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A MODULUS FOR PROPERTY (β) OF ROLEWICZ

BY

J. M. AYERBE, T. DOMÍNGUEZ BENAVIDES
AND S. FRANCISCO CUTILLAS (SEVILLA)

We define a modulus for the property (β) of Rolewicz and study some useful properties in fixed point theory for nonexpansive mappings. Moreover, we calculate this modulus in ℓ^p spaces for the main measures of noncompactness.

0. Introduction. In the geometric theory of Banach spaces, the notion of modulus of convexity plays a very significant role. It allows classifying Banach spaces from the point of view of their geometrical structure. In this regard, the modulus of convexity is a useful tool in fixed point theory. A lot of facts concerning this notion and its applications may be found, for example, in [GK] and [O].

Recently, K. Goebel, T. Sękowski, J. Banaś *et al.* [GS, B1, DL] have proposed several generalizations of the notion of modulus of convexity using some measures of noncompactness. With these moduli (called moduli of noncompact convexity), they proved several interesting facts concerning the geometric theory of Banach spaces. Moreover, these moduli are suitable for nearly uniformly convex spaces (N.U.C.) introduced in [H] in the same sense as the classical modulus of convexity of Clarkson is suitable for uniformly convex spaces (U.C.). Property (β) of Rolewicz, introduced in [R2], is a geometric condition which is situated between U.C. and N.U.C. In [ADF] we defined a modulus for this property utilizing a characterization given by Kutzarova [Ku]. Our results, together with the results obtained in [KMP], permitted obtaining properties not only of the space itself, but also of its dual. However, we were not able to calculate the modulus for Kuratowski's measure of noncompactness in ℓ^p-spaces, and for other measures of noncompactness the functions which we obtained are not simple. In this paper we

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use directly the definition of Rolewicz in order to define the modulus. In this case we have been able to calculate the modulus for the main measures of noncompactness (including Kuratowski's measure) in ℓ^p, obtaining simpler values.

1. Notations, definitions and first results. In this paper X will denote an infinite-dimensional Banach space. Let $\mathcal{B}$ be the family of bounded subsets of X. A map $\mu : \mathcal{B} \to [0,\infty)$ is called a *measure of noncompactness* on X if it has the property that $\mu(A) = 0$ if and only if A is a precompact set. The first measure of noncompactness (set-measure, denoted by $\alpha(A)$) was defined by Kuratowski in [K] as $\inf\{\varepsilon \geq 0 : A$ can be covered by finitely many sets with diameter $\leq \varepsilon\}$. Another measure of noncompactness (ball-measure, denoted by $\beta(A)$) was introduced by several authors (see [GGM] or [S]) as $\inf\{\varepsilon \geq 0 : A$ can be covered by finitely many balls with diameter $\leq \varepsilon\}$. In [WW, p. 91] another measure of noncompactness is defined by $S(A) = \sup\{\varepsilon \geq 0 :$ there exists a sequence $\{x_n\}$ in A with $\operatorname{sep}(\{x_n\}) > \varepsilon\}$, where $\operatorname{sep}(\{x_n\}) = \inf\{\|x_n - x_m\| : n \neq m\}$. It is easy to prove that $S(A) \leq \alpha(A) \leq \beta(A) \leq 2S(A)$ for every bounded subset A of X.

Throughout this paper we denote by μ any of these measures of noncompactness. The main properties of these measures can be found in [AKPRS]. We will use in this paper the following: (1) $\mu(A) = \mu(\overline{A})$, (2) $\mu(A \cup B) = \max\{\mu(A), \mu(B)\}$, (3) $\mu(A) \leq \mu(B)$ if $A \subset B$, (4) $\mu(tA) = |t|\mu(A)$, (5) $\mu(\operatorname{co}(A)) = \mu(A)$, where $\operatorname{co}(A)$ denotes the convex hull of A, (6) $\mu(A) = \mu(R_n(A))$ if X is a Banach space with Schauder basis $\{e_i : i \in \mathbb{N}\}$ and $R_n : X \to X$ is defined by $R_n(\sum_{i=1}^{\infty} x_i e_i) = \sum_{i=n+1}^{\infty} x_i e_i$.

Utilizing these measures of noncompactness an important geometric property of a Banach space, named property (β), has been defined in [R2].

