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TAME MINIMAL NON-POLYNOMIAL GROWTH SIMPLY CONNECTED ALGEBRAS

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1. Introduction. By Drozd's Tame and Wild Theorem [8] the class of finite-dimensional algebras (associative, with identity) over an algebraically closed field may be divided into two disjoint classes. One class consists of the tame algebras, for which the indecomposable modules occur in each dimension d in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory is as complicated as the study of finite-dimensional vector spaces together with two non-commuting endomorphisms, for which the classification of the indecomposable finite-dimensional modules is a well-known difficult problem. Hence we can hope to classify the modules only for tame algebras. Among the tame algebras we may distinguish the class of polynomial growth algebras A for which there exists an integer m (depending on A) such that, in each dimension d, the indecomposable A-modules occur in a finite number of discrete and at most d^m one-parameter families.

Frequently, applying covering techniques, we may reduce the representation theory of a given tame (respectively, polynomial growth) algebra to that of the corresponding simply connected algebra. Recently, the class of polynomial growth simply connected algebras has been extensively investigated. In particular, a rather complete representation theory of polynomial growth strongly simply connected algebras has been established by the second author in [21]. One of the important open problems is to extend this theory to arbitrary simply connected algebras of polynomial growth. We are especially interested in criteria for a simply connected algebra to be of polynomial growth. This leads to the study of tame simply connected algebras which are minimal not of polynomial growth (they themselves are not of polynomial growth but every proper convex subcategory is).

The main aim of this article is to introduce and classify (by quivers and

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relations) a class of tame minimal non-polynomial growth simply connected algebras, which we call (generalized) polynomial growth critical algebras. Moreover, we describe basic properties of polynomial growth critical algebras and the structure of the category of indecomposable finite-dimensional modules over such algebras. It is expected that the class of polynomial growth critical algebras introduced and investigated here will play an important role in the study of arbitrary tame non-polynomial growth simply connected algebras.

The paper is organized as follows. In Section 2 we fix the notations and recall the needed definitions. In Section 3 we introduce the polynomial growth critical algebras and classify them by quivers and relations. In particular, we prove that all such algebras are simply connected and their opposite algebras are also polynomial growth critical. Moreover, applying the main results of [21], we get a handy criterion for a strongly simply connected algebra to be of polynomial growth. Section 4 is devoted to the tilting classes of polynomial growth critical algebras. We prove that two polynomial growth critical algebras with the same number of simple modules belong to the same tilting class. Then we deduce that the Euler form of any polynomial growth critical algebra is positive semi-definite with radical of rank 2. In Section 5 we determine the Coxeter polynomial of any polynomial growth critical algebra and show that the eigenvalues of its Coxeter matrix are roots of unity. In the final Section 6 we investigate the module category of polynomial growth critical algebras. We completely describe the structure of all non-regular components of their Auslander–Reiten quivers and discuss the behaviour of non-regular components in the category of indecomposable finite-dimensional modules. In particular, we show that the Auslander–Reiten quiver of any polynomial growth critical algebra has exactly one preprojective component, exactly one preinjective component, and exactly one component containing both a projective module and an injective module.

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2. Preliminaries. Throughout this article, K will denote a fixed algebraically closed field. By an algebra is meant an associative finite-dimensional K-algebra with an identity, which we shall assume to be basic and connected. An algebra A can be written as a bound quiver algebra $A \cong KQ/I$, where $Q = Q_A$ is the quiver of A and I is an admissible ideal in the path algebra KQ of Q. Equivalently, an algebra $A = KQ/I$ may be considered as a K-category whose object class is the set of vertices of Q, and the set of morphisms $A(x, y)$ from x to y is the quotient of the K-space $KQ(x, y)$

of all K-linear combinations of paths in Q from x to y modulo the subspace $I(x,y) = I \cap KQ(x,y)$. An algebra A with Q_A having no oriented cycle is said to be *triangular*. A full subcategory C of A is said to be *convex* if any path in Q_A with source and target in Q_C lies entirely in Q_C. Following [1] a triangular algebra A is called *simply connected* if, for any presentation $A \cong KQ/I$ of A as a bound quiver algebra, the fundamental group $\Pi_1(Q,I)$ of (Q,I) is trivial. Moreover, following [20] an algebra A is said to be *strongly simply connected* if every convex subcategory of A is simply connected. It was shown in [20] that a triangular algebra is strongly simply connected if and only if every convex subalgebra C of A satisfies the separation condition of Bautista, Larrión and Salmerón [3]. For example, if Q_A is a tree, then A is strongly simply connected.

For an algebra A, we denote by $\operatorname{mod} A$ the category of finite-dimensional right A-modules and by $\operatorname{ind} A$ its full subcategory consisting of the indecomposable modules. We shall denote by Γ_A the Auslander–Reiten quiver of A and by $\tau_A = \mathrm{DTr}$ and $\tau_A^- = \mathrm{TrD}$ the Auslander–Reiten translations. We shall agree to identify an indecomposable A-module with the vertex of Γ_A corresponding to it. For each vertex i of Q_A we denote by $S_A(i)$ the simple A-module having K at the vertex i, by $P_A(i)$ the projective cover of $S_A(i)$, and by $I_A(i)$ the injective envelope of of $S_A(i)$. For a module M in $\operatorname{mod} A$ we shall denote by $\underline{\dim}\, M$ the dimension vector $(\dim_K M(i))_{i \text{ vertex in } Q_A}$. The *support* $\operatorname{supp} M$ of a module M in $\operatorname{mod} A$ is the full subcategory of A given by all vertices i of Q_A such that $M(i) \neq 0$.

Let A be an algebra and $K[X]$ the polynomial algebra in one variable. Following [8], A is said to be *tame* if, for each dimension d, there exists a finite number of $K[X]$-A-bimodules M_i, $1 \le i \le n_d$, which are finitely generated and free as left $K[X]$-modules, and such that all but a finite number of isomorphism classes of indecomposable right A-modules of dimension d are of the form $K[X]/(X-\lambda) \otimes_{K[X]} M_i$ for some $\lambda \in K$ and some i. Let $\mu_A(d)$ be the least number of $K[X]$-A-bimodules satisfying the above conditions. Then A is said to be *of polynomial growth* if there is a positive integer m such that $\mu_A(d) \le d^m$ for any $d \ge 1$ (cf. [19]). Examples of polynomial growth algebras are tilted algebras of Euclidean type and tubular algebras [15].

Let $A = KQ/I$ be a triangular algebra. Denote by $K_0(A)$ the Grothendieck group of A. Then $K_0(A) = \mathbb{Z}^n$, where n is the number of vertices of Q. The *Euler quadratic form* χ_A of A is the integral quadratic form on $K_0(A)$ such that

$$\chi_A(\underline{\dim}\, X) = \sum_{i=0}^{\infty} (-1)^i \dim_K \operatorname{Ext}_A^i(X,X)$$

for any module X in $\operatorname{mod} A$ (see [15, (2.4)]). If $\operatorname{gl.dim} A \le 2$ then χ_A coincides with the *Tits form* q_A of A, defined for $x = (x_i)_{i\in Q_0} \in K_0(A)$ as follows:

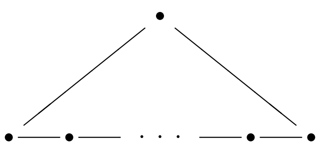
$$q_A(x) = \sum_{i\in Q_0} x_i^2 - \sum_{(i\to j)\in Q_1} x_i x_j + \sum_{i,j\in Q_0} r_{i,j} x_i x_j,$$

where Q_0 and Q_1 are the sets of vertices and arrows of Q, respectively, and $r_{i,j}$ is the cardinality of $L \cap I(i,j)$ for a minimal set of generators $L \subseteq \bigcup_{i,j\in Q_0} I(i,j)$ of the ideal I (see [4]).

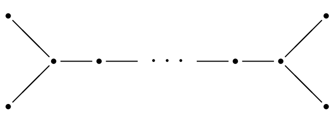
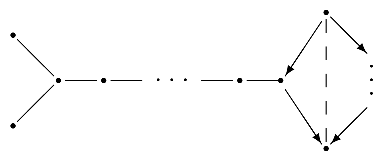
For basic background on the representation theory of finite-dimensional algebras we refer to [15].

