. the ideal I does not contain any variable x_i.

Let C be the weighted projective variety associated with V:

$$C = \operatorname{Proj} V = \operatorname{Proj} K[x_1, \ldots, x_n]/I.$$

By C_2 we denote the zero set of the quasihomogeneous ideal generated by those coordinate functions which have odd weights, i.e. the set of points of C which correspond to the codimension one subschemes of V invariant under the weighted multiplication by -1 (compare s. 2.2).

In the following we fix the second residue homomorphism associated with the valuation v. Namely, we choose a quasihomogeneous fraction $\pi \in K(V)$ of weight 1, i.e. the uniformizer of the quasihomogeneous valuation v, and we put

$$\partial_v^2(\alpha) = \partial_v^1(\langle \pi \rangle \cdot \alpha).$$

Let $Z(\pi)$ denote the set of points of C corresponding to height 1 quasihomogeneous prime ideals of $K[V] = K[x_1, \ldots, x_n]/I$ at which π has an odd weight. In Section 5 we show that the set $Z(\pi)$ contains the set C_2.

We fix the uniformizers for all valuations associated with prime divisors of the projective variety C and denote by $\ker_i$ and coker_i, $i = 1, 2$, respectively the kernel and cokernel of the following homomorphisms:

$$\nabla_1 = \bigoplus_{q \in \mathrm{Proj}_1 \setminus C_2} \partial_q^2 : W(K(C)) \to \bigoplus_{q \in \mathrm{Proj}_1 \setminus C_2} W(K_q);$$

$$\nabla_2 = \bigoplus_{q \in \mathrm{Proj}_1 \setminus Z(\pi)} \partial_q^2 \oplus \bigoplus_{q \in Z(\pi)} \partial_q^1 \oplus \bigoplus_{q \in C_2} \partial_q^2 : \quad W(K(C)) \to \bigoplus_{q \in \mathrm{Proj}_1} W(K_q) \oplus \bigoplus_{q \in C_2} W(K_q).$$

Moreover, we denote by $i^* : W(K(C)) \to W(K(V))$ the mapping induced by the inclusion $i : K(C) = \widetilde{K} \to K(V)$, where $\widetilde{K}$ is the subfield of the function field $K(V)$ consisting of the quasihomogeneous elements of weight 0 and 0 itself.

Theorem 2. *For any choice of uniformizers there exists a homomorphism of $W(K)$-modules δ such that the following sequence is exact*:

$$0 \to \ker_1 \oplus \ker_2 \xrightarrow{\hat{\jmath}} W(K(V))$$
$$\xrightarrow{\partial_V} \bigoplus_{p \in \mathrm{Spec}_1} W(K_p) \xrightarrow{\delta} \mathrm{coker}_1 \oplus \mathrm{coker}_2 \oplus \bigoplus_{r \in \mathrm{Spnh}_2} W(K_r) \to 0$$

where

(i) $\partial_V = \bigoplus_{p \in \mathrm{Spec}_1} \partial_p^2$;
(ii) $\hat{\jmath}(\alpha, \beta) = i^*(\alpha) + \langle \pi \rangle \cdot i^*(\beta)$.

In Section 6 we describe δ in more detail and show how it depends on the choice of uniformizers of valuations v_p, $p \in \mathrm{Spec}_1$, and v_q, $q \in \mathrm{Proj}_1$.

The kernel of ∂_V can also be described in terms of selfdual graded $K[V]$-modules. Let L be a vector space over the field $K(V)$ equipped with a nondegenerate bilinear form $b(\cdot,\cdot)$.

Theorem 3. *If the equivalence class of $(L, b(\cdot,\cdot))$ in the Witt ring $W(K(V))$ belongs to the kernel of ∂_V then L contains a graded $K[V]$-module M such that*

- $L = K(V) \cdot M$,
- $b(\cdot,\cdot)$ *restricted to M is graded and nondegenerate.*

Now let us assume that V is an integral surface. Being normal and quasihomogeneous it has no singular points except the origin and the associated weighted projective variety is a smooth curve. The scheme $V^* = V \setminus \{(x_1, \ldots, x_n)\}$ is quasiprojective and smooth. Hence we have (compare [2]):

Corollary 4. *Let V be a quasihomogeneous normal integral surface. Then*:

- *the Witt ring $W(V^*)$ of the scheme V^* is isomorphic to the direct sum* $\ker_1 \oplus \ker_2$,
- *every vector bundle over V^* with nondegenerate inner product is stably equivalent to a graded reflexive module over the ring $K[V]$.*

Moreover, $\mathrm{Spnh}_2(V)$ consists of all closed points of V, except the origin (the only 0-dimensional quasihomogeneous subvariety). Thus our sequence is an analogue of Pardon exact sequences for algebraic regular local rings (compare [10]).

COROLLARY 5. *Let V be a quasihomogeneous normal integral surface. Then the following sequence of $W(V^*)$-modules is exact*:

$$0 \to W(V^*) \to W(K(V)) \xrightarrow{\partial_V} \bigoplus_{p\in \mathrm{Spec}_1} W(K_p) \xrightarrow{\delta} \mathrm{coker}_1 \oplus \mathrm{coker}_2 \oplus \bigoplus_{r\in \mathrm{Spec}_2 \setminus \{(x_1,\dots,x_n)\}} W(K_r) \to 0$$

where ∂_V and δ are as above.

For the affine surface $V_1 = \operatorname{Spec} K[x,y]$ the above sequence simplifies.

COROLLARY 6. *There exists a mapping δ such that the following sequence of $W(K)$-modules is exact*:

$$0 \to W(K) \xrightarrow{i^*} W(K(x,y)) \xrightarrow{\partial_V} \bigoplus_{p\in \mathrm{Spec}_1} W(K_p) \xrightarrow{\delta} \bigoplus_{r\in \mathrm{Spec}_2} W(K_r) \to 0$$

where

(i) *i^* is induced by the canonical inclusion $i : K \to K(x,y)$;*
(ii) *$\partial_V = \bigoplus_{p\in \mathrm{Spec}_1} \partial_p^2$.*

Analogously, for the surfaces $V_n = \operatorname{Spec} K[x,y,z]/(yz - x^n)$, $n \geq 2$, we obtain:

COROLLARY 7. *For each $m \geq 1$ there exist mappings δ_{o} and δ_{e} such that the following sequences of $W(K)$-modules are exact*:

$$0 \to W(K) \xrightarrow{i^*} W(K(V_{2m+1})) \xrightarrow{\partial_V} \bigoplus_{p\in \mathrm{Spec}_1} W(K_p) \xrightarrow{\delta_{\mathrm{o}}} \bigoplus_{r\in \mathrm{Spec}_2} W(K_r) \to 0,$$

and

$$0 \to W(K) \oplus W(K) \xrightarrow{i_1^*} W(K(V_{2m})) \xrightarrow{\partial_V} \bigoplus_{p\in \mathrm{Spec}_1} W(K_p) \xrightarrow{\delta_{\mathrm{e}}} W(K) \oplus \bigoplus_{r\in \mathrm{Spec}_2} W(K_r) \to 0,$$

where

(i) *i^* is induced by the canonical inclusion $i : K \to K(V_n)$;*
(ii) *$i_1^*(\alpha,\beta) = i^*(\alpha) + \langle y\rangle \cdot i^*(\beta)$;*
(iii) *$\partial_V = \bigoplus_{p\in \mathrm{Spec}_1} \partial_p^2$.*

Note that all the surfaces V_n, $n > 1$, have an isolated singular point at the origin, which in the case $K = \mathbb{C}$ is called A_{n-1}.

