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GENERIC PROPERTIES OF SOME BOUNDARY VALUE PROBLEMS FOR DIFFERENTIAL EQUATIONS

BY

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In this note we consider ordinary differential equations with the Picard, Nicoletti or Floquet boundary value conditions. We are also concerned with some boundary value problems for a hyperbolic equation which were posed in [6]. In [1] we presented results concerning the existence of solutions and the existence and uniqueness of solutions of these problems. These results seem to be the best possible for certain classes of right-hand sides and they develop ideas of Lasota and Olech [3] and of Kasprzyk and Myjak [2].

In the present paper we widen the classes of right-hand sides considered. The above mentioned results are not true for all maps from the extended classes. We show, however, that the subset of maps for which the existence of solutions or the existence and uniqueness of solutions fails is of the first Baire category in an appropriate complete metric space. Recall that a subset of a metric space is called *residual* if its complement is of the first category. A property of elements of a metric space which holds true for every member of a residual set is called *generic*.

This approach to differential equations was introduced by Orlicz [5]. He proved a theorem on genericity of uniqueness for the Cauchy problem. Since then generic properties of functional and differential equations have been studied by many authors. In Myjak's monograph [4] the reader can find several results and a list of references.

1. Preliminaries. In the sequel Δ denotes $[0,p_1]\times\ldots\times[0,p_i]$ with the Lebesgue measure μ. Let E be the Banach space $C(\Delta,\mathbb{R}^n)$ with the usual supremum norm $\|\cdot\|_\infty$. We denote by $\mathcal{K}$ the real Hilbert space of all $x\in L^2(\Delta,\mathbb{R}^n)$ with the scalar product given by $(\gamma\in C(\Delta,(0,\infty)))$

$$(x,y)_{\mathcal{K}}=\int_{\Delta}\gamma(t)\sum_{j=1}^{n}x_j(t)y_j(t)\,d\mu$$

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and the corresponding norm $\|\cdot\|_{\mathcal{K}}$. $\overline{B}_{\infty}(0,r)$ and $\overline{B}_{\mathcal{K}}(0,r)$ stand for the closed balls about 0 with radius r in the spaces E and $\mathcal{K}$ respectively. We denote by J the canonical continuous injection from E to $\mathcal{K}$.

$(\mathcal{Z},\varrho)$ is the Fréchet space $C(\Delta\times(\mathbb{R}^n)^k,\mathbb{R}^n)$ endowed with the metric

$$\varrho(f,g)=\sum_{j=1}^{\infty}\frac{2^{-j}\sup_j}{1+\sup_j},$$

where we have set $\sup_j=\sup\{|f(t,u_0,\ldots,u_{k-1})-g(t,u_0,\ldots,u_{k-1})| : t\in\Delta, |u_0|,\ldots,|u_{k-1}|\le j\}$. Consider two closed subspaces of $\mathcal{Z}$:

$$\mathcal{G}=\{f\in\mathcal{Z} : |f(t,u_0,\ldots,u_{k-1})|\le L_0|u_0|+\ldots+L_{k-1}|u_{k-1}|+M,\quad u_0,\ldots,u_{k-1}\in\mathbb{R}^n\},$$

$$\begin{aligned}\mathcal{F}=\{f\in\mathcal{Z} : |f(t,u_0,\ldots,u_{k-1})&-f(t,v_0,\ldots,v_{k-1})|\\ &\le L_0|u_0-v_0|+\ldots+L_{k-1}|u_{k-1}-v_{k-1}|,\\ &\qquad u_0,\ldots,u_{k-1},v_0,\ldots,v_{k-1}\in\mathbb{R}^n\},\end{aligned}$$

where $L_0,\ldots,L_{k-1},M$ are nonnegative constants.