In a Banach space X with closed unit ball B_X and unit sphere S_X, the *drop* $D(x, B_X)$ defined by an element $x \in X \backslash B_X$ is the set $\operatorname{co}(\{x\} \cup B_X)$, and we write $R_x = D(x, B_X) \backslash B_X$.

X is said to have *property* (β) if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $1 < \|x\| < 1 + \delta$ implies $\mu(R_x) < \varepsilon$.

We now define a modulus for property (β) of Rolewicz.

DEFINITION 1.1. Let X be a Banach space. We define the modulus $P_{X,\mu} : [0, S(B_X)) \to [0,\infty)$ by

$$P_{X,\mu}(\varepsilon) = \inf\{\|x\| - 1 : x \in X,\ \|x\| > 1,\ \mu(R_x) \geq \varepsilon\}.$$

Remark 1.2. (a) It is known that $\alpha(B_X) = \beta(B_X) = 2$ in every Banach space (see [AKPRS, p. 3]) and that $S(B_X)$ depends on the space X. For example, if $1 < p < \infty$ then $S(B_{\ell^p}) = 2^{1/p}$ (see [WW, p. 91].

(b) From Proposition 1 in [KP] it is not difficult to check that this modulus is defined on the whole interval $[0, \mu(B_X))$. Moreover, $0 \leq P_{X,\mu}(\varepsilon) \leq \varepsilon/(\mu(B_X) - \varepsilon)$. It follows that $P_{X,\mu}$ is continuous at zero.

It is easy to prove the following proposition:

PROPOSITION 1.3. *$P_{0,\mu}(X) = \sup\{\varepsilon \geq 0 : P_{X,\mu}(\varepsilon) = 0\} = 0 \Leftrightarrow X$ has property (β).*

Therefore, this modulus is suitable for this property in the same sense as the classical Clarkson modulus is suitable for the uniform convexity.

In [KMP] the following generalization of property (β) is defined.

DEFINITION 1.4. A Banach space X has *property* (β, γ) for some $\gamma > 0$ if for all $\gamma' > \gamma$ there exists a $\delta > 0$ such that for every $x \in B_X$ and every $\{x_n\} \subset B_X$ with $\operatorname{sep}(\{x_n\}) > \gamma'$, there is an $i \in \mathbb{N}$ such that $\|(x + x_i)/2\| < 1 - \delta$.

PROPOSITION 1.5. *Let X be a Banach space. If $P_{0,\mu}(X) < 1/2$, then X has property (β, γ) for some $\gamma < 1$.*

Proof. Since $P_{0,S}(X) \leq P_{0,\alpha}(X) \leq P_{0,\beta}(X)$, it suffices to prove the result for $P_{0,S}(X)$.

Since $P_{0,S}(X) < 1/2$, there exists $0 < \varepsilon < 1/2$ such that $P_{X,S}(\varepsilon) > 0$. Suppose, by contradiction, that X fails property (β, γ) if $\gamma < 1$. Then X fails property $(\beta, 2\varepsilon)$ and thus, there exists $\gamma' > 2\varepsilon$ such that

$$\forall \delta > 0\ \exists x \in B_X\ \exists \{x_n\} \subset B_X \text{ with } \operatorname{sep}(\{x_n\}) > \gamma' \text{ such that} \tag{1}$$
$$\left\| \frac{x + x_i}{2} \right\| \geq 1 - \delta \quad \forall i \in \mathbb{N}.$$

Set $\eta = P_{X,S}(\varepsilon)$ and let η' be a real number, $0 < \eta' < \eta$, $\{x_n\}$ a sequence in B_X with $\operatorname{sep}(\{x_n\}) \geq 2\varepsilon$ and x an arbitrary point of B_X.

We claim that there exists $n \in \mathbb{N}$ such that

$$[x, x_n] \cap \left(1 - \frac{\eta'}{\eta + 2}\right) B_X \neq \emptyset,$$

where $[x, x_n]$ denotes the line segment between x and x_n. Indeed, by contradiction, assume that

$$[x, x_n] \cap \left(1 - \frac{\eta'}{\eta + 2}\right) B_X = \emptyset$$

and take $y = (1 + \eta')x$. Then

$$\|y\| = (1 + \eta')\|x\| \leq 1 + \eta' < 1 + \eta$$

and

$$\|y\| > (1 + \eta')\left(1 - \frac{\eta'}{\eta + 2}\right) = 1 + \frac{\eta'}{\eta + 2}(\eta + 1 - \eta') > 1.$$

