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ON THE FUNDAMENTAL THEOREM OF ALGEBRA

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In most traditional textbooks on complex variables, the Fundamental Theorem of Algebra is obtained as a corollary of Liouville's theorem using elementary topological arguments.

The difficulty presented by such a scheme is that the proofs of Liouville's theorem involve complex integration which makes the reader believe that a proof of the Fundamental Theorem of Algebra is too involved, even when topological arguments are used.

In this note we show that such a difficulty can be avoided by giving a simple proof of the Maximum Modulus Theorem for rational functions and then obtaining the Fundamental Theorem of Algebra as a corollary. The proof obtained in this way is intuitive and mnemotechnic in contrast to the usual elementary proofs of the Fundamental Theorem of Algebra.

As usual we use $\mathbb{C}$ to denote the set of complex numbers. By $D(a,\varepsilon)$ we denote the set $\{z\in\mathbb{C}:|z-a|<\varepsilon\}$.

LEMMA. *Let f be a function such that $f(D(a,\varepsilon))$ is contained in a half plane whose defining straight line contains* 0. *Let $k\geq 1$. Then if the limit $\lim_{z\to a}f(z)/(z-a)^k$ exists*, *it is* 0.

Proof. Suppose $\lim_{z\to a}f(z)/(z-a)^k=b\neq 0$. Without loss of generality we can suppose that $b=1$ and that $f(D(a,\varepsilon))$ is contained in the half plane $\{z:\mathrm{Re}(z)\geq 0\}$ (take $\widetilde{f}=cf$ for a suitable $c\in\mathbb{C}$). Let $\{z_n:n\geq 1\}$ be a sequence such that $\lim_{n\to\infty}z_n=a$ and $(z_n-a)^k$ is a negative real number, for every $n\geq 1$. Thus we have

$$\begin{aligned}1&=\lim_{n\to\infty}f(z_n)/(z_n-a)^k=\mathrm{Re}\lim_{n\to\infty}f(z_n)/(z_n-a)^k\\&=\lim_{n\to\infty}\mathrm{Re}\,f(z_n)/(z_n-a)^k\leq 0,\end{aligned}$$

which is absurd. ■

MAXIMUM MODULUS THEOREM FOR RATIONAL FUNCTIONS. *Let $R(z)=p(z)/q(z)$, with p,q complex polynomials without common factors. Suppose*

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there exists $a \in \mathbb{C}$ *such that* $q(a) \neq 0$ *and* $|R(z)| \leq |R(a)|$ *for every* $z \in D(a, \varepsilon)$, *with* $\varepsilon > 0$. *Then* R *is a constant function.*

Proof. Suppose that R is not constant. Since $p_1(z) = q(a)p(z) - p(a)q(z)$ has a zero at $z = a$, there exist an integer $k \geq 1$ and a polynomial $c(z)$ such that $p_1(z) = (z-a)^k c(z)$ and $c(a) \neq 0$. Thus

$$\frac{R(z) - R(a)}{(z-a)^k} = \frac{p_1(z)}{q(a)q(z)(z-a)^k} = \frac{c(z)}{q(a)q(z)}$$

and therefore

$$\lim_{z \to a} \frac{R(z) - R(a)}{(z-a)^k} \neq 0.$$

Since $|R(z)| \leq |R(a)|$ for every $z \in D(a, \varepsilon)$, $f(z) = R(z) - R(a)$ satisfies the hypothesis of the above lemma (make a picture). Thus we arrive at a contradiction. ∎

FUNDAMENTAL THEOREM OF ALGEBRA. *A polynomial with no zeros is constant.*

Proof. Suppose that $p(z)$ is not constant and $p(z) \neq 0$ for every $z \in \mathbb{C}$. Since $\lim_{z \to \infty} |p(z)| = \infty$, there exists $a \in \mathbb{C}$ such that $|p(a)| \leq |p(z)|$ for every $z \in \mathbb{C}$. Thus applying the Maximum Modulus Theorem to the rational function $1/p(z)$, we arrive at a contradiction. ∎

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