Let $T_0,\ldots,T_{k-1}:\mathcal{K}\to E$ be completely continuous and linear. We denote by θ_j the norm of the bounded linear operator $J\circ T_j:\mathcal{K}\to\mathcal{K}$, where $j=0,\ldots,k-1$. For any $f\in\mathcal{Z}$ and $\xi_0,\ldots,\xi_{k-1}\in E$ define a completely continuous operator $h_f:E\to E$ by

$$h_f(x)(t)=f(t,\xi_0(t)+T_0\circ J(x)(t),\ldots,\xi_{k-1}(t)+T_{k-1}\circ J(x)(t)),\quad t\in\Delta.$$

THEOREM 1. *Suppose that $\theta_0L_0+\ldots+\theta_{k-1}L_{k-1}\le 1$. Then the set $\mathcal{G}^*$ of all $f\in\mathcal{G}$ such that for all $\xi_0,\ldots,\xi_{k-1}\in E$ the map h_f has at least one fixed point is residual in the complete space $(\mathcal{G},\varrho)$.*

Proof. Define $\mathcal{N}=\{\mu f : \mu\in[0,1), f\in\mathcal{G}\}$. Observe that $\mathcal{N}$ is dense in $\mathcal{G}$. For every $g\in\mathcal{N}$ and $l\in\mathbb{N}$ we will find $\eta(g,l)>0$ such that

$$\bigcap_{l=1}^{\infty}\bigcup_{g\in\mathcal{N}}B(g,\eta(g,l))\subset\mathcal{G}^*.\tag{1.1}$$

Hence $\mathcal{G}^*$ contains a dense G_δ-set, and so it is residual in $(\mathcal{G},\varrho)$.

Suppose that $l\in\mathbb{N}$ and $g\in\mathcal{N}$. Choose $\mu(g)\in[0,1)$ such that $g\in\mu(g)\mathcal{G}$. Set

$$R(g,l)=\|J\|(1+M+L_0l+\ldots+L_{k-1}l)/(1-\mu(g)),$$
$$r(g,l)=M+L_0l+\ldots+L_{k-1}l+(L_0\|T_0\|+\ldots+L_{k-1}\|T_{k-1}\|)R(g,l).$$

Define the nonempty, closed, convex and bounded set $C(g,l)\subset E$ by

$$C(g,l)=J^{-1}(\overline{B}_{\mathcal{K}}(0,R(g,l)))\cap\overline{B}_{\infty}(0,r(g,l)).$$

From the definition of ϱ and the continuity of $T_0,\ldots,T_{k-1}$ it follows that one can find $\eta(g,l)>0$ such that for any $f\in B(g,\eta(g,l))$ and $x,\xi_0,\ldots,\xi_{k-1}\in E$

we have

$$(1.2)\qquad \|\xi_0\|_\infty \le l, \ldots, \|\xi_{k-1}\|_\infty \le l, \|J(x)\|_{\mathcal{K}} \le R(g,l) \Rightarrow \|h_f(x) - h_g(x)\|_\infty \le 1.$$

Now we prove (1.1). Suppose that f belongs to the left side of that inclusion. Then there exists a sequence $\{g_l\}$ such that $g_l \in \mathcal{N}$ and $\varrho(g_l, f) < \eta(g_l, l)$ for $l \in \mathbb{N}$. Observe that for all $\xi_0, \ldots, \xi_{k-1} \in E$ such that $\|\xi_0\|_\infty, \ldots, \|\xi_{k-1}\|_\infty \le l$ we have $h_f(C(g_l, l)) \subset C(g_l, l)$. In fact, $g_l \in \mu(g_l)\mathcal{G}$. Furthermore, from (1.2) it follows that for $x \in C(g_l, l)$ we have

$$\begin{aligned}
&\|J \circ h_f(x)\|_{\mathcal{K}} \\
&\quad \le \|J \circ (h_f - h_{g_l})(x)\|_{\mathcal{K}} + \|J \circ h_{g_l}(x)\|_{\mathcal{K}} \\
&\quad \le \|J\| + \|M\|_{\mathcal{K}} + L_0\|J(\xi_0)\|_{\mathcal{K}} + \ldots + L_{k-1}\|J(\xi_{k-1})\|_{\mathcal{K}} + \mu(g_l)\|J(x)\|_{\mathcal{K}} \\
&\quad \le R(g_l, l)
\end{aligned}$$