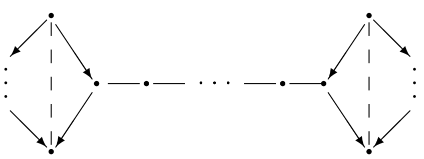
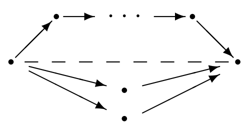
3. Tame minimal non-polynomial growth algebras. In this section we give a complete description of a class of tame minimal non-polynomial growth algebras by quivers and relations. For an algebra $A = kQ/I$, generators of I are usually called *relations*. In our bound quivers, a dashed line indicates a relation being the sum of all paths from the starting point to the end point. Moreover, a dotted line indicates a zero-relation along a path of length 2.

Recall that a concealed algebra is of *concealed type* Δ if it is an algebra C of the form $C = \mathrm{End}_H(T)$ where H is a hereditary algebra of type Δ and T is a preprojective tilting H-module. We know from [5], [10] that there is only one family of concealed algebras of type $\widetilde{\mathbf{A}}_n$, $n \geq 1$, given by the quivers



and four families of concealed algebras of type $\widetilde{\mathbf{D}}_n$, $n \geq 4$, given by the following quivers and relations:

(1)  (2) 

(3)  (4) 

where the number of vertices is equal to $n+1$ and $\bullet\!-\!\bullet$ means $\bullet\!\to\!\bullet$ or $\bullet\!\leftarrow\!\bullet$. It is known that if C is a concealed algebra of Euclidean type then Γ_C consists of a preprojective component $\mathcal{P}$, a preinjective component $\mathcal{Q}$ and a $\mathbb{P}_1(K)$-family $\mathcal{T} = (\mathcal{T}_\lambda)_{\lambda\in\mathbb{P}_1(K)}$ of stable tubes. It is shown in [15, (4.9)] that an algebra B is a tilted algebra $\operatorname{End}_H(T)$, where H is a hereditary algebra of type $\widetilde{\mathbf{D}}_n$ and T a tilting H-module without preinjective (respectively, preprojective) direct summands, if and only if B is a tubular extension (respectively, coextension) of tubular type $(2,2,n-2)$ of a concealed algebra C of type $\widetilde{\mathbf{A}}_m$ or $\widetilde{\mathbf{D}}_m$, $m \le n$. In the case when B is a tubular extension of C (of type $(2,2,n-2)$), Γ_B consists of a preprojective component $\mathcal{P}'$ (which is the preprojective component of Γ_C), a preinjective component $\mathcal{Q}'$ having a complete slice of type $\widetilde{\mathbf{D}}_n$, and a $\mathbb{P}_1(K)$-family $\mathcal{T}' = (\mathcal{T}'_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ of ray tubes. Two tubes in $\mathcal{T}'$ have 2 rays, one has $n-2$ rays, and the remaining ones are stable tubes of rank 1 (homogeneous tubes). We have the dual structure for Γ_B in the case when B is a tubular coextension of C (of type $(2,2,n-2)$). Finally, we note that any representation-infinite tilted algebra of type $\widetilde{\mathbf{D}}_n$ is of one of the above types.

The main objective of this article is to investigate the following class of algebras. By a *polynomial growth critical algebra*, briefly *pg-critical algebra*, we mean an algebra A satisfying the following conditions:

(i) A is of one of the forms:

$$B[M] = \begin{bmatrix} K & M \\ 0 & B \end{bmatrix}, \qquad B[N,t] = \begin{bmatrix} K & K & \cdots & K & K & K & N \\ & K & \cdots & K & K & K & 0 \\ & & \ddots & \vdots & \vdots & \vdots & \vdots \\ & & & K & K & K & 0 \\ & 0 & & & K & 0 & 0 \\ & & & & & K & 0 \\ & & & & & & B \end{bmatrix},$$

where B is a representation-infinite tilted algebra of the form $\operatorname{End}_H(T)$, for a hereditary algebra H of type $\widetilde{\mathbf{D}}_n$ and a titlting H-module T without non-zero preinjective direct summands, $M = \operatorname{Hom}_H(T,R)$ (respectively, $N = \operatorname{Hom}_H(T,S)$) for an indecomposable regular H-module R of regular length 2 (respectively, indecomposable regular H-module S of regular length 1) lying in a tube of Γ_H with $n-2$ rays, and $t+1$ ($t \ge 2$) is the number of objects of $B[N,t]$ which are not in B.

(ii) Every proper convex subcategory of A is of polynomial growth.

If $A = B[M]$ then the quiver Q_A of A consists of the quiver Q_B of B and an extension vertex w (which is a source of Q_A) such that M is the restriction of $P_A(w)$ to B. In the case when $A = B[N,t]$ the quiver Q_A consists of Q_B and the quiver

$$\begin{tikzcd}[row sep=small] & & & & \bullet\, a \\ w\ \bullet \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r] & \underset{c}{\bullet} \arrow[ur] \arrow[dr] & \\ & & & & \bullet\, b \end{tikzcd}$$

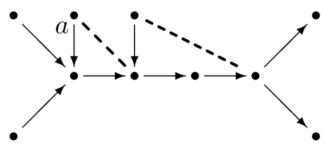
and N is the restriction of $P_A(w)$ to B.

The following proposition motivates the name "pg-critical algebra".

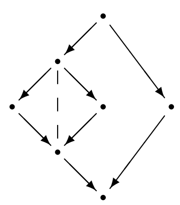
PROPOSITION 3.1. *Let A be a pg-critical algebra. Then A is tame but not of polynomial growth.*

Proof. We may assume that A is of the form $B[M]$ or $B[N,t]$. If $A = B[M]$ then the claim follows from [14] and [17]. In the case when $A = B[N,t]$, applying the APR-tilting module [2] induced by the simple projective A-module given by one of the vertices a or b, we get an algebra of type $B'[M']$ for a tubular extension of a tilted algebra B' of type $\widetilde{\mathbf{D}}_{n+t}$ and a regular indecomposable B'-module M' of regular length 2 lying in the tube of $\Gamma_{B'}$ having $n+t-2$ rays.

We note that the use of the term "pg-critical algebra" in the present paper slightly deviates from its use in an earlier publication by the authors [12]. Here, we consider a more general class of algebras which seems to be crucial for studying arbitrary tame simply connected algebras which are not of polynomial growth. Observe also that in the above definition of a pg-critical algebra both conditions (i) and (ii) are essential. Indeed, if Λ is an algebra given by the following quiver and relations:



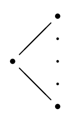
then Λ satisfies (i) but not (ii), as the convex subcategory of Λ formed by all vertices except a is still not of polynomial growth. The algebra Γ given by the following quiver and relations:



is tame (see [14, (3.9)]) with all proper convex subcategories representation-finite (hence of polynomial growth) but does not satisfy (i).

In order to save space in the theorem below and to make the list below more accessible we write down only the possible frames. Given such a frame, we allow the following admissible operations:

(i) Replacing each subgraph



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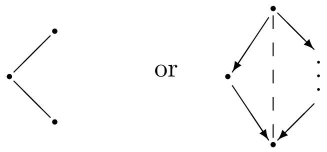
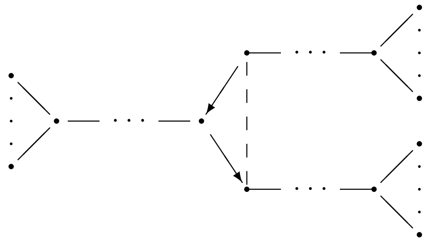
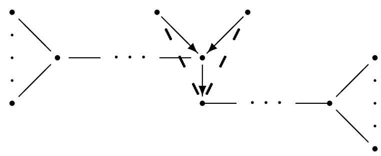
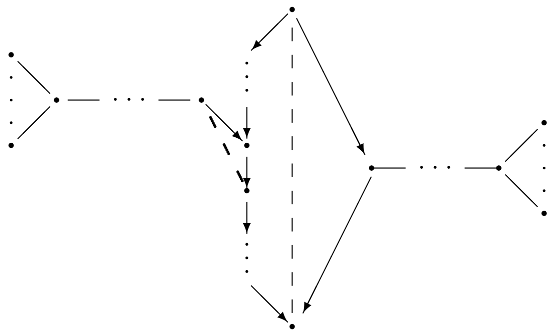


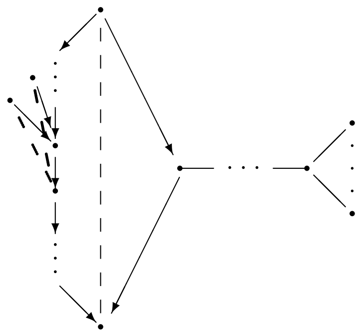
(ii) Choice of arbitrary orientations in non-oriented edges.

