ON THE FUNDAMENTAL THEOREM OF ALGEBRA

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In most traditional textbooks on complex variables, the Fundamental Theorem of Algebra is obtained as a corollary of Liouville’s theorem using elementary topological arguments.

The difficulty presented by such a scheme is that the proofs of Liouville’s theorem involve complex integration which makes the reader believe that a proof of the Fundamental Theorem of Algebra is too involved, even when topological arguments are used.

In this note we show that such a difficulty can be avoided by giving a simple proof of the Maximum Modulus Theorem for rational functions and then obtaining the Fundamental Theorem of Algebra as a corollary. The proof obtained in this way is intuitive and mnemotechnic in contrast to the usual elementary proofs of the Fundamental Theorem of Algebra.

As usual we use \( \mathbb{C} \) to denote the set of complex numbers. By \( D(a, \varepsilon) \) we denote the set \( \{ z \in \mathbb{C} : |z - a| < \varepsilon \} \).

**Lemma.** Let \( f \) be a function such that \( f(D(a, \varepsilon)) \) is contained in a half plane whose defining straight line contains 0. Let \( k \geq 1 \). Then if the limit \( \lim_{z \to a} f(z)/(z - a)^k \) exists, it is 0.

**Proof.** Suppose \( \lim_{z \to a} f(z)/(z - a)^k = b \neq 0 \). Without loss of generality we can suppose that \( b = 1 \) and that \( f(D(a, \varepsilon)) \) is contained in the half plane \( \{ z : \text{Re}(z) \geq 0 \} \) (take \( \tilde{f} = cf \) for a suitable \( c \in \mathbb{C} \)). Let \( \{ z_n : n \geq 1 \} \) be a sequence such that \( \lim_{n \to \infty} z_n = a \) and \( (z_n - a)^k \) is a negative real number, for every \( n \geq 1 \). Thus we have

\[
1 = \lim_{n \to \infty} f(z_n)/(z_n - a)^k = \text{Re} \lim_{n \to \infty} f(z_n)/(z_n - a)^k
\]

\[
= \lim_{n \to \infty} \text{Re} f(z_n)/(z_n - a)^k \leq 0,
\]

which is absurd. \( \blacksquare \)

**Maximum Modulus Theorem for Rational Functions.** Let \( R(z) = p(z)/q(z) \), with \( p, q \) complex polynomials without common factors. Suppose

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there exists $a \in \mathbb{C}$ such that $q(a) \neq 0$ and $|R(z)| \leq |R(a)|$ for every $z \in D(a, \varepsilon)$, with $\varepsilon > 0$. Then $R$ is a constant function.

Proof. Suppose that $R$ is not constant. Since $p_1(z) = q(a)p(z) - p(a)q(z)$ has a zero at $z = a$, there exist an integer $k \geq 1$ and a polynomial $c(z)$ such that $p_1(z) = (z - a)^k c(z)$ and $c(a) \neq 0$. Thus

$$ \frac{R(z) - R(a)}{(z - a)^k} = \frac{p_1(z)}{q(a)q(z)(z - a)^k} = \frac{c(z)}{q(a)q(z)} $$

and therefore

$$ \lim_{z \to a} \frac{R(z) - R(a)}{(z - a)^k} \neq 0. $$

Since $|R(z)| \leq |R(a)|$ for every $z \in D(a, \varepsilon)$, $f(z) = R(z) - R(a)$ satisfies the hypothesis of the above lemma. Thus we arrive at a contradiction.

Fundamental Theorem of Algebra. A polynomial with no zeros is constant.

Proof. Suppose that $p(z)$ is not constant and $p(z) \neq 0$ for every $z \in \mathbb{C}$. Since $\lim_{z \to \infty} |p(z)| = \infty$, there exists $a \in \mathbb{C}$ such that $|p(a)| \leq |p(z)|$ for every $z \in \mathbb{C}$. Thus applying the Maximum Modulus Theorem to the rational function $1/p(z)$, we arrive at a contradiction.

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