and

$$\begin{aligned}
\|h_f(x)\|_\infty &\le M + L_0\|\xi_0\|_\infty + \ldots + L_{k-1}\|\xi_{k-1}\|_\infty \\
&\qquad + (L_0\|T_0\| + \ldots + L_{k-1}\|T_{k-1}\|)\|J(x)\|_{\mathcal{K}} \\
&\le r(g_l, l),
\end{aligned}$$

which proves $h_f(C(g_l, l)) \subset C(g_l, l)$. The map $h_f : E \to E$ is completely continuous and so it has a fixed point. One can choose $l \in \mathbb{N}$ arbitrarily and consequently $f \in \mathcal{G}^*$. This completes the proof.

THEOREM 2. *Suppose that $\theta_0 L_0 + \ldots + \theta_{k-1} L_{k-1} \le 1$. Then the set $\mathcal{F}^*$ of all $f \in \mathcal{F}$ such that for all $\xi_0, \ldots, \xi_{k-1} \in E$ the map h_f has exactly one fixed point is residual in the complete space $(\mathcal{F}, \varrho)$.*

Proof. As in the proof of Theorem 1 one can show that the set of all $f \in \mathcal{F}$ such that for all $\xi_0, \ldots, \xi_{k-1} \in E$ the map h_f has at least one fixed point is residual in $\mathcal{F}$. Therefore, it suffices to prove that the set $\widetilde{\mathcal{F}}$ of all $f \in \mathcal{F}$ such that for all $\xi_0, \ldots, \xi_{k-1} \in E$ the map h_f has at most one fixed point is residual in $\mathcal{F}$.

Define $\mathcal{M} = \{\mu f : \mu \in [0,1), f \in \mathcal{F}\}$. Observe that $\mathcal{M}$ is dense in $\mathcal{F}$. For every $g \in \mathcal{M}$ and $l \in \mathbb{N}$ we will find $\varepsilon(g,l) > 0$ such that

$$(1.3)\qquad \bigcap_{l=1}^{\infty} \bigcup_{g \in \mathcal{M}} B(g, \varepsilon(g,l)) \subset \widetilde{\mathcal{F}}.$$

Then $\widetilde{\mathcal{F}}$ is residual in $\mathcal{F}$. Suppose that $l = 1, 2, \ldots$ and $g \in \mathcal{M}$. Choose $\mu(g) \in [0,1)$ such that $g \in \mu(g)\mathcal{F}$. From the definition of ϱ and the continuity of $J, T_0, \ldots, T_{k-1}$ it follows that one can find $\varepsilon(g,l) > 0$ such that for any $f \in B(g, \varepsilon(g,l))$ and $x, \xi_0, \ldots, \xi_{k-1} \in E$ we have

$$\|\xi_0\|_\infty \le l, \ldots, \|\xi_{k-1}\|_\infty \le l, \|x\|_\infty \le l \quad \Rightarrow \|h_f(x) - h_g(x)\|_\infty \le \frac{1-\mu(g)}{2l\|J\|}. \tag{1.4}$$