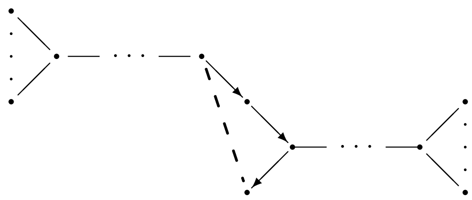
(iii) Constructing the opposite algebra.

THEOREM 3.2. *An algebra A is pg-critical if and only if it is obtained from a frame in the following list by admissible operations:*

(1)  (2) 

(3) 

(4) 

(5) 

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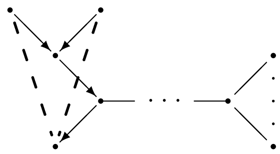

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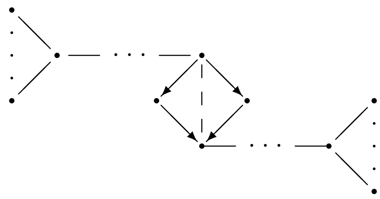

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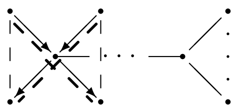

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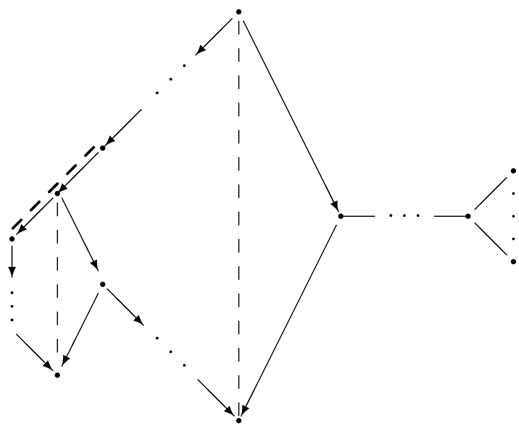

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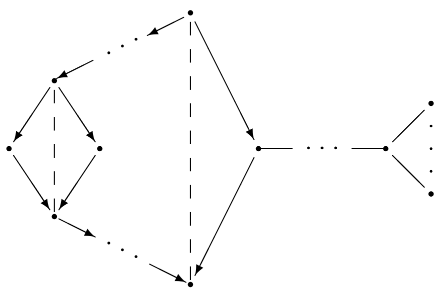

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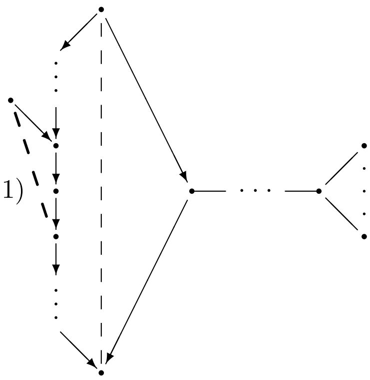

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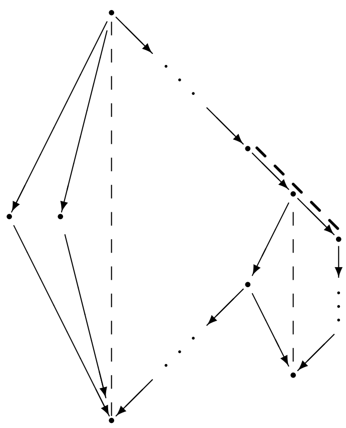

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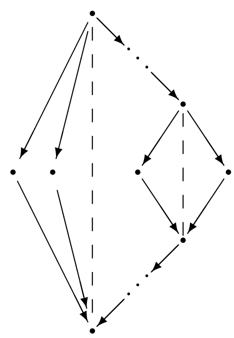
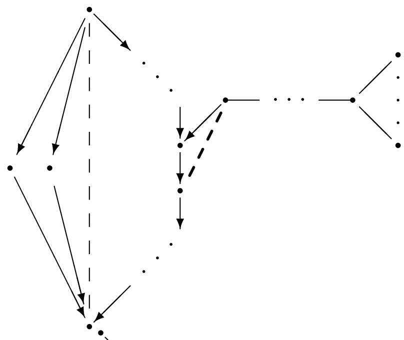
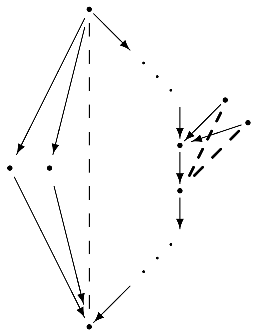
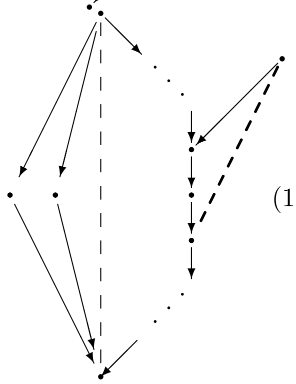
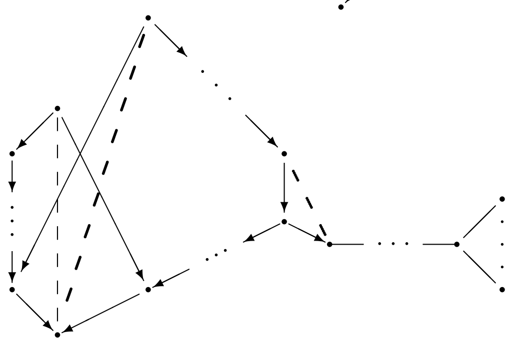
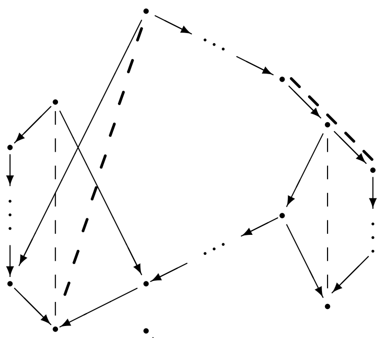


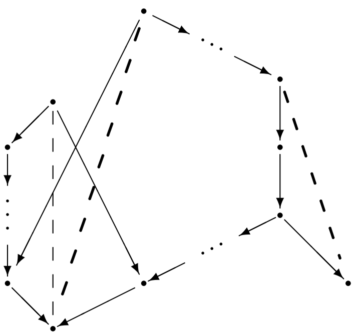
(14) 

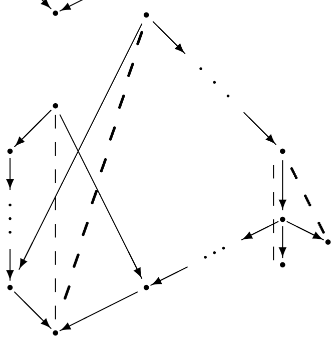
(15) 

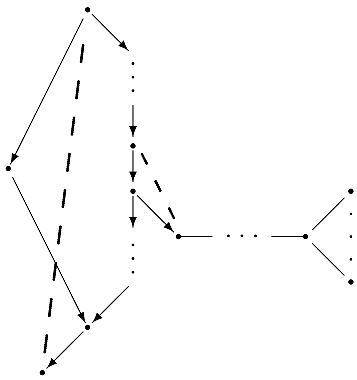
(16) 

(17) 

(18) 

(19) 

(20) 

(21) 

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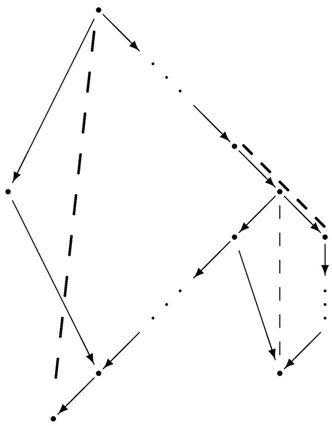

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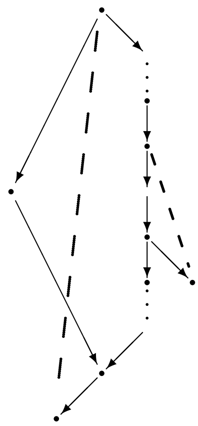

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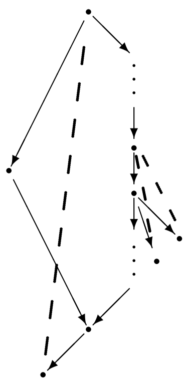