Therefore $1 < \|y\| < 1 + \eta$. Let $y_n = (y + x_n)/2$ for each $n \in \mathbb{N}$. We can write

$$y_n = \left(1 + \frac{\eta'}{2}\right)\left[\frac{1+\eta'}{2+\eta'}x + \frac{1}{2+\eta'}x_n\right]$$

and so

$$\|y_n\| > \left(1 + \frac{\eta'}{2}\right)\left(1 - \frac{\eta'}{\eta+2}\right) = 1 + \frac{\eta'}{2(\eta+2)}(\eta - \eta') > 1.$$

This implies that $\{y_n\}$ is contained in R_y. Since $S(R_y) < \varepsilon$ we conclude that $\operatorname{sep}(\{y_n\}) < \varepsilon$ and hence $\operatorname{sep}(\{x_n\}) < 2\varepsilon$, contradicting our hypothesis.

Therefore, there exist $i \in \mathbb{N}$, $\lambda_0, \lambda_1 \in \mathbb{R}$, $\lambda_0 \geq 0$, $\lambda_1 \geq 0$, $\lambda_0 + \lambda_1 = 1$, such that $\|\lambda_0 x + \lambda_1 x_i\| \leq 1 - \eta'/(\eta + 2)$. Suppose $\lambda_0 \geq \lambda_1$ (the opposite case is analogous). Then

$$\frac{x + x_i}{2} = \frac{1}{2}\left[\frac{1}{\lambda_0}(\lambda_0 x + \lambda_1 x_i) + \left(1 - \frac{\lambda_1}{\lambda_0}\right)x_i\right]$$

and thus

$$\left\|\frac{x + x_i}{2}\right\| \leq \frac{1}{2}\left[\frac{1}{\lambda_0}\left(1 - \frac{\eta'}{\eta+2}\right) + \left(1 - \frac{\lambda_1}{\lambda_0}\right)\right] \leq 1 - \frac{\eta'}{2(\eta+2)},$$

contradicting (1). ∎

This proposition, together with Theorem 2 bis in [KMP], yields the following useful result in metric fixed point theory:

COROLLARY 1.6. *If $P_{0,\mu}(X)$ is less than $1/2$, then the spaces X and X^* are reflexive and have normal structure.*

Remark 1.7. Let us recall that normal structure is not a self-dual property. Actually Bynum [By] proved that the space $X = (\ell^2, \|\cdot\|_{2,1})$, where $\|x\|_{2,1} = \|x^+\|_2 + \|x^-\|_2$ ($\|\cdot\|_2$ is the ℓ^2-norm and x^+ and x^- are the positive and negative parts of x), has normal structure, while its dual lacks it.

2. Computation of the modulus in ℓ^p-spaces. We start proving several technical lemmas.

LEMMA 2.1. *Let X be a Banach space, $x \in X$ with $\|x\| > 1$ and $Y_x = \{y \in S_X : \|\lambda y + (1-\lambda)x\| > 1\ \forall \lambda \in [0,1)\}$. Then $R_x = \operatorname{co}(Y_x \cup \{x\}) \setminus B_X$.*

Proof. Since $Y_x \subset B_X$, it follows that $\operatorname{co}(Y_x \cup \{x\}) \subset \operatorname{co}(B_X \cup \{x\})$ and so $\operatorname{co}(Y_x \cup \{x\}) \setminus B_X \subset R_x$.

Conversely, let $z \in R_x$. Then there is $y \in B_X$ such that $z = \mu y + (1-\mu)x$ for some $\mu \in [0,1]$. Consider the continuous function $\varphi(\lambda) = \|\lambda y + (1-\lambda)x\|$ defined on $[0,1]$. Since $\varphi(1) \leq 1 < \varphi(0)$ we know that the set $L = \{\lambda \in$

$[0,1] : \varphi(\lambda) \le 1\}$ is nonempty and $\varphi(\lambda_0) = 1$ if λ_0 is the infimum of this set. Let $y_0 = \lambda_0 y + (1-\lambda_0)x$. Then $y_0 \in Y_x$ because if $\lambda \in [0,1)$ then

$$\|\lambda y_0 + (1-\lambda)x\| = \|\lambda\lambda_0 y + (1-\lambda\lambda_0)x\| = \varphi(\lambda\lambda_0) > 1.$$

Since $z = (\mu/\lambda_0)y_0 + (1-(\mu/\lambda_0))x$ with $\mu/\lambda_0 \in [0,1)$ we have $z \in \mathrm{co}(Y_x \cup \{x\})$ and the proof is complete. ■

The following lemma was proved by Rolewicz in [R1] and it will play a crucial role in our reasoning.