Now we prove (1.3). Suppose that f belongs to the left side of that inclusion. Then there exists a sequence $\{g_l\}$ such that $g_l \in \mathcal{M}$ and $\varrho(g_l, f) < \varepsilon(g_l, l)$ for $l = 1, 2 \ldots$ Assume that $x, y, \xi_0, \ldots, \xi_{k-1} \in E$, $x = h_f(x)$, $y = h_f(y)$ and $\|x\|_\infty \le l$, $\|y\|_\infty \le l$, $\|\xi_0\|_\infty \le l$, $\ldots$, $\|\xi_{k-1}\|_\infty \le l$. Observe that $g_l \in \mu(g_l)\mathcal{F}$. From (1.4) we get

$$\begin{aligned}
&\|J(x-y)\|_{\mathcal{K}} \\
&\quad = \|J \circ h_f(x) - J \circ h_f(y)\|_{\mathcal{K}} \\
&\quad \le \|J \circ (h_f - h_{g_l})(x)\|_{\mathcal{K}} + \|J \circ h_{g_l}(x) - J \circ h_{g_l}(y)\|_{\mathcal{K}} + \|J \circ (h_{g_l} - h_f)(y)\|_{\mathcal{K}} \\
&\quad \le \mu(g_l)(L_0\theta_0\|J(x-y)\|_{\mathcal{K}} + \ldots + L_{k-1}\theta_{k-1}\|J(x-y)\|_{\mathcal{K}}) + (1-\mu(g_l))/l \\
&\quad \le \mu(g_l)\|J(x-y)\|_{\mathcal{K}} + (1-\mu(g_l))/l.
\end{aligned}$$

Thus, $\|J(x-y)\|_{\mathcal{K}} \le 1/l$. Since one can choose $l = 1, 2, \ldots$ arbitrarily we get $x = y$. This means that h_f has at most one fixed point and completes the proof.

2. Generic properties of the Picard problem. Consider the Picard boundary value problem

$$\begin{cases} x''(t) = f(t, x(t)), & t \in [0,p], \\ x(0) = a, \quad x(p) = b, \end{cases} \tag{2.1}$$

where $a, b \in \mathbb{R}^n$ and $f : [0,p] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous. By a solution of the problem (2.1) we mean any $x \in C^2([0,p], \mathbb{R}^n)$ satisfying (2.1).

Let Δ be $[0,p]$. Therefore, E is the Banach space $C([0,p], \mathbb{R}^n)$ and $\mathcal{K}$ is the Hilbert space $L^2([0,p], \mathbb{R}^n)$ (we assume that $\gamma \equiv 1$). Let $T : \mathcal{K} \to E$ be given by

$$T(x)(t) = -\frac{1}{p}\Big((p-t)\int_0^t s x(s)\, ds + t\int_t^p (p-s)x(s)\, ds\Big).$$

It is known (see [1]) that $\|J \circ T\| = (p/\pi)^2$. The map $h_f : E \to E$ is defined by

$$h_f(y)(t) = f(t, \xi(t) + T \circ J(y)(t)), \qquad t \in [0,p],$$

where $\xi(t) = (1 - (t/p))a + (t/p)b$ for $t \in [0,p]$. After setting $y = x''$ the problem (2.1) may be written as $y = h_f(y)$, where $y \in E$. If y is a fixed point of h_f, then $\xi + T \circ J(y)$ is a solution of (2.1). From Theorems 1 and 2 respectively we obtain the following two theorems.

THEOREM 3. *Suppose that* $M \geq 0$. *Then the set of all* f *such that for every* $a, b \in \mathbb{R}^n$ *the problem* (2.1) *has at least one solution*, *is residual in the complete space*

$$\{f \in C([0,p] \times \mathbb{R}^n, \mathbb{R}^n) : |f(t,u)| \leq (\pi^2/p^2)|u| + M \text{ for } t \in [0,p], u \in \mathbb{R}^n\}$$

with the metric ϱ.

THEOREM 4. *The set of all* f *such that for every* $a, b \in \mathbb{R}^n$ *the problem* (2.1) *has exactly one solution is residual in the complete space*

$$\{f \in C([0,p] \times \mathbb{R}^n, \mathbb{R}^n) : |f(t,u) - f(t,v)| \leq (\pi^2/p^2)|u - v| \quad \text{for } t \in [0,p], u, v \in \mathbb{R}^n\}$$

with the metric ϱ.