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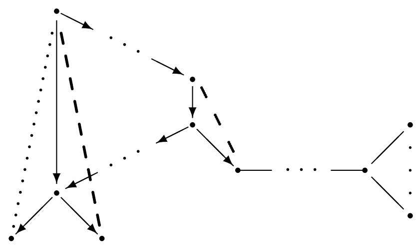

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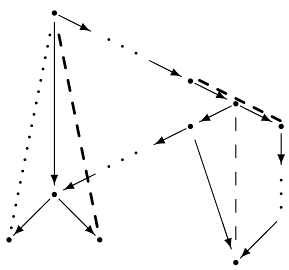

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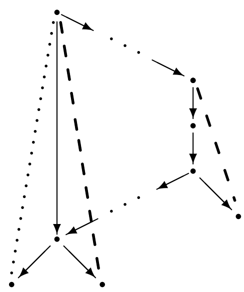

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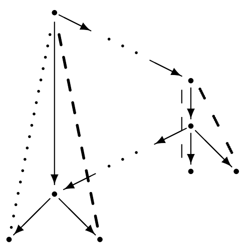

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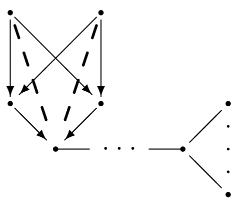
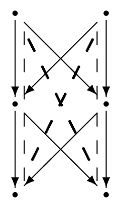
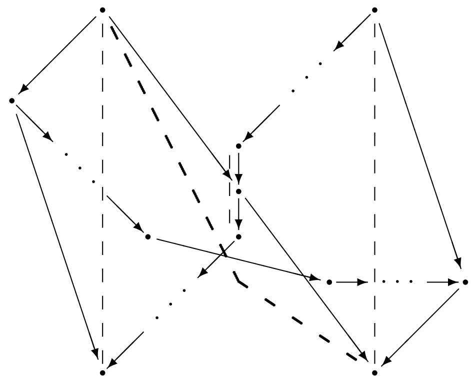
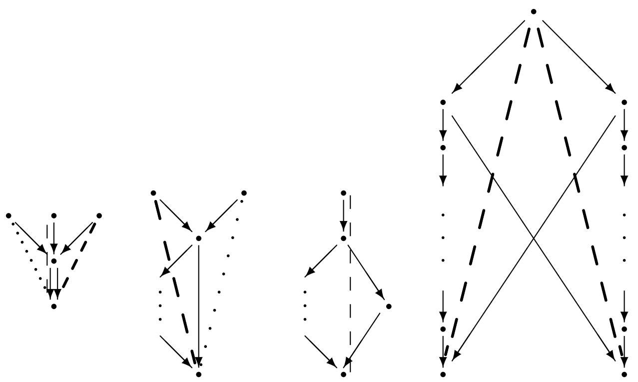


(30)  (31) 

P r o o f. Let B be a representation-infinite tilted algebra of type $\widetilde{\mathbf{D}}_n$, $n \geq 4$, with a complete slice in its preinjective component. Then B can be obtained from a concealed algebra C (of type $\widetilde{\mathbf{A}}_m$ or type $\widetilde{\mathbf{D}}_m$, $m \leq n$) by adding branches $L_1, \ldots, L_r$ in the extension vertices $\omega_1, \ldots, \omega_r$ of a multiple one-point extension $C[E_1][E_2] \ldots [E_r]$ of C by pairwise non-isomorphic simple regular C-modules $E_1, \ldots, E_r$. In this process we create two tubes with 2 rays, one tube with $n-2$ rays and the remaining tubes (of rank 1) are not changed (see [15, Section 4]). Moreover, if C is of type $\widetilde{\mathbf{A}}_m$, then B contains a convex, tilted subcategory $\overline{C}$ of type $\widetilde{\mathbf{D}}_s$, $m < s \leq n$, isomorphic to one of the following:



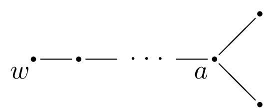
In this case, the structure of indecomposable regular $\overline{C}$-modules of regular length at most 2 is well known. In the case when C is of type $\widetilde{\mathbf{D}}_m$, the indecomposable regular C-modules of regular length at most 2 are completely described in [12].

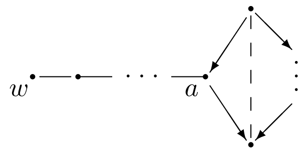
Let now A be a pg-critical algebra of one of the forms $B[M]$ or $B[N,t]$. Then a direct analysis shows that every proper convex subcategory of B is of polynomial growth if and only if A is a minimal non-polynomial growth algebra of one of the forms

$$D(U)=\begin{bmatrix} K & U \\ 0 & \overline{C} \end{bmatrix},\quad E(V)=\begin{bmatrix} K & 0 & V \\ 0 & K & V \\ 0 & 0 & \overline{C} \end{bmatrix},\quad F(V)=\begin{bmatrix} \Lambda & V^r \\ 0 & \overline{C} \end{bmatrix},\quad \text{or}$$

$$G(V)=\begin{bmatrix} K & K & K & \dots & K & W \\ & K & K & \dots & K & V \\ & & K & \dots & K & V \\ & & & \ddots & \vdots & \vdots \\ & 0 & & & K & V \\ & & & & & \overline{C} \end{bmatrix},$$

where $\overline{C}=C$ if C is of type $\widetilde{\mathbf{D}}_m$, U is an indecomposable regular $\overline{C}$-module of regular length 2 lying in a tube with $s-2$ rays, V is a simple regular $\overline{C}$-module lying in a tube with $s-2$ rays, W is the direct successor of V (in $\Gamma_{\overline{C}}$), Λ is given by one of the following quivers:

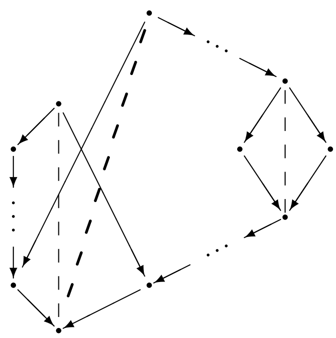




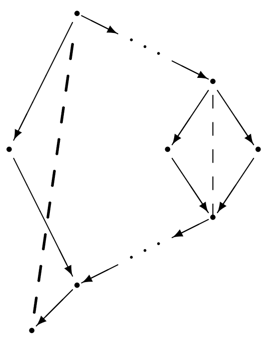
(possibly $w=a$, but no loop) and such that $F(N)(x,y)=N(y)\otimes_K \Lambda(x,w)$ for any object x in Λ and any object y in $\overline{C}$. Therefore it remains to describe the bound quivers of minimal non-polynomial growth algebras of the above types $D(U)$, $E(V)$, $F(V)$, $G(V)$ and their duals.

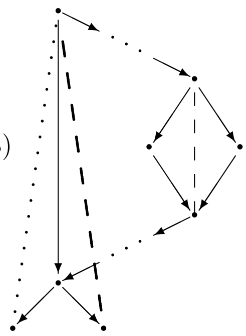
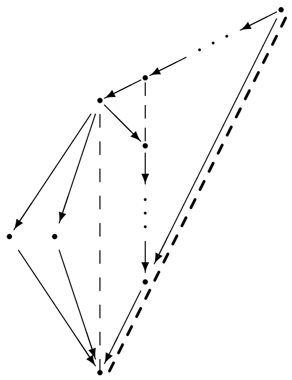
The strongly simply connected algebras of the form $D(U)$, $E(V)$, $F(V)$, and their duals, have been described by the authors in [12]. All such algebras are minimal non-polynomial growth, hence pg-critical, and appear in the families (1)–(16). Further, a direct analysis of the remaining possibilities, using the known structure of the indecomposable regular $\overline{C}$-modules of regular length at most 2, leads to A being given by one of the frames (1)–(31) or by one of the following forms:

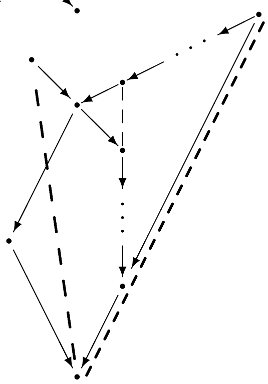
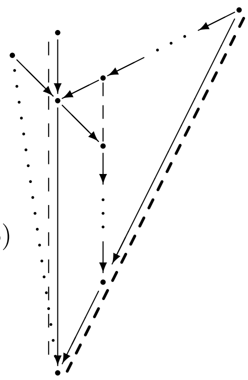
(r1)



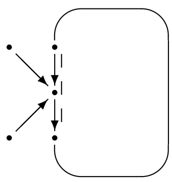
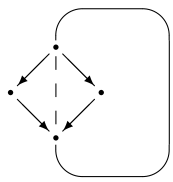
(r2)



(r3)

(r4)


(r5)

(r6)


The above algebras (r1)–(r6) contain a proper convex subcategory of one of the forms

(s1)

(s2)


which are not of polynomial growth. Indeed, the universal Galois coverings of these algebras (with infinite cyclic group) admit convex subcategories given by pg-critical trees. Therefore, applying Proposition 3.1 and the properties of the associated push down functors [9], we deduce that (s1) and (s2) are not of polynomial growth. Hence the algebras (r1)–(r6) are not pg-critical. This finishes the proof.