LEMMA 2.2. *Let X be a uniformly convex Banach space. Then there is a positive increasing function $f(r)$, defined for positive r, such that* $\lim_{r\to 0^+} f(r) = 0$ *and* $\mathrm{diam}(R_x) \le f(\|x\|-1)$ *for every* $x \in X\backslash B_X$.

LEMMA 2.3. *Let X be a uniformly convex Banach space*, $x \in X\backslash B_X$, $0 < \varepsilon < \|x\| - 1$ *and*

$$\widehat{R}_x = \{z = \lambda y + (1-\lambda)x : 0 \le \lambda < 1,\ \|y\| \le 1,\ \|z\| \ge 1+\varepsilon\}.$$

Then $R_x \subset \widehat{R}_x + f(\varepsilon)B_X$, *where f is the function obtained in the above lemma.*

Proof. Let $z \in R_x$, $1 < \|z\| < 1+\varepsilon$, and consider the line segment $[z,x]$. Then there exists $z' \in (z,x)$ such that $\|z'\| = 1+\varepsilon$. Thus $z' \in \widehat{R}_x$ and it can be easily shown that $z \in R_{z'}$. Therefore $\|z - z'\| \le \mathrm{diam}(R_{z'}) \le f(\|z'\|-1) = f(\varepsilon)$. Hence $z \in \widehat{R}_x + f(\varepsilon)B_X$. ■

THEOREM 2.4. *Let* $1 < p < \infty$ *and* $x \in \ell^p$, $\|x\|_p > 1$. *Then*

$$S(R_x) = (2(1 - 1/\|x\|_p^q))^{1/p}$$

where q is the conjugate exponent of p, that is, $1/p + 1/q = 1$.

Proof. We split the proof in two steps. In the first step, we suppose that x is an eventually null sequence, that is, $x = (x_1, \ldots, x_n, 0, \ldots)$ with $\|x\|_p > 1$.

Let Y_x be the set defined in Lemma 2.1, $y \in Y_x$ and write $y = a + b$ with $a = (a_1, \ldots, a_n, 0, \ldots)$, $b = (0, \overset{(n)}{\ldots}, 0, b_1, b_2, \ldots)$ and so $\|y\|_p^p = \|a\|_p^p + \|b\|_p^p = 1$.

Consider the norming functional f_y of y, given by

$$f_y = (a_1|a_1|^{p-2}, a_2|a_2|^{p-2}, \ldots, a_n|a_n|^{p-2}, b_1|b_1|^{p-2}, b_2|b_2|^{p-2}, \ldots)$$

(see [M, p. 48]) and define

$$J_a = (a_1|a_1|^{p-2}, a_2|a_2|^{p-2}, \ldots, a_n|a_n|^{p-2}, 0, 0, \ldots).$$

Since $\|\lambda y + (1-\lambda)x\|_p > 1$ for all $0 \le \lambda < 1$ and f_y is the directional derivative of the norm (see [B, p. 182]), we have $f_y(x-y) \ge 0$, which implies

$f_y(x) \ge 1$. Therefore

$$1 \le f_y(x) = J_a(x) \le \|x\|_p \|J_a\|_q.$$

Keeping in mind that $\|J_a\|_q = \|a\|_p^{p/q}$ we conclude that $1 \le \|a\|_p^{p/q}\|x\|_p$, that is, $1 \le \|a\|_p^p \|x\|_p^q$. It follows that

$$\|b\|_p^p \le 1 - 1/\|x\|_p^q.$$

Hence $R_n(Y_x) \subset (1 - 1/\|x\|_p^q)^{1/p} B_{\ell^p}$, where $R_n : \ell^p \to \ell^p$ is the projection defined by $R_n(\sum_{i=1}^{\infty} x_i e_i) = \sum_{i=n+1}^{\infty} x_i e_i$ and $\{e_i : i \in \mathbb{N}\}$ is the standard Schauder basis of ℓ^p. Thus from Lemma 2.1,

$$\begin{aligned} S(R_x) &= S(\mathrm{co}(Y_x \cup \{x\}) \backslash B_{\ell^p}) \le S(\mathrm{co}(Y_x \cup \{x\})) = S(Y_x) = S(R_n Y_x) \\ &\le S((1 - 1/\|x\|_p^q)^{1/p} B_{\ell^p}) = (2(1 - 1/\|x\|_p^q))^{1/p}. \end{aligned}$$