3. Generic properties of the Floquet problem. Consider the Floquet boundary value problem

$$\begin{cases} x^{(k)}(t) = f(t, x(t), x'(t), \ldots, x^{(k-1)}(t)), & t \in [0,p], \\ x(0) + \lambda x(p) = r_0, \ \ldots, \ x^{(k-1)}(0) + \lambda x^{(k-1)}(p) = r_{k-1}, \end{cases} \tag{3.1}$$

where $\lambda > 0$, $r_0, \ldots, r_{k-1} \in \mathbb{R}^n$ and $f : [0,p] \times (\mathbb{R}^n)^k \to \mathbb{R}^n$ is continuous. By a solution of the problem (3.1) we mean any $x \in C^k([0,p], \mathbb{R}^n)$ satisfying (3.1).

In this section $\Delta = [0,p]$. Consequently, $E = C([0,p], \mathbb{R}^n)$. Here $\mathcal{K}$ is the Hilbert space of all $x \in L^2([0,p], \mathbb{R}^n)$ with the scalar product

$$(x,y)_{\mathcal{K}} = \int_0^p \lambda^{2t/p} \sum_{j=1}^n x_j(t) y_j(t)\, dt.$$

Set $\theta = p/\sqrt{\pi^2 + \ln^2 \lambda}$. Let $T : \mathcal{K} \to E$ be given by

$$T(x)(t) = \frac{1}{1+\lambda} \int_0^t x(s)\, ds - \frac{\lambda}{1+\lambda} \int_t^p x(s)\, ds.$$

It is proved in [1] that $\|J \circ T\| \leq \theta$.

Define $\xi \in C^\infty([0,p], \mathbb{R}^n)$ and $h_f : E \to E$ by

$$\xi(t) = (r_0 + T \circ J(r_1)(t) + \ldots + (T \circ J)^{k-1}(r_{k-1})(t))/(1+\lambda),$$
$$h_f(y)(t) = f(t, \xi(t) + (T \circ J)^k(y)(t), \ldots, \xi^{(k-1)}(t) + T \circ J(y)(t)).$$

After setting $y = x^{(k)}$ the problem (3.1) may be written as $y = h_f(y)$, where $y \in E$. If y is a fixed point of h_f, then $\xi + (T \circ J)^k(y)$ is a solution of (3.1). From Theorems 1 and 2 we obtain

THEOREM 5. *Suppose that* $M, L_0, L_1, \ldots, L_{k-1} \geq 0$ *satisfy*

$$\theta^k L_0 + \ldots + \theta L_{k-1} \leq 1. \tag{3.2}$$

Then the set of all $f \in \mathcal{G}$ such that for every $r_0, \ldots, r_{k-1} \in \mathbb{R}^n$ the problem (3.1) *has at least one solution, is residual in the complete space* $(\mathcal{G}, \varrho)$.

THEOREM 6. *Suppose that $L_0, L_1, \ldots, L_{k-1} \geq 0$ satisfy* (3.2). *Then the set of all $f \in \mathcal{F}$ such that for every $r_0, \ldots, r_{k-1} \in \mathbb{R}^n$ the problem* (3.1) *has exactly one solution, is residual in the complete space* $(\mathcal{F}, \varrho)$.

4. Generic properties of the Nicoletti problem. Consider the Nicoletti boundary value problem

$$\begin{cases} x^{(k)}(t) = f(t, x(t), x'(t), \ldots, x^{(k-1)}(t)), & t \in [0,p], \\ x_j^{(l)}(t_{lj}) = r_{lj}, \quad l = 0, \ldots, k-1, \ j = 1, \ldots, n, \end{cases} \tag{4.1}$$

where $t_l = (t_{l1}, \ldots, t_{ln}) \in [0,p]^n$, $r_l = (r_{l1}, \ldots, r_{ln}) \in \mathbb{R}^n$ for $l = 0, \ldots, k-1$ and $f : [0,p] \times (\mathbb{R}^n)^k \to \mathbb{R}^n$ is continuous. By a solution of (4.1) we mean any $x \in C^k([0,p], \mathbb{R}^n)$ satisfying (4.1).