We remark that similar exact sequences were constructed by the author also for certain algebroid surfaces (see [4]).

Furthermore, basing on the above we can describe the Witt ring of inner products on V_n^*-vector bundles.

COROLLARY 8. (a) $W(V^*_{2m+1})$, $m = 0, 1, \ldots,$ *is a* $W(K)$*-module generated by a trivial line bundle with the inner product*

$$b(e, e) = 1,$$

where e *is a nonvanishing global section.*

(b) $W(V^*_{2m})$, $m = 1, 2, \ldots,$ *is a* $W(K)$*-module generated by two elements: a trivial line bundle with the inner product*

$$b(e, e) = 1,$$

where e *is a nonvanishing global section, and a line bundle generated by two global sections* e_1, e_2, $x^m e_2 = z e_1$, *with the inner product*

$$b(e_1, e_1) = y, \quad b(e_2, e_2) = z, \quad b(e_1, e_2) = x^m.$$

4. Witt rings of fields of quotients of graded rings. The proof of Theorem 1 is based on the following construction (see [1], Ch. V, §1.8L.4):

Let the homogeneous fraction π be a uniformizer of the valuation v induced by the graded structure; i.e.

$$\pi = \frac{a}{b}, \quad a \in S^*_{i+d},\ b \in S^*_i$$

where d is the greatest common divisor of all nontrivial indices:

$$d = \mathrm{GCD}\{i > 0 : S_i \neq \{0\}\}.$$

PROPOSITION 1. *The* $\widetilde{K}$*-linear mapping*

$$\Psi : \widetilde{K}[t]_t \to S_T, \qquad \Psi(t^k) = \pi^k,\ k \in \mathbb{Z},$$

is a ring isomorphism.

Proof. Since both π and π^{-1} belong to the ring S_T and $\widetilde{K}$ is a subfield of this ring, the mapping Ψ is well defined. On the other hand, any homogeneous element a of S_T is a product

$$a = b\pi^k, \quad b \in \widetilde{K},\ k \in \mathbb{Z},$$

thus Ψ is onto. ∎

COROLLARY 9. Ψ *induces a* 1-1 *correspondence between the prime nonquasihomogeneous height* 1 *ideals of the ring* S *and the prime ideals of the ring* $\widetilde{K}[t]$ *different from the one generated by* t:

$$\Psi^\sharp : \mathrm{Spnh}_1(S) \to \mathrm{Spec}_1(\widetilde{K}[t]) \setminus \{(t)\}.$$

COROLLARY 10. *For any nonhomogeneous ideal* $p \in \mathrm{Spnh}_1(S)$ *the local rings* S_p *and* $\widetilde{K}[t]_{\Psi^\sharp(p)}$ *are isomorphic.*

To finish the proof of Theorem 1 we apply Ψ to the Milnor exact sequence of the ring $\widetilde{K}[t]$ (see [7], [11]). We have

$$0 \to W(\widetilde{K}) \xrightarrow{i^*} W(\widetilde{K}(t)) \xrightarrow{\partial} \bigoplus_{p\in \mathrm{Spec}_1} W(K_p) \to 0$$

where

$$\partial = \bigoplus_{p\in \mathrm{Spec}_1} \partial_p^2.$$

We remark that:

(a) the field $\widetilde{K}(t)$ is isomorphic to the field $K(S)$,

(b) the residue field of the ideal (t) is isomorphic to the field $\widetilde{K}$ and $\partial_{(t)}^2 = \partial_v^2 \circ \Psi^*$ (v is the valuation induced by the grading),

(c) the other ideals of $\widetilde{K}[t]$ correspond to nonquasihomogeneous ideals of height 1 of the ring S and the residue fields of the corresponding ideals are isomorphic.

Hence the following sequence is split exact:

$$0 \to W(\widetilde{K}) \xrightarrow{i^*} W(K(S)) \xrightarrow{\partial} W(\widetilde{K}) \oplus \bigoplus_{p\in \mathrm{Spnh}_1} W(K_p) \to 0$$

where

$$\partial = \partial_v^2 \oplus \bigoplus_{p\in \mathrm{Spnh}_1} \partial_p^2.$$

Next we observe that the mapping i^* is a right inverse of the first residue homomorphism ∂_v^1. Thus the mapping

$$\Delta_S = \partial_v^1 \oplus \partial_v^2 \oplus \bigoplus_{p\in \mathrm{Spnh}_1} \partial_p^2 : W(K(S)) \to W(\widetilde{K}) \oplus W(\widetilde{K}) \oplus \bigoplus_{p\in \mathrm{Spnh}_1} W(K_p)$$

is a group isomorphism. Moreover, it commutes with the multiplication by the forms defined over the field $\widetilde{K}$, hence it is an isomorphism of $W(\widetilde{K})$-modules.

Since the mapping

$$\hat{\jmath} : W(\widetilde{K}) \oplus W(\widetilde{K}) \to \mathcal{W}, \quad \hat{\jmath}(\alpha, \beta) = i^*(\alpha) + \langle \pi \rangle i^*(\beta),$$

is a group isomorphism and

$$j^* = \hat{\jmath} \circ (\partial_v^1 \oplus \partial_v^2),$$

we conclude that the mapping $\Delta = j^* \oplus \bigoplus_{p\in \mathrm{Spnh}_1} \partial_p^2$ is an isomorphism too, which proves Corollary 1.

To prove the next corollary we have first to choose an orthogonal base of L such that $b = \langle a_1, \ldots, a_n \rangle$, where all a_i belong to S. Due to Corollary 1, (L, b) is Witt equivalent to (L', b') where $b' = \langle j(a_1), \ldots, j(a_n) \rangle$. Since the dimensions of both spaces are equal, it follows that b and b' are linearly equivalent, i.e. we may choose an orthogonal base of L in which b has homogeneous coefficients (namely $j(a_1), \dots, j(a_n)$).