Actually, $S(R_x) = (2(1 - 1/\|x\|_p^q))^{1/p}$. Indeed, consider the set

$$A = \{(x_1/\|x\|_p^q, \ldots, x_n/\|x\|_p^q, c_1, c_2, \ldots) : \|(c_1, c_2, \ldots)\|_p = (1 - 1/\|x\|_p^q)^{1/p}\}.$$

This set has the following properties:

(i) $S(A) = (2(1 - 1/\|x\|_p^q))^{1/p}$ because $S(A) = S((1 - 1/\|x\|_p^q)^{1/p} S_{\ell^p})$.
(ii) $A \subset Y_x$.

Indeed, if $y \in A$ then

$$\|y\|_p^p = \sum_{i=1}^{n} |x_i|^p / \|x\|_p^{pq} + 1 - 1/\|x\|_p^q = 1.$$

Moreover, for all $0 \le \lambda < 1$ we have

$$\begin{aligned} \|\lambda y + (1-\lambda)x\|_p^p &= \sum_{i=1}^{n} \left| \frac{\lambda x_i}{\|x\|_p^q} + (1-\lambda) x_i \right|^p + \left(1 - \frac{1}{\|x\|_p^q}\right)\lambda^p \\ &= \frac{1}{\|x\|_p^q} (\lambda + \|x\|_p^q (1-\lambda))^p + \left(1 - \frac{1}{\|x\|_p^q}\right)\lambda^p > 1. \end{aligned}$$

Since $Y_x \subset \overline{R}_x$ we obtain $S(A) \le S(Y_x) \le S(\overline{R}_x) = S(R_x)$ and therefore $(2(1 - 1/\|x\|_p^q))^{1/p} \le S(R_x)$.

Now, we prove the general case. From the finite-dimensional case, we know that for every $n \in \mathbb{N}$ large enough such that $\|P_n x\|_p > 1$, we have

$$S(R_{P_n x}) = (2(1 - 1/\|P_n x\|_p^q))^{1/p}$$

where P_n are the natural projections associated with the standard basis of ℓ^p. Therefore, since

$$\lim_{n\to\infty} (2(1 - 1/\|P_n x\|_p^q))^{1/p} = (2(1 - 1/\|x\|_p^q))^{1/p}$$

it suffices to show that $\lim_{n\to\infty} S(R_{P_n x}) = S(R_x)$ in order to obtain the required result.

Let $0 < \varepsilon < (\|x\|_p - 1)/2$ and $n_0 \in \mathbb{N}$ be such that $\|x - P_n x\|_p < \varepsilon$ for all $n \geq n_0$. Then $\|x\|_p$ and $\|P_n x\|_p$ are greater than $1 + \varepsilon$.

Let us see that $\widehat{R}_{P_n x} \subset R_x + \varepsilon B_{\ell^p}$ for all $n \geq n_0$ (for the definition of $\widehat{R}_x$ see Lemma 2.3). Take $\widehat{z} \in \widehat{R}_{P_n x}$. Then $\widehat{z} = \lambda y + (1-\lambda) P_n x$ with $\|z\|_p \geq 1 + \varepsilon$, and let $z = \lambda y + (1-\lambda)x$. Then

$$\|z - \widehat{z}\|_p = (1-\lambda)\|x - P_n x\|_p < (1-\lambda)\varepsilon < \varepsilon$$

and

$$\|z\|_p \geq \|\widehat{z}\|_p - \|\widehat{z} - z\|_p > 1 + \varepsilon - \varepsilon = 1.$$

Hence $z \in R_x$ and so $\widehat{R}_{P_n x} \subset R_x + \varepsilon B_{\ell^p}$.