In this section $\Delta = [0,p]$. Consequently, $E = C([0,p], \mathbb{R}^n)$. Now $\mathcal{K}$ is the Hilbert space of all $x \in L^2([0,p], \mathbb{R}^n)$ with the scalar product

$$(x,y)_{\mathcal{K}} = \int_0^p \sum_{j=1}^n x_j(t) y_j(t)\, dt.$$

Set $\omega = 2p/\pi$. Define $T_0, \ldots, T_{k-1} : \mathcal{K} \to E$ by

$$T_l(x)(t)_j = \int_{t_{lj}}^t x_j(s)\, ds, \qquad l = 0, \ldots, k-1, \ j = 1, \ldots, n.$$

In [1] it was shown that for $l = 0, \ldots, k-1$ we have $\|J \circ T_l\| \leq \omega$.

Define $\xi \in C^\infty([0,p], \mathbb{R}^n)$ and $h_f : E \to E$ by

$$\xi = r_0 + T_0 \circ J(r_1) + \ldots + T_0 \circ J \circ \ldots \circ T_{k-2} \circ J(r_{k-1}),$$
$$h_f(y)(t) = f(t, \xi(t) + T_0 \circ J \circ \ldots \circ T_{k-1} \circ J(y)(t), \ldots, \xi^{(k-1)}(t) + T_{k-1} \circ J(y)(t)).$$

After setting $y = x^{(k)}$ the problem (4.1) may be written as $y = h_f(y)$, where $y \in E$. If y is a fixed point of h_f, then $\xi + T_0 \circ J \circ \ldots \circ T_{k-1} \circ J(y)$ is a solution of (4.1). From Theorems 1 and 2 we obtain

THEOREM 7. *Suppose that $M, L_0, L_1, \ldots, L_{k-1} \geq 0$ satisfy*

$$\omega^k L_0 + \ldots + \omega L_{k-1} \leq 1. \tag{4.2}$$

Then the set of all $f \in \mathcal{G}$ such that for every $r_0, \ldots, r_{k-1} \in \mathbb{R}^n$ the problem (4.1) *has at least one solution, is residual in the complete space* $(\mathcal{G}, \varrho)$.

THEOREM 8. *Suppose that $L_0, L_1, \ldots, L_{k-1} \geq 0$ satisfy* (4.2). *Then the set of all $f \in \mathcal{F}$ such that for every $r_0, \ldots, r_{k-1} \in \mathbb{R}^n$ the problem* (4.1) *has exactly one solution, is residual in the complete space* $(\mathcal{F}, \varrho)$.

5. Generic properties of a boundary value problem for a hyperbolic equation. Consider the boundary value problem

$$\begin{cases} u_{xy} = f(x, y, u), \quad x \in [0, p],\ y \in [0, q], \\ u(0, y) + \lambda_1 u(p, y) = \sigma(y), \quad u(x, 0) + \lambda_2 u(x, q) = \chi(x), \end{cases} \tag{5.1}$$

where $\lambda_1, \lambda_2 > 0$, $f : [0, p] \times [0, q] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, $\chi : [0, p] \to \mathbb{R}^n$ and $\sigma : [0, q] \to \mathbb{R}^n$ are continuously differentiable and $\chi(0) + \lambda_1\chi(p) = \sigma(0) + \lambda_2\sigma(q)$. By a solution of (5.1) we mean any continuous map $u : [0, p] \times [0, q] \to \mathbb{R}^n$ with continuous derivatives $u_x, u_y, u_{xy} = u_{yx}$ satisfying (5.1) on $[0, p] \times [0, q]$.