COROLLARY 3.3. *Let A be a pg-critical algebra. Then*

(i) *A is simply connected.*
(ii) $\mathrm{gl.dim}\, A = 2$.
(iii) *A^{op} is pg-critical.*

P r o o f. This follows from the shape of the frames (1)–(31) and the fact that the algebras $B[M]$ and $B[N,t]$, defining the pg-critical algebras, have global dimension 2.

It is shown in [21] that a strongly simply connected algebra A is of polynomial growth if and only if A does not contain a convex subcategory which is hypercritical or pg-critical. The hypercritical algebras (which are the preprojective tilts of minimal wild hereditary tree algebras) have been classified by quivers and relations (cf. [22]). This together with the corollary below gives a handy criterion for a strongly simply connected algebra to be of polynomial growth.

COROLLARY 3.4. *Let A be an algebra. The following are equivalent*:

(i) *A is tame minimal non-polynomial growth strongly simply connected.*
(ii) *A is strongly simply connected pg-critical.*
(iii) *A is obtained from one of the frames* (1)–(16) *by admissible operations.*

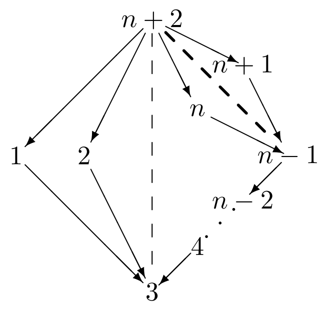
P r o o f. This follows from Theorem 3.2, and Theorem 4.1 in [21].

Following [7] a triangular algebra A is called *completely separating* if it is Schurian (that is, $\dim_K(P_A(x), P_A(y)) \leq 1$ for all vertices x and y of Q_A) and every convex subcategory of A has the separation property. We then get the following consequence of the above corollary:

COROLLARY 3.5. *Let A be a completely separating algebra. Then A is tame minimal of non-polynomial growth if and only if A is obtained from one of the frames* (1)–(11) *by applying admissible operations.*

4. The tilting classes of pg-critical algebras. Recall that the Euler form χ_A of an algebra A is called *positive semi-definite* if $\chi_A(z) \geq 0$ for all $z \in K_0(A)$. In this case, the radical $\operatorname{rad}\chi_A$ of χ_A is the set of all $z \in K_0(A)$ satisfying $\chi_A(z) = 0$. Moreover, $\operatorname{rad}\chi_A$ is then a subgroup of $K_0(A)$ and its rank is said to be the *radical rank* of χ_A.

For every $n \geq 4$ denote by Λ_n the following pg-critical algebra of type (13):



which we call a *canonical* pg-critical algebra. Since $\operatorname{gl.dim} \Lambda_n = 2$, for $x \in K_0(\Lambda_n) = \mathbb{Z}^{n+2}$, we have

$$\begin{aligned}
\chi_{\Lambda_n}(x) &= \sum_{i=1}^{n+2} x_i^2 - \sum_{i=3}^{n-2} x_i x_{i+1} \\
&\quad - x_1x_3 - x_2x_3 - x_1x_{n+2} - x_2x_{n+2} - x_{n-1}x_n - x_{n-1}x_{n+1} \\
&\quad - x_nx_{n+2} - x_{n+1}x_{n+2} + x_3x_{n+2} + x_{n-1}x_{n+2} \\
&= \left(x_1 - \tfrac{1}{2}x_3 - \tfrac{1}{2}x_{n+2}\right)^2 + \left(x_2 - \tfrac{1}{2}x_3 - \tfrac{1}{2}x_{n+2}\right)^2 \\
&\quad + \frac{1}{2}\sum_{i=3}^{n-2}(x_i - x_{i+1})^2 \\
&\quad + \frac{1}{2}(x_{n-1} - x_n - x_{n+1} + x_{n+2})^2 + \frac{1}{2}(x_n - x_{n+1})^2.
\end{aligned}$$

Hence χ_{Λ_n} is positive semi-definite and $\operatorname{rad}\chi_{\Lambda_n}$ is the free abelian subgroup of $K_0(\Lambda_n)$ generated by the vectors $h_\infty = (1,1,0,0,\dots,0,0,1,1,2)$ and $h = (1,1,1,1,\dots,1,1,1,1,1)$.

Observe that h_∞ is the positive generator of $\operatorname{rad}\chi_{H_\infty}$, where H_∞ is the tame hereditary convex subcategory of Λ_n given by the vertices $1,2,n,n+1,n+2$. Further, consider also the tame hereditary convex subcategory H_0 of Λ_n given by the vertices $1,2,3,\dots,n-1,n,n+1$, and the positive generator $h_0 = (1,1,2,2,\dots,2,2,1,1,0)$ of $\operatorname{rad}\chi_{H_0}$. Then $2h = h_0 + h_\infty$, and hence h_0 and h_∞ generate a subgroup of $\operatorname{rad}\chi_{\Lambda_n}$ of index 2.

We say that an algebra Λ can be obtained from an algebra Γ by a *sequence of tilts* if there is a finite sequence of algebras $\Gamma = \Gamma_0, \Gamma_1, \dots, \Gamma_{r+1} = \Lambda$ and tilting Γ_i-modules T_i, $0 \le i \le r$, such that, for each i, Γ_{i+1} is isomorphic to $\operatorname{End}_{\Gamma_i}(T_i)$.

The aim of this section is to prove the following theorem:

THEOREM 4.1. *Let A be a pg-critical algebra and n be the rank of $K_0(A)$. Then*

(i) *Λ_{n-2} can be obtained from A by a sequence of tilts.*
(ii) *A can be obtained from Λ_{n-2} by a sequence of tilts.*
(iii) *χ_A is positive semi-definite with radical rank 2.*

In order to prove the theorem we need the following two lemmas.

LEMMA 4.2. *Let H be a hereditary algebra of type $\widetilde{\mathbf{D}}_n$, $n \ge 4$, let T be a tilting H-module without non-zero preinjective direct summands, and $B = \operatorname{End}_H(T)$. Let M be an indecomposable regular B-module of regular length 2 lying in a tube of Γ_B with $n-2$ rays, let R be the indecomposable regular H-module of regular length 2 such that $M = \operatorname{Hom}_H(T,R)$, and let Λ be the one-point extension $\Lambda = H[R]$ of H by R, say with the extension* *vertex w. Then $T' = T \oplus P_\Lambda(w)$ is a tilting Λ-module and $\operatorname{End}_\Lambda(T')$ is isomorphic to the one-point extension $B[M]$ of B by M.*

Proof. Observe that the number of pairwise non-isomorphic indecomposable direct summands of T' is equal to the rank of the Grothendieck group $K_0(\Lambda) \cong K_0(H) \oplus \mathbb{Z}$. Clearly, $\operatorname{pd}_\Lambda T' \le 1$, because $\operatorname{pd}_\Lambda T = \operatorname{pd}_H T \le 1$. Moreover, $\operatorname{Ext}^1_\Lambda(T', T') = 0$ since $\operatorname{Ext}^1_\Lambda(T, T) = \operatorname{Ext}^1_H(T, T) = 0$, $P_\Lambda(w)$ is projective, and $\operatorname{Ext}^1_\Lambda(T, P_\Lambda(w)) = \operatorname{Ext}^1_H(T, R) = 0$, because R belongs to the torsion class of the tilting theory in $\operatorname{mod} H$ determined by T. Finally, we get

$$\operatorname{End}_\Lambda(T') \cong \begin{pmatrix} K & \operatorname{Hom}_H(T,R) \\ 0 & \operatorname{Hom}_H(T,T) \end{pmatrix} = \begin{pmatrix} K & M \\ 0 & B \end{pmatrix} = B[M].$$