Furthermore, if $i_v : K(S) \to \widehat{K}_v$ is the completion with respect to a discrete valuation v then

$$\partial_v^k = \partial_{\hat{v}}^k \circ i_v^*, \quad k = 1, 2,$$

where $\hat{v}$ is an extension of v to $\widehat{K}_v$. Therefore for any $\alpha \in W(K(S))$ if $i_v^*(\alpha) = 0$ then $\partial_v^1(\alpha) = \partial_v^2(\alpha) = 0$.

Now all valuations induced by prime nonhomogeneous ideals of height 1 map $\widetilde{K}$ to 0, i.e. they are $\widetilde{K}$-valuations. Hence if for every $\widetilde{K}$-valuation v, $i_v^*(\alpha) = 0$ then $\Delta_S(\alpha) = 0$ and due to Theorem 1, $\alpha = 0$.

Later we shall need one more consequence of Proposition 1.

COROLLARY 11. *Every nonhomogeneous height* 2 *prime ideal of the graded ring contains just one homogeneous prime ideal of height* 1.

Proof. Let r be any nonhomogeneous height 2 prime ideal of the graded ring S. Since the quotient ring S_T has no proper prime ideals of height 2 (see Proposition 1), it follows that the extension r_T of the ideal r is equal to the whole ring. Thus there are some homogeneous elements in the ideal r and it contains a homogeneous prime ideal of smaller height, i.e. of height 1.

On the other hand, if the ideal r contained two different homogeneous ideals of height 1, say p_1 and p_2, then it would contain their union. But the primary decomposition of the homogeneous ideal generated by $p_1 \cup p_2$ consists only of homogeneous ideals of height greater than 1, a the contradiction. ∎

5. Relations between residue homomorphisms on $W(K(V))$ and $W(K(C))$. Let p be a prime quasihomogeneous ideal, $p \in \mathrm{Speh}_1(K[V])$, and $q = \psi(p)$ the associated prime divisor of the projective variety C. We shall compare four valuations, namely, the quasihomogeneous valuation v on $K(V)$, the induced quasihomogeneous valuation v_* on K_p, the valuation v_p on $K(V)$ and the valuation v_q on $K(C) = \widetilde{K}$.

Let the quasihomogeneous fractions π, π_*, π_p and π_q or respectively their images be uniformizers of the valuations v, v_*,v_p and v_q. They are related in the following way:

$$\pi_q = \pi_p^{d_1} \pi_*^{d_2} \omega_1, \qquad \pi = \pi_p^{d_3} \pi_*^{d_4} \omega_2,$$

where d_i, $i = 1, 2, 3, 4$, are nonzero integers and the fractions ω_i, $i = 1, 2$, are quasihomogeneous of weight 0 and regular but not zero at the point q (or equivalently at p). We shall call the above fractions ω_i the *linking elements* of the uniformizers.

We start with the following fact:

Lemma 1. *The integers* d_i, $i = 1, 2, 3, 4$, *fulfil the following conditions*:

(i) $d_1 = v(\pi_*)$, $d_2 = -v(\pi_p)$;
(ii) $d_1 d_4 - d_2 d_3 = 1$.

Proof. Step 1. *The integers* d_1 *and* d_2 *are relatively prime, moreover*, d_1 *is positive.*

The quasihomogeneous fraction π_p is a generator of the maximal ideal of the ring $K[V]_p$. On the other hand, π_q is a generator of the maximal ideal of the ring $K(C) \cap K[V]_p$. Therefore π_q belongs to the maximal ideal of the ring $K[V]_p$ and d_1 must be positive.

Now, if d_1 and d_2 had a common factor $d > 1$,

$$d_1 = d \cdot d_1', \quad d_2 = d \cdot d_2',$$

then the product $\pi_q' = \pi_p^{d_1'} \cdot \pi_*^{d_2'}$ would be quasihomogeneous of weight 0 and $0 < v_p(\pi_q') < v_p(\pi_q)$. This would contradict the assumption that π_q is a uniformizer.

Step 2. We compare the v-weights of the fractions π_p and π_*:

$$\pi_q = \pi_p^{d_1} \pi_*^{d_2} \omega_1, \quad \pi = \pi_p^{d_3} \pi_*^{d_4} \omega_2.$$

Hence we have

$$d_1 v(\pi_p) + d_2 v(\pi_*) = 0, \quad d_3 v(\pi_p) + d_4 v(\pi_*) = 1.$$

Thus $v(\pi_p)$ and $v(\pi_*)$ are relatively prime. Moreover, $v(\pi_*)$ is positive hence from the first equation we obtain

$$d_1 = v(\pi_*), \quad d_2 = -v(\pi_p).$$

We substitute the above to the second equation and obtain $d_1 d_4 - d_2 d_3 = 1$. ∎

Next we consider the restriction of the second residue homomorphism associated with the prime quasihomogeneous ideal $p \in \mathrm{Speh}_1$ to the subgroup

$$\mathcal{W} = W(\widetilde{K}) \oplus W(\widetilde{K}) \cdot \langle \pi \rangle$$

of the Witt ring $W(K(V))$ generated by all forms with quasihomogeneous coefficients.

Lemma 2. *Let* p *be a quasihomogeneous prime ideal of height* 1 *of the ring* $K[V]$ *and* $q = \psi(p)$. *Then for any choice of the uniformizers*,

$$(\partial^1_{v_*} \oplus \partial^2_{v_*}) \circ (\partial^2_p)_{|\mathcal{W}} = \begin{cases} \langle\overline{\omega}_1\rangle\partial^2_q \circ (\partial^1_v)_{|\mathcal{W}} \oplus \langle\overline{\omega}_1\overline{\omega}_2\rangle\partial^2_q \circ (\partial^2_v)_{|\mathcal{W}} \\ \qquad \textit{if } q \notin Z(\pi) \textit{ and } v(\pi_p) \textit{ even}, \\ \langle\overline{\omega}_1\overline{\omega}_2\rangle\partial^2_q \circ (\partial^2_v)_{|\mathcal{W}} \oplus \langle\overline{\omega}_1\rangle\partial^2_q \circ (\partial^1_v)_{|\mathcal{W}} \\ \qquad \textit{if } q \notin Z(\pi) \textit{ and } v(\pi_p) \textit{ odd}, \\ \langle\overline{\omega}_1\rangle\partial^2_q \circ (\partial^1_v)_{|\mathcal{W}} \oplus \langle\overline{\omega}_2\rangle\partial^1_q \circ (\partial^2_v)_{|\mathcal{W}} \\ \qquad \textit{if } q \in Z(\pi) \setminus C_2, v(\pi_p) \textit{ even}, \\ \langle\overline{\omega}_2\rangle\partial^1_q \circ (\partial^2_v)_{|\mathcal{W}} \oplus \langle\overline{\omega}_1\rangle\partial^2_q \circ (\partial^1_v)_{|\mathcal{W}} \\ \qquad \textit{if } q \in Z(\pi) \setminus C_2,\ v(\pi_p) \textit{ odd}, \\ \langle\overline{\omega}_2\rangle\partial^1_q \circ (\partial^2_v)_{|\mathcal{W}} \oplus \langle\overline{\omega}_1\overline{\omega}_2\rangle\partial^2_q \circ (\partial^2_v)_{|\mathcal{W}} \\ \qquad \textit{if } q \in C_2 \textit{ and } d_4 \textit{ is even}, \\ \langle\overline{\omega}_1\overline{\omega}_2\rangle\partial^2_q \circ (\partial^2_v)_{|\mathcal{W}} \oplus \langle\overline{\omega}_2\rangle\partial^1_q \circ (\partial^2_v)_{|\mathcal{W}} \\ \qquad \textit{if } q \in C_2 \textit{ and } d_4 \textit{ is odd}, \end{cases}$$