Let Δ be $[0, p] \times [0, q]$. Then E is the Banach space $C([0, p] \times [0, q], \mathbb{R}^n)$. Let $\mathcal{K}$ be the Hilbert space of all $v \in L^2([0, p] \times [0, q], \mathbb{R}^n)$ with the scalar product

$$(u, v)_{\mathcal{K}} = \int_0^p\int_0^q \sum_{j=1}^n \lambda_1^{2s/p}\lambda_2^{2t/q} u_j(s, t)v_j(s, t)\, ds\, dt.$$

Write $\widetilde{\theta} = p/\sqrt{\pi^2 + \ln^2\lambda_1}$ and $\widehat{\theta} = q/\sqrt{\pi^2 + \ln^2\lambda_2}$. Define two operators $\widetilde{S}, \widehat{S} : \mathcal{K} \to E$ by

$$\widetilde{S}(v)(x, y) = \frac{1}{1 + \lambda_1}\int_0^x v(s, y)\, ds - \frac{\lambda_1}{1 + \lambda_1}\int_x^p v(s, y)\, ds,$$

$$\widehat{S}(v)(x, y) = \frac{1}{1 + \lambda_2}\int_0^y v(x, t)\, dt - \frac{\lambda_2}{1 + \lambda_2}\int_y^q v(x, t)\, dt.$$

Observe that $\widehat{S} \circ J \circ \widetilde{S} : \mathcal{K} \to E$ is completely continuous. It was proved in [1] that $\|J \circ \widetilde{S}\| \le \widetilde{\theta}$ and $\|J \circ \widehat{S}\| \le \widehat{\theta}$.

Define $\xi : [0, p] \times [0, q] \to \mathbb{R}^n$ and $h_f : E \to E$ by

$$\xi(x, y) = \frac{\chi(x)}{1 + \lambda_2} + \frac{\sigma(y)}{1 + \lambda_1} - \frac{\chi(0) + \lambda_1\chi(p)}{(1 + \lambda_1)(1 + \lambda_2)},$$

$$h_f(v)(x, y) = f(x, y, \xi(x, y) + \widehat{S} \circ J \circ \widetilde{S} \circ J(v)(x, y)).$$

After setting $v = u_{xy}$ the problem (5.1) may be written as $v = h_f(v)$, where $v \in E$. If $h_f(v) = v$, then $\xi + \widehat{S} \circ J \circ \widetilde{S} \circ J(v)$ is a solution of (5.1). From Theorems 1 and 2 we obtain

THEOREM 9. *Suppose that $M \ge 0$. Then the set of all f such that for every $\chi \in C^1([0, p], \mathbb{R}^n)$ and $\sigma \in C^1([0, q], \mathbb{R}^n)$ satisfying $\chi(0) + \lambda_1\chi(p) = \sigma(0) + \lambda_2\sigma(q)$ the problem* (5.1) *has at least one solution, is residual in the complete space*

$$\{f \in \mathcal{Z} : |f(x, y, z)| \le (\widetilde{\theta}\,\widehat{\theta})^{-1}|z| + M,\ x \in [0, p],\ y \in [0, q],\ z \in \mathbb{R}^n\}$$

with the metric ϱ.

THEOREM 10. *The set of all f such that for every $\chi \in C^1([0,p],\mathbb{R}^n)$ and $\sigma \in C^1([0,q],\mathbb{R}^n)$ satisfying $\chi(0)+\lambda_1\chi(p) = \sigma(0)+\lambda_2\sigma(q)$ the problem* (5.1) *has exactly one solution, is residual in the complete space*

$$\{f \in \mathcal{Z} : |f(x,y,z) - f(x,y,\widetilde{z})| \le (\widetilde{\theta}\,\widehat{\theta})^{-1}|z-\widetilde{z}|,\\ x \in [0,p],\ y \in [0,q],\ z,\widetilde{z} \in \mathbb{R}^n\}$$

with the metric ϱ.