LEMMA 4.3. *Let H be a hereditary algebra of Euclidean type and R_1, R_2 two indecomposable regular H-modules lying in the same τ_H-orbit of Γ_H, say $R_2 = \tau_H^{-m} R_1$ for some $m \ge 0$. Consider the one-point extensions $\Lambda_1 = H[R_1]$ and $\Lambda_2 = H[R_2]$, say with the extension vertices w_1 and w_2, respectively. Then $T = \tau_H^{-m} H \oplus P_{\Lambda_2}(w_2)$ is a tilting Λ_2-module and $\operatorname{End}_{\Lambda_2}(T) \cong \Lambda_1$.*

Proof. Clearly, $\operatorname{pd}_{\Lambda_2}(T) \le 1$ and T has the correct number of pairwise non-isomorphic indecomposable direct summands. Moreover,

$$\operatorname{Ext}^1_{\Lambda_2}(\tau_H^{-m} H, \tau_H^{-m} H) \cong \operatorname{Ext}^1_H(H, H) = 0,$$

$P_{\Lambda_2}(w_2)$ is projective and

$$\begin{aligned} \operatorname{Ext}^1_{\Lambda_2}(\tau_H^{-m} H, P_{\Lambda_2}(w_2)) &\cong D\overline{\operatorname{Hom}}_{\Lambda_2}(P_{\Lambda_2}(w_2), \tau_{\Lambda_2}(\tau_H^{-m} H)) \\ &\cong D\overline{\operatorname{Hom}}_{\Lambda_2}(P_{\Lambda_2}(w_2), \tau_H(\tau_H^{-m} H)) = 0. \end{aligned}$$

Further,

$$\begin{aligned} R_1 = \operatorname{Hom}_H(H, R_1) &\cong \operatorname{Hom}_H(\tau_H^{-m} H, \tau_H^{-m} R_1) \\ &= \operatorname{Hom}_H(\tau_H^{-m} H, R_2) = \operatorname{Hom}_H(\tau_H^{-m} H, P_{\Lambda_2}(w_2)). \end{aligned}$$

Therefore,

$$\operatorname{End}_{\Lambda_2}(T) \cong \begin{pmatrix} K & R_1 \\ 0 & H \end{pmatrix} = \Lambda_1.$$

Proof of Theorem 4.1. (i) First observe that if A is of type $B[N,t]$ then applying the APR-tilting module [2] associated with one of the two simple projective A-modules which are not B-modules, we get a pg-critical algebra of type $B'[M']$. Hence we may assume that A is of type $B[M]$. Now applying Lemmas 4.2 and 4.3 we infer that after one or two tilts we may pass to an algebra $H[R]$, where H is the hereditary algebra given by the quiver

$$\begin{tikzcd}
2 \arrow[dr] & & & & & n \arrow[dl] \\
& 3 & 4 \arrow[l] & \cdots \arrow[l] & n-1 \arrow[l] & \\
1 \arrow[ur] & & & & & n+1 \arrow[ul]
\end{tikzcd}$$

and R is an arbitrary indecomposable regular H-module of regular length 2 lying in a tube of Γ_H of rank $n-2$. For $n \geq 5$ we have only one tube of rank $n-2$ in Γ_H and this contains the following indecomposable module of regular length 2:

$$\begin{tikzcd}
K \arrow[dr, "\binom{-1}{-1}"] & & & & & K \arrow[dl, "1"'] \\
& K^2 & K \arrow[l, "\binom{0}{1}"'] & \cdots \arrow[l, "1"'] & K \arrow[l, "1"'] & \\
K \arrow[ur, "\binom{1}{0}"'] & & & & & K \arrow[ul, "1"]
\end{tikzcd}$$

Taking this module as R we find that $H[R]$ is isomorphic to Λ_n.

For $n = 4$, Γ_H has three tubes of rank 2, which contain the following indecomposable modules of regular length 2:

$$\begin{tikzcd}
K \arrow[dr, "\binom{-1}{-1}"] & & K \arrow[dl, "\binom{0}{1}"'] \\
& K^2 & \\
K \arrow[ur, "\binom{1}{0}"'] & & K \arrow[ul, "\binom{0}{1}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[dr, "\binom{0}{1}"] & & K \arrow[dl, "\binom{-1}{-1}"'] \\
& K^2 & \\
K \arrow[ur, "\binom{1}{0}"'] & & K \arrow[ul, "\binom{0}{1}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
K \arrow[dr, "\binom{0}{1}"] & & K \arrow[dl, "\binom{0}{1}"'] \\
& K^2 & \\
K \arrow[ur, "\binom{1}{0}"'] & & K \arrow[ul, "\binom{-1}{-1}"]
\end{tikzcd}$$

Taking any of these modules as R we deduce that $H[R]$ is isomorphic to Λ_4. This proves (i).

(ii) From Corollary 3.3 we know that A^{op} is also a pg-critical algebra. Further, by (i), there exists a sequence of algebras $A^{\mathrm{op}} = \Gamma_0, \Gamma_1, \ldots, \Gamma_{r+1} = \Lambda_{n-2}$ and tilting Γ_i-modules T_i such that $\Gamma_{i+1} = \mathrm{End}_{\Gamma_i}(T_i)$ for $0 \leq i \leq r$. Then each T_i is also a tilting $\Gamma_{i+1}^{\mathrm{op}}$-module and $\Gamma_i^{\mathrm{op}} = \mathrm{End}_{\Gamma_{i+1}^{\mathrm{op}}}(T_i)$. Therefore we get a sequence of tilts $\Lambda_{n-2} = \Lambda_{n-2}^{\mathrm{op}} = \Gamma_{r+1}^{\mathrm{op}}, \Gamma_r^{\mathrm{op}}, \ldots, \Gamma_1^{\mathrm{op}}, \Gamma_0^{\mathrm{op}} = (A^{\mathrm{op}})^{\mathrm{op}} = A$ leading from Λ_{n-2} to A.

(iii) Since A can be obtained from Λ_{n-2} by a sequence of tilts, the forms χ_A and $\chi_{\Lambda_{n-2}}$ are $\mathbb{Z}$-congruent (see [15, (4.1)(7)]). Hence χ_A is positive semi-definite with radical rank 2, by the above description of the properties of $\chi_{\Lambda_{n-2}}$.

5. The Coxeter matrix. Let A be a triangular algebra and $P_1, \ldots, P_n$ a complete set of pairwise non-isomorphic indecomposable A-modules. The *Cartan matrix* C_A of A is the $(n \times n)$-matrix whose (i,j)-entry is given by $\dim_K \mathrm{Hom}_A(P_i, P_j)$. Then C_A is invertible over $\mathbb{Z}$ and we get a symmetric bilinear form $(-,-)_A$ on $K_0(A)$, given by $(x,y)_A = \frac{1}{2}x(C_A^{-1} + C_A^{-T})y^T$, such that $\chi_A(x) = (x,x)_A$. Further, the matrix $\Phi_A = -C_A^{-T} C_A$ is called the *Coxeter matrix* of A. The characteristic polynomial of Φ_A is called the *Coxeter polynomial* of A. Note also that, for χ_A positive semi-definite, we have $\{x \in K_0(A) \mid \chi_A(x) = 0\} = \{x \in K_0(A) \mid x\Phi_A = x\}$.

THEOREM 5.1. *Let A be a pg-critical algebra and n be the rank of $K_0(A)$. Then*

(i) $\Phi_A^{2(n-5)}$ *is the identity matrix.*

(ii) *The Coxeter polynomial of A is of the form $(T^{n-5}+1)(T-1)^2(T+1)^3$.*

In particular, the spectral radius of Φ_A is 1.