where v (resp. v_*) is the quasihomogeneous valuation on $K(V)$ (resp. on K_p) and $\overline{\omega}_1$, $\overline{\omega}_2$ are the images in K_q of the linking elements ω_1 and ω_2.

Proof. We compare the restrictions of compositions of the residue ring homomorphisms

$$\partial_{v_*} \circ \partial_p : \mathcal{W} \to W(K_q)[\xi, \eta], \qquad \xi^2 = 1,\ \eta^2 = 1,$$

and

$$\partial_{v_q} \circ \partial_v : \mathcal{W} \to W(K_q)[\xi, \eta], \qquad \xi^2 = 1,\ \eta^2 = 1.$$

We have

$$\partial_{v_*} \circ \partial_p \begin{pmatrix} \langle 1\rangle \\ \langle\pi_q\rangle \\ \langle\pi\rangle \\ \langle\pi\pi_q\rangle \end{pmatrix} = \begin{pmatrix} \langle 1\rangle \\ \langle\overline{\omega}_1\rangle\xi^{d_1}\eta^{d_2} \\ \langle\overline{\omega}_2\rangle\xi^{d_3}\eta^{d_4} \\ \langle\overline{\omega}_1\overline{\omega}_2\rangle\xi^{d_1+d_3}\eta^{d_2+d_4} \end{pmatrix}$$

and

$$\partial_{v_q} \circ \partial_v \begin{pmatrix} \langle 1\rangle \\ \langle\pi_q\rangle \\ \langle\pi\rangle \\ \langle\pi\pi_q\rangle \end{pmatrix} = \begin{pmatrix} \langle 1\rangle \\ \langle 1\rangle\eta \\ \langle 1\rangle\xi \\ \langle 1\rangle\xi\eta \end{pmatrix}.$$

The image of $\partial_{v_*} \circ \partial_p$ depends on the parity of d_i's. It follows from Lemma 4 that there are only six possibilities. Namely:

(i,ii) d_1 odd, d_3 even, then d_4 must be odd and d_2 may be either even or odd;

(iii) d_1 odd, d_3 odd, d_2 even, then d_4 must be odd;

(iv) d_1 odd, d_3 odd, d_2 odd, then d_4 must be even;

(v,vi) d_1 even, then d_2 and d_3 must be odd and d_4 may be either even or odd.

We remark that $q \in C_2$ if and only if d_1 is even (i.e. the value group of v_* is contained in $2\mathbb{Z}$) and $q \in Z(\pi)$ if and only if d_3 is odd. Thus C_2 is a subset of $Z(\pi)$.

Moreover, $d_2 = -v(\pi_p)$, therefore to finish the proof it is enough to check the above six cases. ∎

6. The proof of Theorem 2. We shall obtain the exact sequence from Theorem 2 by *glueing* other exact sequences.

We start the construction of the exact sequence from Theorem 2, taking the direct sum of the exact sequences induced by the mappings ∇_i, $i = 1, 2$:

$$\nabla_1 = \bigoplus_{q\in \mathrm{Proj}_1 \setminus C_2} \partial_q^2 : W(K(C)) \to \bigoplus_{q\in \mathrm{Proj}_1 \setminus C_2} W(K_q);$$

$$\nabla_2 = \bigoplus_{q\in \mathrm{Proj}_1 \setminus Z(\pi)} \partial_q^2 \oplus \bigoplus_{q\in Z(\pi)} \partial_q^1 \oplus \bigoplus_{q\in C_2} \partial_q^2 : \qquad W(K(C)) \to \bigoplus_{q\in \mathrm{Proj}_1} W(K_q) \oplus \bigoplus_{q\in C_2} W(K_q).$$

We obtain the following exact sequence:

$$0 \to \ker_1 \oplus \ker_2 \to W(K(C)) \oplus W(K(C)) \xrightarrow{\nabla_1\oplus\nabla_2} \bigoplus_{q\in \mathrm{Proj}_1 \setminus C_2} W(K_q) \oplus \Big(\bigoplus_{q\in \mathrm{Proj}_1} W(K_q) \oplus \bigoplus_{q\in C_2} W(K_q)\Big) \xrightarrow{\delta_1\oplus\delta_2} \mathrm{coker}_1 \oplus \mathrm{coker}_2 \to 0,$$

where $\ker_i$ and coker_i denote the kernel and the cokernel of the homomorphism ∇_i, and δ_i is the corresponding factor mapping.

Next, we add to it the trivial exact sequence

$$0 \to 0 \to 0 \to \bigoplus_{p\in \mathrm{Speh}_1} \bigoplus_{r\in \mathrm{Spnh}_2 \cap V(p)} W(K_r) \xrightarrow{\mathrm{id}} \bigoplus_{p\in \mathrm{Speh}_1} \bigoplus_{r\in \mathrm{Spnh}_2 \cap V(p)} W(K_r) \to 0.$$

We obtain the exact sequence

$$(1) \qquad 0 \to \ker_1 \oplus \ker_2 \to W(K(C)) \oplus W(K(C)) \xrightarrow{\nabla_1\oplus\nabla_2} \bigoplus_{q\in \mathrm{Proj}_1 \setminus C_2} W(K_q) \oplus \Big(\bigoplus_{q\in \mathrm{Proj}_1} W(K_q) \bigoplus_{q\in C_2} W(K_q)\Big) \oplus \mathcal{N} \xrightarrow{\delta_1\oplus\delta_2\oplus\mathrm{id}_{\mathcal{N}}} \mathrm{coker}_1 \oplus \mathrm{coker}_2 \oplus \mathcal{N} \to 0,$$

where

$$\mathcal{N} = \bigoplus_{p\in \mathrm{Speh}_1} \bigoplus_{r\in \mathrm{Spnh}_2 \cap V(p)} W(K_r).$$