6. Generic properties of a hyperbolic equation with another boundary value condition. Consider the boundary value problem

$$\begin{cases} u_{xy} = f(x,y,u), & x \in [0,p],\ y \in [0,q],\\ u_j(s_j,y) = \sigma_j(y), & u_j(x,t_j) = \chi_j(x), \quad j = 1,\dots,n, \end{cases} \tag{6.1}$$

where $f : [0,p] \times [0,q] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous, $\chi : [0,p] \to \mathbb{R}^n$ and $\sigma : [0,q] \to \mathbb{R}^n$ are continuously differentiable, $s_j \in [0,p]$, $t_j \in [0,q]$ and $\chi_j(s_j) = \sigma_j(t_j)$ for $j = 1,\dots,n$. By a solution of the problem (6.1) we mean any continuous map $u : [0,p] \times [0,q] \to \mathbb{R}^n$ with continuous partial derivatives $u_x, u_y, u_{xy} = u_{yx}$ satisfying (6.1) on $[0,p] \times [0,q]$.

Let $\Delta = [0,p] \times [0,q]$. Thus, E is the Banach space $C([0,p] \times [0,q],\mathbb{R}^n)$. Let $\mathcal{K}$ be the Hilbert space $L^2([0,p] \times [0,q],\mathbb{R}^n)$. Define two operators $\widetilde{S}, \widehat{S} : \mathcal{K} \to E$ by

$$\widetilde{S}(v)(x,y)_j = \int_{s_j}^{x} v_j(s,y)\,ds, \qquad \widehat{S}(v)(x,y)_j = \int_{t_j}^{y} v_j(x,t)\,dt.$$

Notice that $\widehat{S} \circ J \circ \widetilde{S} : \mathcal{K} \to E$ is completely continuous. In [1] it was shown that $\|J \circ \widetilde{S}\| \le 2p/\pi$ and $\|J \circ \widehat{S}\| \le 2q/\pi$. Define $\xi : [0,p] \times [0,q] \to \mathbb{R}^n$ and $h_f : E \to E$ by

$$\xi_j(x,y) = \chi_j(x) + \sigma_j(y) - \chi_j(s_j), \qquad j = 1,\dots,n,$$
$$h_f(v)(x,y) = f(x,y,\xi(x,y) + \widehat{S} \circ J \circ \widetilde{S} \circ J(v)(x,y)),$$

After setting $v = u_{xy}$ the problem (6.1) may be written as $v = h_f(v)$, where $v \in E$. If $h_f(v) = v$, then $\xi + \widehat{S} \circ J \circ \widetilde{S} \circ J(v)$ is a solution of (6.1). From Theorems 1 and 2 we obtain.

THEOREM 11. *Let $M \ge 0$. Then the set of all f such that for every $\chi \in C^1([0,p],\mathbb{R}^n)$ and $\sigma \in C^1([0,q],\mathbb{R}^n)$ satisfying $\chi_j(s_j) = \sigma_j(t_j)$ $(j = 1,\dots,n)$ the problem* (6.1) *has at least one solution, is residual in the complete space*

$$\left\{f \in \mathcal{Z} : |f(x,y,z)| \le \frac{\pi^2}{4pq}|z| + M,\ x \in [0,p],\ y \in [0,q],\ z \in \mathbb{R}^n\right\}$$

with the metric ϱ.

THEOREM 12. *The set of all f such that for every $\chi \in C^1([0,p],\mathbb{R}^n)$ and $\sigma \in C^1([0,q],\mathbb{R}^n)$ satisfying $\chi_j(s_j) = \sigma_j(t_j)$ $(j = 1,\ldots,n)$ the problem* (6.1) *has exactly one solution, is residual in the complete space*

$$\left\{f \in \mathcal{Z} : |f(x,y,z) - f(x,y,\widetilde{z})| \le \frac{\pi^2}{4pq}|z - \widetilde{z}|,\ x \in [0,p],\ y \in [0,q],\ z,\widetilde{z} \in \mathbb{R}^n\right\}$$

with the metric ϱ.

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