Proof. It follows from Section 4 that A is in the same tilting class as an algebra $\Lambda = \Lambda_m$, $m \geq 4$. Then, by [15, (4.1)(7)], there exists an invertible matrix Ψ such that $\Phi_A = \Psi \Phi_\Lambda \Psi^{-1}$. Therefore it is sufficient to prove the claim for Λ. This can be done by elementary calculations. In fact, the Cartan matrix C_{Λ_m} is by definition given by

$$C_{\Lambda_m} = \left[\begin{array}{ccc|ccccc|ccc}
1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 2 \\
\hline
 & & & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & 1 \\
 & & & & 1 & \cdots & 1 & 1 & 1 & 1 & 1 \\
 & 0 & & & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & & & & 0 & & 1 & 1 & 1 & 1 & 1 \\
 & & & & & & & 1 & 1 & 1 & 1 \\
\hline
 & & & & & & & & 1 & 0 & 1 \\
 & 0 & & & & 0 & & & 0 & 1 & 1 \\
 & & & & & & & & 0 & 0 & 1
\end{array}\right].$$

Its inverse $C_{\Lambda_m}^{-1}$, depending on m, is, for $m = 4$,

$$C_{\Lambda_4}^{-1} = \left[\begin{array}{ccc|ccc}
1 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & -1 \\
-1 & -1 & 1 & -1 & -1 & 2 \\
\hline
 & & & 1 & 0 & -1 \\
 & 0 & & 0 & 1 & -1 \\
 & & & 0 & 0 & 1
\end{array}\right],$$

and for $m > 4$,

$$C_{\Lambda_m}^{-1} = \left[\begin{array}{ccc|ccccc|ccc}
1 & 0 & 0 & 0 & 0 & & \cdots & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & & \cdots & 0 & 0 & 0 & -1 \\
-1 & -1 & 1 & -1 & 0 & & \cdots & 0 & 0 & 0 & 1 \\
\hline
 & & & 1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\
 & & & & 1 & -1 & \ddots & \vdots & \vdots & \vdots & \vdots \\
 & 0 & & & & \ddots & \ddots & 0 & 0 & 0 & 0 \\
 & & & & 0 & & 1 & -1 & 0 & 0 & 0 \\
 & & & & & & & 1 & -1 & -1 & 1 \\
\hline
 & & & & & & & & 1 & 0 & -1 \\
 & 0 & & & & 0 & & & 0 & 1 & -1 \\
 & & & & & & & & 0 & 0 & 1
\end{array}\right].$$

Thus the Coxeter matrix $\Phi_{\Lambda_m} = -C_{\Lambda_m}^{-T} C_{\Lambda_m}$ is, for $m = 4$,

$$\Phi_{\Lambda_4} = \left[\begin{array}{ccc|ccc} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -1 & -2 \\ \hline 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 \\ -1 & -1 & -2 & -1 & -1 & -1 \end{array}\right],$$

and for $m > 4$,

$$\Phi_{\Lambda_m} = \left[\begin{array}{ccc|cccc|ccc} 0 & 1 & 1 & 1 & & \cdots & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & & \cdots & 1 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & & \cdots & -1 & -1 & -1 & -2 \\ \hline 1 & 1 & 1 & 0 & & \cdots & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \text{\Large 0} & & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & \cdots & -1 & -2 & -1 & -1 & 0 \end{array}\right].$$

Now (i) is a matter of direct verification, while (ii) follows by induction and expansion, using the $(m-1)$st row of the corresponding determinant

$$\left|\begin{array}{ccc|cccc|ccc} T & -1 & -1 & -1 & & \cdots & -1 & -1 & -1 & -1 \\ -1 & T & -1 & -1 & & \cdots & -1 & -1 & -1 & -1 \\ 1 & 1 & T+1 & 1 & & \cdots & 1 & 1 & 1 & 2 \\ \hline -1 & -1 & -1 & T & & \cdots & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & T & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \text{\Large 0} & & -1 & T & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & \cdots & 0 & -1 & T & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & -1 & -1 & T & 0 \\ 0 & 0 & 1 & 1 & \cdots & 1 & 2 & 1 & 1 & T \end{array}\right|.$$

6. The Auslander–Reiten quiver. The main aim of this section is to describe the structure of non-regular components of the Auslander–Reiten quiver of a pg-critical algebra. Moreover, we give a view on the structure of the category of indecomposable modules over such an algebra.

Let A be a pg-critical algebra. If $A = B[M]$ we put $B_0 = B$ and denote by w the extension vertex of $A = B_0[M]$. Assume now $A = B[N,t]$. Then Q_A consists of Q_B and the quiver

$$\begin{tikzcd}[row sep=small] & & & & \bullet a \\ w \ \bullet \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r] & \underset{c}{\bullet} \arrow[ur] \arrow[dr] & \\ & & & & \bullet b \end{tikzcd}$$

and N is the restriction of $P_A(w)$ to B. Denote by B_0 the convex subcategory of A given by all objects of A except a. Then B_0 is a tilted algebra of type $\widetilde{\mathbf{D}}_{n+t}$ having a complete slice in the preinjective component, containing B as a convex subcategory, and A is the one-point coextension of B_0 by the injective module $I_{B_0}(c)$. In both cases, B_0 is a tubular extension of tubular type $(2,2,r)$, with $r = n$ or $r + n + t$, of its unique tame concealed convex subcategory C_0 (of type $\widetilde{\mathbf{A}}_m$ or $\widetilde{\mathbf{D}}_m$). It follows from [15, (4.9)] that Γ_{B_0} consists of a preprojective component $\mathcal{P}_0$, formed by the indecomposable preprojective C_0-modules, a $\mathbb{P}_1(K)$-family $\mathcal{T}_0^{(\lambda)}, \lambda \in \mathbb{P}_1(K)$, of pairwise orthogonal ray tubes, and a preinjective component $\mathcal{I}_0$, containing all indecomposable injective B_0-modules. In $(\mathcal{T}_0^{(\lambda)})_{\lambda\in\mathbb{P}_1(K)}$ two tubes have 2 rays, one has $r-2$ rays, and the remaining ones are homogeneous (stable tubes of rank 1). Without loss of generality, we may assume that $\mathcal{T}_0^{(0)}$ and $\mathcal{T}_0^{(1)}$ are tubes with 2 rays and $\mathcal{T}_0^{(\infty)}$ is the tube with $r-2$ rays containing a module which is a direct predecessor (if $A = B[M]$) or direct successor (if $A = B[N,t]$) of $P_A(a)$ in Γ_A.

Since A^{op} is also pg-critical, by Corollary 3.3, we conclude that A is also of one of the forms

$$[M']B' = \begin{bmatrix} B' & \mathrm{D}(M') \\ 0 & K \end{bmatrix}, \qquad [t',N']B' = \begin{bmatrix} B' & 0 & 0 & 0 & \cdots & 0 & \mathrm{D}(N') \\ & K & 0 & K & \cdots & K & K \\ & & K & K & \cdots & K & K \\ & & & K & \cdots & K & K \\ & & 0 & & \ddots & \vdots & \vdots \\ & & & & & K & K \\ & & & & & & K \end{bmatrix},$$

where B' is a representation-infinite tilted algebra of type $\widetilde{\mathbf{D}}_{n'}$ with a complete slice in the preprojective component, M' (respectively, N') is an indecomposable regular B'-module of regular length 2 (respectively, 1) lying in a tube of $\Gamma_{B'}$ with $n'-2$ corays, and $t'+1$ $(t \geq 2)$ is the number of objects of $[t',N']B'$ which are not in B'.

If $A = [M']B'$ we put $B_\infty = B'$ and denote by a' the coextension vertex of $A = [M']B_\infty$. Assume now that $A = [t',N']B'$. Then the quiver Q_A of A consists of $Q_{B'}$ and the quiver,

$$\begin{tikzcd}[row sep=small] & & & & & a' \arrow[dl] \\ w' & \bullet \arrow[l] & \bullet \arrow[l] & \cdots \arrow[l] & c' \arrow[l] & \\ & & & & & b' \arrow[ul] \end{tikzcd}$$

and N' is the restriction of $I_A(w')$ to B'. Denote by B_∞ the convex subcategory of A given by all objects of A except a'. Then B_∞ is a tilted algebra of type $\widetilde{\mathbf{D}}_{n'+t'}$ having a complete slice in the preprojective component, containing B' as a convex subcategory, and A' is the one-point extension of B_∞ by the projective module $P_{B_\infty}(c')$. In both cases, B_∞ is a tubular coextension of tubular type $(2,2,r)$, with $r = n'$ or $r = n' + t'$, of its unique tame concealed convex subcategory C_∞ (of type $\widetilde{\mathbf{A}}_{m'}$ or $\widetilde{\mathbf{D}}_{m'}$).