Analogously we can verify that $\widehat{R}_x \subset R_{P_n x} + \varepsilon B_{\ell^p}$ for all $n \geq n_0$ and thus

$$R_{P_n x} \subset \widehat{R}_{P_n x} + f(\varepsilon) B_{\ell^p} \subset R_x + (\varepsilon + f(\varepsilon)) B_{\ell^p}$$

for all $n \geq n_0$. Therefore

$$S(R_{P_n x}) \leq S(R_x) + (\varepsilon + f(\varepsilon)) 2^{1/p}.$$

Similarly, for all $n \geq n_0$ we have

$$R_x \subset \widehat{R}_x + f(\varepsilon) B_{\ell^p} \subset R_{P_n x} + (\varepsilon + f(\varepsilon)) B_{\ell^p}$$

and thus

$$S(R_x) \leq S(R_{P_n x}) + (\varepsilon + f(\varepsilon)) 2^{1/p}.$$

It follows that for all $n \geq n_0$,

$$|S(R_x) - S(R_{P_n x})| \leq (\varepsilon + f(\varepsilon)) 2^{1/p}$$

and since $f(\varepsilon) \to 0$ as $\varepsilon \to 0^+$ we conclude that $\lim_{n\to\infty} S(R_{P_n x}) = S(R_x)$. ∎

With similar arguments we can prove the following result:

Corollary 2.5. *Let $1 < p < \infty$ and $x \in \ell^p$, $\|x\|_p > 1$. Then*

$$\alpha(R_x) = \beta(R_x) = 2(1 - 1/\|x\|_p^q)^{1/p}.$$

Theorem 2.6. *If $1 < p < \infty$ and $0 \leq \varepsilon < 2^{1/p}$, then*

$$P_{\ell^p, S}(\varepsilon) = \left(\frac{2}{2 - \varepsilon^p}\right)^{1/q} - 1.$$

Proof. According to Theorem 2.4,

$$\inf\{\|x\|_p > 1 : S(R_x) \geq \varepsilon\} = \inf\{\delta > 1 : (2(1 - 1/\delta^q))^{1/p} \geq \varepsilon\}$$

and since the function $\delta \to (2(1-1/\delta^q))^{1/p}$ is strictly increasing, the infimum is attained when $(2(1-1/\delta^q))^{1/p} = \varepsilon$, that is, when

$$\delta = \left(\frac{2}{2 - \varepsilon^p}\right)^{1/q}. \ ∎$$

Similarly, we can prove the following result:

COROLLARY 2.7. *If* $1 < p < \infty$ *and* $0 \le \varepsilon < 2$, *then*

$$P_{\ell^p,\alpha}(\varepsilon) = P_{\ell^p,\beta}(\varepsilon) = \left(\frac{2^p}{2^p - \varepsilon^p}\right)^{1/q} - 1.$$

Remark 2.8. The cases $p = 1$ and $p = \infty$ are much easier. Indeed:

(a) $P_{\ell^\infty,\mu}(\varepsilon) = 0$ for all $\varepsilon \in [0, 2]$. In order to obtain this result we are going to show that for all $\eta > 0$ there exists $x \in \ell^\infty$ with $\|x\|_\infty = 1 + \eta$ such that $\mu(R_x) = 2$.

Consider the sequence $\{y_n\}$ in ℓ^∞ given by

$$y_n = (1, \overset{(n)}{\ldots}, 1, -1, 0, 0, \ldots)$$

for every $n \in \mathbb{N}$ and $x = (1 + \eta, 0, 0, \ldots)$. Obviously $\|y_n\|_\infty = 1$ for all $n \in \mathbb{N}$ and $\operatorname{sep}(\{y_n\}) = 2$. Moreover, for every $0 \le \lambda < 1$ and $n \in \mathbb{N}$ we have

$$\|\lambda y_n + (1 - \lambda)x\|_\infty = \lambda + (1 - \lambda)(1 + \eta) > 1.$$

Hence $y_n \in Y_x = \{y \in S_{\ell^\infty} : \|\lambda y + (1 - \lambda)x\| > 1 \ \forall \lambda \in [0, 1)\}$ and since $Y_x \subset \overline{R}_x$ we can conclude that $\mu(R_x) = 2$.

(b) $P_{\ell^1,\mu}(\varepsilon) = 0$ for all $\varepsilon \in [0, 2]$. The argument is the same taking as $\{y_n\}$ the basis sequence.

(c) Analogously $P_{c,\mu}(\varepsilon) = P_{c_0,\mu}(\varepsilon) = 0$ for all $\varepsilon \in [0, 2]$.

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Departamento de Análisis Matemático
Facultad de Matemáticas
Universidad de Sevilla
Aptdo. 1160
Sevilla 41080, Spain
E-mail: ayerbe@cica.es

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