We remark that every nonquasihomogeneous prime ideal of height 2 contains just one quasihomogeneous prime ideal of height 1 (see Corollary 11). Thus as a matter of fact

$$\mathcal{N} = \bigoplus_{r\in \mathrm{Spnh}_2} W(K_r).$$

We apply Theorem 1 to each quasihomogeneous variety $P = \operatorname{Spec} K[V]/p$, $p \in \mathrm{Speh}_1$. We put $S = K[V]/p$, and $K(S) = K_p$. Thus there is a group isomorphism

$$\partial^1_{v_*} \oplus \partial^2_{v_*} \oplus \bigoplus_{r\in \mathrm{Spnh}_1} \partial^2_r : W(K_p) \to W(\widetilde{K}_P) \oplus W(\widetilde{K}_P) \oplus \bigoplus_{r\in \mathrm{Spnh}_1} W(K_r),$$

where v_* is the quasihomogeneous valuation on K_p. Moreover, the field $\widetilde{K}_P$, consisting of quasihomogeneous fractions of weight 0, is isomorphic to the field K_q for $q = \psi(p)$ and

$$\mathrm{Spnh}_1 K[V]/p \approx V(p) \cap \mathrm{Spnh}_2 K[V].$$

We remark that the images $\overline{\omega}_1$ and $\overline{\omega}_2$ (in K_q), of the linking elements ω_1 and ω_2 introduced in the previous section, are nonzero. Therefore the mapping

$$\Theta^1_p \oplus \Theta^2_p \oplus \bigoplus_{r\in \mathrm{Spnh}_1} \partial^2_r : W(K_p) \to W(K_q) \oplus W(K_q) \oplus \bigoplus_{r\in \mathrm{Spnh}_1} W(K_r),$$

where

$$\Theta^1_p \oplus \Theta^2_p = \begin{cases} \langle\overline{\omega}_1\rangle\partial^1_{v_*} \oplus \langle\overline{\omega}_1\overline{\omega}_2\rangle\partial^2_{v_*} & \text{if } \psi(p) \notin Z(\pi) \text{ and } v(\pi_p) \text{ is even,}\\ \langle\overline{\omega}_1\rangle\partial^2_{v_*} \oplus \langle\overline{\omega}_1\overline{\omega}_2\rangle\partial^1_{v_*} & \text{if } \psi(p) \notin Z(\pi) \text{ and } v(\pi_p) \text{ is odd,}\\ \langle\overline{\omega}_1\rangle\partial^1_{v_*} \oplus \langle\overline{\omega}_2\rangle\partial^2_{v_*} & \text{if } \psi(p) \in Z(\pi)\setminus C_2 \text{ and } v(\pi_p) \text{ is even,}\\ \langle\overline{\omega}_1\rangle\partial^2_{v_*} \oplus \langle\overline{\omega}_2\rangle\partial^1_{v_*} & \text{if } \psi(p) \in Z(\pi)\setminus C_2 \text{ and } v(\pi_p) \text{ is odd,}\\ \langle\overline{\omega}_2\rangle\partial^1_{v_*} \oplus \langle\overline{\omega}_1\overline{\omega}_2\rangle\partial^2_{v_*} & \text{if } \psi(p) \in C_2 \text{ and } d_4 \text{ is even,}\\ \langle\overline{\omega}_2\rangle\partial^2_{v_*} \oplus \langle\overline{\omega}_1\overline{\omega}_2\rangle\partial^1_{v_*} & \text{if } \psi(p) \in C_2 \text{ and } d_4 \text{ is odd,} \end{cases}$$

is a group isomorphism for each $p \in \mathrm{Speh}_1$.

Let Θ be the direct sum of Θ_p's:

$$\Theta = \bigoplus_{\psi(p)\in \mathrm{Proj}_1 \setminus C_2} \Theta^1_p \oplus \Big(\bigoplus_{\psi(p)\in \mathrm{Proj}_1} \Theta^2_p \oplus \bigoplus_{\psi(p)\in C_2} \Theta^1_p \Big) :$$

$$\bigoplus_{p\in \mathrm{Spec}_1} W(K_p) \to \bigoplus_{q\in \mathrm{Proj}_1 \setminus C_2} W(K_q) \oplus \Big(\bigoplus_{q\in \mathrm{Proj}_1} W(K_q) \oplus \bigoplus_{q\in C_2} W(K_q) \Big).$$

Taking into account the commutation rules from Lemma 2 we have

$$\varTheta \circ \Big(\bigoplus_{p\in \mathrm{Speh}_1} \partial_p^2 \Big)_{|\mathcal{W}} = (\nabla_1 \circ \partial_v^1 \oplus \nabla_2 \circ \partial_v^2)_{|\mathcal{W}},$$

where

$$\mathcal{W} = W(K(C)) \oplus \langle \pi \rangle W(K(C)).$$

Moreover, for any $r \in \mathrm{Spnh}_2$ and $p \in \mathrm{Speh}_1$,

$$\partial_r^2 \circ \partial_p^2(\mathcal{W}) = \{0\}.$$

Hence the following diagram is commutative:

$$\begin{array}{ccc}
 & & \ker_1 \oplus \ker_2 \\
 & & \downarrow \\
\mathcal{W} & \xrightarrow{\partial_v^1 \oplus \partial_v^2} & W(K(C)) \oplus W(K(C)) \\
\Big\downarrow \bigoplus_{p\in\mathrm{Speh}_1} \partial_p^2 & & \Big\downarrow \nabla_1 \oplus \nabla_2 \oplus 0 \\
\bigoplus_{p\in\mathrm{Speh}_1} W(K_p) & \xrightarrow{\varTheta \oplus \varOmega} & \bigoplus_{q\in\mathrm{Proj}_1 \setminus C_2} W(K_q) \oplus \Big(\bigoplus_{q\in\mathrm{Proj}_1} W(K_q) \oplus \bigoplus_{q\in C_2} W(K_q) \Big) \oplus \mathcal{N} \\
 & & \Big\downarrow \delta_1 \oplus \delta_2 \oplus \mathrm{id}_{\mathcal{N}} \\
 & & \mathrm{coker}_1 \oplus \mathrm{coker}_2 \oplus \mathcal{N}
\end{array}$$

where

$$\varOmega = \bigoplus_{p\in\mathrm{Speh}_1} \bigoplus_{r\in\mathrm{Spnh}_2 \cap V(p)} \partial_r^2 : \bigoplus_{p\in\mathrm{Speh}_1} W(K_p) \to \mathcal{N}.$$

The horizontal arrows are isomorphisms, the homomorphism i^*, induced by the inclusion $i : K(C) = \widetilde{K} \to K(V)$, is a right inverse of the residue homomorphism ∂_v^1 and the homomorphism $\langle \pi \rangle \cdot i^*$ is a right inverse of ∂_v^2. Thus the following sequence is exact:

$$0 \to \ker_1 \oplus \ker_2 \xrightarrow{\hat{\jmath}} \mathcal{W} \xrightarrow{\bigoplus_{p\in\mathrm{Speh}_1} \partial_p^2} \bigoplus_{p\in\mathrm{Speh}_1} W(K_p)$$
$$\xrightarrow{(\delta_1 \oplus \delta_2)\circ\varTheta\oplus\varOmega} \mathrm{coker}_1 \oplus \mathrm{coker}_2 \oplus \mathcal{N} \to 0,$$

where $\hat{\jmath}(\alpha, \beta) = i^*(\alpha) + \langle \pi \rangle \cdot i^*(\beta)$.

We have an isomorphism $\varDelta$ (see Corollary 1):

$$\varDelta = j^* \oplus \bigoplus_{p\in\mathrm{Spnh}_1} \partial_p^2 : W(K(V)) \to \mathcal{W} \oplus \bigoplus_{p\in\mathrm{Spnh}_1} W(K_p),$$
$$j^*(\alpha) = i^* \circ \partial_v^1(\alpha) + \langle \pi \rangle i^* \circ \partial_v^2(\alpha).$$

We remark that for $\alpha \in \mathcal{W}$ and $\beta \in \bigoplus_{p\in\mathrm{Spnh}_1} W(K_p)$,

$$\varDelta^{-1}(\alpha, \beta) = \alpha + \varDelta^{-1}(0, \beta).$$

Moreover,

$$\begin{aligned}\partial_V \circ \Delta^{-1}(\alpha,\beta) &= \Big(\bigoplus_{p\in\mathrm{Speh}_1} \partial_p^2 \circ \Delta^{-1}(\alpha,\beta), \bigoplus_{p\in\mathrm{Spnh}_1} \partial_p^2 \circ \Delta^{-1}(\alpha,\beta)\Big)\\ &= \Big(\bigoplus_{p\in\mathrm{Speh}_1} \partial_p^2(\alpha + \Delta^{-1}(0,\beta)), \beta\Big).\end{aligned}$$

Furthermore, the following sequence is split exact:

$$0 \to \bigoplus_{p\in\mathrm{Spnh}_1} W(K_p) \xrightarrow{\Delta^{-1}(0,\cdot)} W(K((V))) \xrightarrow{j^*} \mathcal{W} \to 0.$$

We use the above sequence to extend the sequence (1) by the trivial exact sequence:

$$0 \to \bigoplus_{p\in\mathrm{Spnh}_1} W(K_p) \xrightarrow{\mathrm{id}} \bigoplus_{p\in\mathrm{Spnh}_1} W(K_p) \to 0.$$

We obtain the following diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[d] & \mathrm{Ker}(\partial_V) \arrow[d] & \mathrm{ker}_1 \oplus \mathrm{ker}_2 \arrow[d, "\hat{\jmath}"] & \\
0 \arrow[r] & \bigoplus_{p\in\mathrm{Spnh}_1} W(K_p) \arrow[r, "\Delta^{-1}(0,\cdot)"] \arrow[d, "\mathrm{id}"] & W(K((V))) \arrow[r, "j^*"] \arrow[d, "\partial_V"] & \mathcal{W} \arrow[r] \arrow[d, "\bigoplus_{p\in\mathrm{Speh}_1} \partial_p^2"] & 0 \\
0 \arrow[r] & \bigoplus_{p\in\mathrm{Spnh}_1} W(K_p) \arrow[r, "\varphi_1"] \arrow[d] & \bigoplus_{p\in\mathrm{Spec}_1} W(K_p) \arrow[r, "\varphi_2"] \arrow[d] & \bigoplus_{p\in\mathrm{Speh}_1} W(K_p) \arrow[r] \arrow[d, "(\delta_1\oplus\delta_2)\circ\Theta\oplus\Omega"] & 0 \\
 & 0 & \mathrm{Coker}(\partial_V) \arrow[d] & \mathrm{coker}_1 \oplus \mathrm{coker}_2 \oplus \mathcal{N} \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where φ_1 and φ_2 are defined by

$$\varphi_1 : \bigoplus_{p\in\mathrm{Spnh}_1} W(K_p) \to \bigoplus_{p\in\mathrm{Speh}_1} W(K_p) \oplus \bigoplus_{p\in\mathrm{Spnh}_1} W(K_p),$$

$$\varphi_1(\beta) = \Big(\bigoplus_{p\in\mathrm{Speh}_1} \partial_p^2 \circ \Delta^{-1}(0,\beta), \beta\Big);$$

$$\varphi_2 : \bigoplus_{p\in\mathrm{Speh}_1} W(K_p) \oplus \bigoplus_{p\in\mathrm{Spnh}_1} W(K_p) \to \bigoplus_{p\in\mathrm{Speh}_1} W(K_p),$$

$$\varphi_2(\alpha,\beta) = \alpha - \bigoplus_{p\in\mathrm{Speh}_1} \partial_p^2 \circ \Delta^{-1}(0,\beta).$$

LEMMA 3. *The above diagram is commutative and furthermore* $\mathrm{Ker}(\partial_V)$ *and* $\mathrm{Coker}(\partial_V)$ *are isomorphic respectively to* $\mathrm{ker}_1 \oplus \mathrm{ker}_2$ *and* $\mathrm{coker}_1 \oplus \mathrm{coker}_2 \oplus \mathcal{N}$.

Proof. The commutativity of the left hand square is obvious. Indeed,

$$\begin{aligned}\partial_V \circ \Delta^{-1}(0,\beta) &= \bigoplus_{p\in\mathrm{Spec}_1} \partial_p^2 \circ \Delta^{-1}(0,\beta)\\ &= \Big(\bigoplus_{p\in\mathrm{Speh}_1} \partial_p^2(\Delta^{-1}(0,\beta)),\beta\Big) = \varphi_1(\beta).\end{aligned}$$

The commutativity of the right hand square is more complicated. We have

$$\bigoplus_{p\in\mathrm{Speh}_1} \partial_p^2 \circ j^* \circ \Delta^{-1}(\alpha,\beta) = \bigoplus_{p\in\mathrm{Speh}_1} \partial_p^2(\alpha).$$

On the other hand,

$$\begin{aligned}\varphi_2 \circ \partial_V \circ \Delta^{-1}(\alpha,\beta) &= \varphi_2\Big(\bigoplus_{p\in\mathrm{Speh}_1} \partial_p^2(\alpha+\Delta^{-1}(0,\beta)),\beta\Big)\\ &= \bigoplus_{p\in\mathrm{Speh}_1} \partial_p^2(\alpha+\Delta^{-1}(0,\beta)) - \bigoplus_{p\in\mathrm{Speh}_1} \partial_p^2(\Delta^{-1}(0,\beta)) = \bigoplus_{p\in\mathrm{Speh}_1} \partial_p^2(\alpha).\end{aligned}$$

Since Δ^{-1} is an isomorphism, we conclude that $\bigoplus_{p\in\mathrm{Speh}_1} \partial_p^2 \circ j^* = \varphi_2 \circ \partial_V$.

The second part of the lemma follows from the fact that the horizontal rows of the above diagram are split exact. ■

COROLLARY 12. *The following sequence is exact*:

$$\begin{aligned}0 \to \mathrm{ker}_1 \oplus \mathrm{ker}_2 \xrightarrow{\hat{\jmath}} W(K((V))) \xrightarrow{\partial_V} \bigoplus_{p\in\mathrm{Spec}_1} W(K_p)&\\ \xrightarrow{\delta} \mathrm{coker}_1 \oplus \mathrm{coker}_2 \oplus \mathcal{N} \to 0&,\end{aligned}$$

where $\delta(\alpha,\beta) = (\delta_1 \oplus \delta_2)\circ(\Theta \oplus \Omega)(\alpha - \bigoplus_{p\in\mathrm{Speh}_1} \partial_p^2(\Delta^{-1}(0,\beta)))$, *for* $\alpha \in \bigoplus_{p\in\mathrm{Speh}_1} W(K_p)$ *and* $\beta \in \bigoplus_{p\in\mathrm{Spnh}_1} W(K_p)$.

This finishes the proof of Theorem 2, because all the above homomorphisms commute with the multiplication by elements from $W(K)$.

7. The proof of Theorem 3. Due to Corollary 2 there is a base $e_1,\ldots,e_n$ of the $K(V)$-vector space L such that

$$b(e_i,e_j) = \begin{cases} a_i & \text{if } i=j,\\ 0 & \text{otherwise,}\end{cases}$$

where all a_i are nonzero, quasihomogeneous and belong to $K[V]$. We put $a = a_1 \cdot \ldots \cdot a_n$.

Let N be a $K[V]$-sublattice of L generated by $e_1, \ldots, e_n$ and N^* a dual sublattice, i.e. the one spanned by $a_1^{-1}e_1, \ldots, a_n^{-1}e_n$. N and N^* become graded $K[V]$-modules when we assume that e_i are homogeneous of weight $v(e_i) = -v(a_i)/2$. We remark that the bilinear form $b(\cdot,\cdot)$ restricted to N or N^* preserves the grading and has values in respectively $K[V]$ or $(a^{-1})K[V]$. We shall denote by T_N the set of homogeneous elements of N.

Generally $b(\cdot,\cdot)$ restricted to N is degenerate, therefore we seek some greater module.

LEMMA 4. *There exists a graded $K[V]$-module Q contained in N^* such that*

(i) *the bilinear form $b(\cdot,\cdot)$ restricted to Q takes values in $K[V]$ and for every prime height* 1 *ideal* p:

(ii) *the bilinear form $b(\cdot,\cdot)$ restricted to $K[V]_p$-module $Q_p = K[V]_pQ$ is nondegenerate*;

(iii) $K[V]_pN \subset Q_p \subset K[V]_pN^*$;

(iv) *if no a_i belongs to p then $K[V]_pN = Q_p = K[V]_pN^*$.*

Proof. Let p be a prime quasihomogeneous height 1 ideal containing at least one a_i. We fix a quasihomogeneous uniformizer π of the associated valuation such that in all other valuations associated with prime height 1 ideals the weight of π is nonpositive. After a slight modification of the method used in [8] in the proof of Theorem 3.1 (Ch. IV, §3) we obtain a positive integer k and homogeneous vectors $f_1, \ldots, f_n \in (\pi^{-k})T_N$ such that $b(\cdot,\cdot)$ restricted to the $K[V]_p$-module $N(p)$ generated by f_i's is $K[V]_p$-nondegenerate. After multiplying by some invertible elements of $K[V]_p$ we see that the products of f_i's and the products of f_i's and e_i's belong to $K[V]$.

We repeat the same procedure for other prime quasihomogeneous height 1 ideals containing at least one a_i in such a way that the products of f_i's defined for different ideals also belong to $K[V]$.

Let Q be a $K[V]$-module generated by e_i's and f_i's constructed for all prime ideals of height 1 containing a_i's.

Obviously Q fulfils the condition (i) of the lemma. The remaining conditions (ii) and (iii) follow directly from the construction. ■

The module Q is usually not selfdual, therefore we put

$$M = \bigcap_{p \in \mathrm{Spec}_1 K[V]} Q_p.$$

LEMMA 5. *The module M defined above is a graded $K[V]$-module and the bilinear form $b(\cdot,\cdot)$ restricted to M is nondegenerate.*

P r o o f. As V is normal, $K[V]$ is a Krull domain and we may apply [1], §VII.4, Theorem 3, to show that M is a reflexive $K[V]$-module. Thus $M_p = Q_p$ for all $p \in \operatorname{Spec}_1 K[V]$.

Next we show that the bilinear form $b(\cdot,\cdot)$ restricted to the module M is nondegenerate, i.e. M is selfdual.

Let α be any $K[V]$-functional on M. Since α can be extended to a $K(V)$-functional on L, it follows that there exists $\eta \in L$ such that for every $x \in L$, $\alpha(x) = (\eta, x)$.

But, on the other hand, α can be extended to a $K[V]_p$-functional on $M_p = Q_p$, for every prime divisor p. Q_p is selfdual, hence η belongs to every Q_p. Therefore it belongs to their intersection, i.e. to M.

Let Q^* be the dual of Q, i.e. the set of $\eta \in L$ such that for every $x \in Q$ $b(\eta, x) \in K[V]$. Obviously it is graded.

Now, as M is selfdual and contains Q, it is contained in Q^*. But, in the same way as above we can show that if α is a $K[V]$-functional on Q then the corresponding η belongs to M. Therefore M is equal to Q^*, hence it is graded. ■

8. The two-dimensional case. When the variety V is a surface then every point of codimension 2 is closed and is contained in some quasihomogeneous subvariety $V(p)$. Moreover, there is only one quasihomogeneous closed point—the origin. Thus

$$\operatorname{Speh}_2 = \bigcup_{p \in \operatorname{Speh}_1} (\operatorname{Spnh}_2 \cap V(p)) = \operatorname{Spec}_2 \setminus \{(x_1, \ldots, x_n)\}.$$

The open subscheme $V^* = V \setminus \{(x_1, \ldots, x_n)\}$ is quasiprojective and smooth hence the kernel of the direct sum of all second residue homomorphisms corresponding to prime divisors is isomorphic to the Witt ring of this subscheme (see [2]).

Comparing the above with Theorems 2 and 3 we obtain the first two corollaries.

Now if $V_1 = \operatorname{Spec} K[x, y]$ then we may choose the quasihomogeneous weight in several ways. For example we may put

$$v(x) = v(y) = 1.$$

In this case the weight coincides with the order of an element at the origin. The associated projective variety C is just the projective line and the function field $K(C)$ is the field of rational functions in one variable. We choose $\pi = x$. Thus $Z(\pi)$ consists of one point (x). Moreover, C_2 is empty. Thus ∇_1 is the direct sum of all second residue homomorphisms associated with points of C. Hence its kernel and cokernel are isomorphic to $W(K)$ (see [7, 8, 11]). ∇_2 is the direct sum of the first residue homomorphism associated with (x) and the second residue homomorphisms associated with all points distinct from (x). Hence it is an isomorphism (compare [7, 8, 11]).

Moreover,

(i) the function field corresponding to the maximal ideal (x,y) is K,

(ii) the forms from $W(K(x,y))$ defined over K belong to the kernel of any second residue homomorphism.

Therefore we may rewrite the exact sequence from Theorem 2 in the following way:

$$0 \to W(K) \xrightarrow{i^*} W(K(x,y)) \xrightarrow{\partial_V} \bigoplus_{p\in \mathrm{Spec}_1} W(K_p) \xrightarrow{\delta} \bigoplus_{r\in \mathrm{Spec}_2} W(K_r) \to 0$$

where

(i) i^* is induced by the canonical inclusion $i : K \to K(x,y)$;

(ii) $\partial_V = \bigoplus_{p\in \mathrm{Spec}_1} \partial^2_p$.

Analogously if $V_n = \operatorname{Spec} K[x,y,z]/(yz - x^n)$ then we may also choose the quasihomogeneous weight in several ways. For example we may put

$$v(x) = 1,\quad v(y) = v(z) = m \quad \text{for } n = 2m,$$
$$v(x) = 1,\quad v(y) = m,\quad v(z) = m+1 \quad \text{for } n = 2m+1.$$

The associated projective variety C is just the projective line and the function field $K(C)$ is the field of rational functions in one variable.

$$\widetilde{K} = K\left(\frac{y}{x^m}\right).$$

We choose $\pi = x$. Thus $Z(\pi)$ consists of two points (x,y) and (x,z). Moreover,

$$C_2 = \begin{cases} Z(\pi) & \text{if } n = 0 \pmod 4,\\ \emptyset & \text{if } n = 2 \pmod 4,\\ (y,x) & \text{if } n = 1 \pmod 4,\\ (z,x) & \text{if } n = 3 \pmod 4, \end{cases}$$
$$\nabla_1 = \bigoplus_{q\in \mathrm{Proj}_1 \setminus C_2} \partial^2_q.$$

Thus

$$\ker \nabla_1 = \begin{cases} W(K)\langle 1\rangle \oplus W(K)\langle y\rangle & \text{if } n = 0 \pmod 4,\\ W(K)\langle 1\rangle & \text{if } n = 2 \pmod 4,\\ W(K)\langle 1\rangle & \text{if } n = 1 \pmod 2; \end{cases}$$

and

$$\operatorname{coker} \nabla_1 = \begin{cases} 0 & \text{if } n = 0 \pmod 4,\\ W(K) & \text{if } n = 2 \pmod 4,\\ 0 & \text{if } n = 1 \pmod 2, \end{cases}$$

$$\nabla_2 = \bigoplus_{q\in \mathrm{Proj}_1 \setminus Z(\pi)} \partial_q^2 \oplus \bigoplus_{q\in Z(\pi)} \partial_q^1 \oplus \bigoplus_{q\in C_2} \partial_q^2 .$$

Thus

$$\ker \nabla_2 = \begin{cases} 0 & \text{if } n = 0 \pmod 4, \\ W(K)\langle y\rangle & \text{if } n = 2 \pmod 4, \\ 0 & \text{if } n = 1 \pmod 2, \end{cases}$$

and

$$\operatorname{coker} \nabla_2 = \begin{cases} W(K) \oplus W(K) & \text{if } n = 0 \pmod 4, \\ W(K) & \text{if } n = 2 \pmod 4, \\ W(K) & \text{if } n = 1 \pmod 2. \end{cases}$$

As before, the function field corresponding to the origin, i.e. to the maximal ideal (x, y, z), is K. Therefore we may rewrite the exact sequence from Theorem 2 in the following way. For n odd,

$$0 \to W(K) \xrightarrow{i^*} W(K(V_n)) \xrightarrow{\partial_V} \bigoplus_{p\in \mathrm{Spec}_1} W(K_p) \xrightarrow{\delta} \bigoplus_{r\in \mathrm{Spec}_2} W(K_r) \to 0,$$

and for n even,

$$\begin{aligned} 0 \to W(K) \oplus W(K) &\xrightarrow{i_1^*} W(K(V_n)) \\ &\xrightarrow{\partial_V} \bigoplus_{p\in \mathrm{Spec}_1} W(K_p) \xrightarrow{\delta} W(K) \oplus \bigoplus_{r\in \mathrm{Spec}_2} W(K_r) \to 0 \end{aligned}$$

where

(i) i^* is induced by the canonical inclusion $i : K \to K(V_n)$;
(ii) $i_1^*(\alpha, \beta) = i^*(\alpha) + \langle y\rangle i^*(\beta)$;
(iii) $\partial_V = \bigoplus_{p\in \mathrm{Spec}_1} \partial_p^2$.

This finishes the proof of Corollary 7. Corollary 8 follows from the following two facts.

• Let $\mathcal{L}$ be a line bundle over V_n^* with a nonvanishing section e. Then the inner product $b(e, e) = 1$ is nondegenerate and corresponds to $\langle 1\rangle$ in $W(K(V))$.

• Let $\mathcal{L}$ be a line bundle over V_n^* generated by two sections e_1, e_2, $x^m \cdot e_1 = z \cdot e_2$. The matrix

$$\begin{pmatrix} y & x^m \\ x^m & z \end{pmatrix}$$

has rank one at every point of $V_{2m}^* = V_{2m} \setminus \{(x, y, z)\}$. Therefore the inner product on $\mathcal{L}$

$$b(e_1, e_1) = y, \quad b(e_2, e_2) = z, \quad b(e_1, e_2) = x^m$$

is nondegenerate. Moreover, the image of $(\mathcal{L}, b)$ in $W(K(V))$ is equal to $\langle y\rangle$.

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Institute of Mathematics
University of Warsaw
Banacha 2
02-097 Warszawa, Poland
E-mail: jwptxa@mimuw.edu.pl

Received 28 May 1996;
revised 9 September 1996