It follows from [15, (4.9)] that Γ_{B_∞} consists of a preinjective component $\mathcal{Q}_\infty$, formed by the indecomposable preinjective C_∞-modules, a $\mathbb{P}_1(K)$-family $\mathcal{T}_\infty^{(\lambda)}$, $\lambda \in \mathbb{P}_1(K)$, of pairwise orthogonal coray tubes, and a preprojective component $\mathcal{P}'_\infty$ containing all indecomposable projective B_∞-modules. In $(\mathcal{T}_\infty^{(\lambda)})_{\lambda\in\mathbb{P}_1(K)}$ two tubes have 2 corays, one has $r-2$ corays and the remaining ones are homogeneous. We may assume that $\mathcal{T}_\infty^{(0)}$, $\mathcal{T}_\infty^{(1)}$ are tubes with 2 corays and $\mathcal{T}_\infty^{(\infty)}$ is the tube with $r-2$ corays containing a module which is a direct successor (if $A = [M']B'$) or a direct predecessor (if $A = [t', N']B'$) of $I_A(a')$ in Γ_A.

Further, denote by $\mathcal{T}_0$ the tubular K-family $\mathcal{T}_0^{(\lambda)}$, $\lambda \in K$, by $\mathcal{T}_\infty$ the tubular K-family $\mathcal{T}_\infty^{(\lambda)}$, $\lambda \in K$, by $\mathcal{Q}_0$ the class of indecomposable A-modules whose restrictions to B_0 have no non-zero direct summands from $\mathcal{P}_0$ and $\mathcal{T}_0$, and by $\mathcal{P}_\infty$ the class of indecomposable A-modules whose restrictions to B_∞ have no non-zero direct summands from $\mathcal{T}_\infty$ and $\mathcal{Q}_\infty$. Finally, denote by Δ the following quiver of type $\mathbf{D}_\infty$:

$$\begin{tikzcd}[row sep=small] \bullet & & & & \\ & \bullet \arrow[ul] \arrow[dl] \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \cdots \\ \bullet & & & & \end{tikzcd}$$

THEOREM 6.1. *Let A be a pg-critical algebra. Then*

(i) *$\Gamma_A = \mathcal{P}_0 \vee \mathcal{T}_0 \vee \mathcal{Q}_0 \cap \mathcal{P}_\infty \vee \mathcal{T}_\infty \vee \mathcal{Q}_\infty$, where $\mathcal{Q}_0 \cap \mathcal{P}_\infty$ is a disjoint union of regular components and one non-regular component $\mathcal{C}$, and the ordering from left to right indicates that there are non-zero maps* (*in* $\operatorname{mod} A$) *only from any of these families to itself or to the families to its right.*

(ii) *The regular components in $\mathcal{Q}_0 \cap \mathcal{P}_\infty$ consist entirely of modules whose restrictions to B_0 have non-zero preinjective direct summands and whose restrictions to B_∞ have non-zero preprojective direct summands.*

(iii) *The component $\mathcal{C}$ has the following properties*:

(a) *$\mathcal{C}$ contains all modules of $\mathcal{T}_0^{(\infty)}$ and $\mathcal{T}_\infty^{(\infty)}$.*

(b) *The stable part of $\mathcal{C}$ is of the form $\mathbb{Z}\mathbf{A}_\infty$.*

(c) *$\mathcal{C}$ admits a full translation subquiver $\mathcal{R} = (-\mathbb{N})\Delta$ which is closed under successors in $\mathcal{C}$ and consists of modules whose restrictions to B_0 are direct sums of modules from $\mathcal{T}_0^{(\infty)}$ and whose restrictions to B_∞ are direct sums of preprojective modules.*

(d) *$\mathcal{C}$ admits a full translation subquiver $\mathcal{L} = \mathbb{N}\Delta^{\mathrm{op}}$ which is closed under predecessors in $\mathcal{C}$ and consists of modules whose restrictions to B_0 are direct sums of preinjective modules and whose restrictions to B_∞ are direct sums of modules from $\mathcal{T}_\infty^{(\infty)}$.*

(e) $\mathrm{Hom}_A(\mathcal{L},\mathcal{R}) = 0$ *and* $\mathrm{Hom}_A(\mathcal{R},\mathcal{L}) \neq 0$.

Proof. Assume first that $A = B[N,t]$. Then, in the above notation, $A_A = P' \oplus S_A(a)$. Consider the APR-tilting A-module $T = P' \oplus \tau_A^- S_A(a)$ and the algebra A' obtained from A by reversing the arrow $c \to a$ to $c \leftarrow a$. Then $A' = \mathrm{End}_A(T)$ is a pg-critical algebra of the form $B_0[P_{B_0}(c)]$. Moreover, by [2], the functor $\mathrm{Hom}_A(T,-) : \mathrm{mod}\, A \to \mathrm{mod}\, A'$ induces an equivalence between the full subcategory of $\mathrm{mod}\, A$ formed by all modules without direct summands isomorphic to $S_A(a)$ and the full subcategory of $\mathrm{mod}\, A'$ formed by all modules without direct summands isomorphic to $S_{A'}(a)$. Moreover, $\Gamma_{A'}$ is obtained from Γ_A by replacing

$$\begin{tikzcd}
S_A(b) \arrow[dr] & & \bullet \arrow[dr] & \\
S_A(a) \arrow[r] & P_A(c) \arrow[ur] \arrow[r] \arrow[dr] & \bullet \arrow[r] & \bullet \\
{} \arrow[ur] & & \bullet \arrow[ur] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & I_A(b) \arrow[dr] & & \\
\bullet \arrow[ur] \arrow[r] \arrow[dr] & I_A(a) \arrow[r] & I_A(c) \arrow[dr, dash] & \\
 & \bullet \arrow[ur] & & {}
\end{tikzcd}$$

by

$$\begin{tikzcd}
S_{A'}(b) \arrow[dr] & & \bullet \arrow[dr] & \\
 & P_{A'}(c) \arrow[ur] \arrow[r] \arrow[dr] & P_{A'}(a) \arrow[r] & \bullet \\
{} \arrow[ur] & & \bullet \arrow[ur] &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
 & I_{A'}(b) \arrow[dr] & & \\
\bullet \arrow[ur] \arrow[r] \arrow[dr] & \bullet \arrow[r] & I_{A'}(c) \arrow[r] \arrow[dr, dash] & I_{A'}(a) \\
 & \bullet \arrow[ur] & & {}
\end{tikzcd}$$

respectively. Therefore, in order to prove the theorem, we may assume that A is of the form $B[M]$. We identify the objects of $\mathrm{mod}\, B[M] = \mathrm{mod}\, A$ with the triples (V, X, φ), where V is a (finite-dimensional) vector space over K, X an object of $\mathrm{mod}\, B$ and $\varphi : V \to \mathrm{Hom}_B(M, X)$ is a K-linear map. Then the B-modules X are the triples $(0, X, 0)$. Moreover, a $B[M]$-homomorphism $(V, X, \varphi) \to (W, Y, \psi)$ consists of a pair (α, f), where $\alpha : V \to W$ is a K-linear map, $f : X \to Y$ a B-homomorphism, and $\psi\alpha = \mathrm{Hom}_B(M, f)\varphi$. It is known that if $0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$ is an Auslander–Reiten sequence in $\mathrm{mod}\, B$ then in $\mathrm{mod}\, B[M]$ we have an Auslander–Reiten sequence

$$0 \to (|X|, X, 1_{|X|}) \xrightarrow{(1_{|X|}, f)} (|X|, Y, |f|) \to Z \to 0,$$

where $|X| = \mathrm{Hom}_B(M, X)$ and $|f| = \mathrm{Hom}_B(M, f)$ (see [15, (2.5)(6)]). Since in our situation $\mathrm{Hom}_B(M, \mathcal{P}_0 \vee \mathcal{T}_0) = 0$, we conclude that $\Gamma_A = \mathcal{P}_0 \vee \mathcal{T}_0 \vee \mathcal{Q}_0$.

Denote by $\mathcal{C}_0$ the connected component of Γ_A containing $P_A(a)$. Observe that the restriction of the vector space category $\operatorname{Hom}_B(M, \operatorname{mod} B)$ to the tube $\mathcal{T}_0^{(\infty)}$ is the K-linear category of the partially ordered set given by the following full translation subquiver:

$$\begin{tikzcd}
 & Y_1 \arrow[r] & Y_2 \arrow[r] & Y_3 \arrow[r] & \cdots \\
M = & X_0 \arrow[r] \arrow[u] & X_1 \arrow[r] \arrow[u] & X_2 \arrow[r] \arrow[u] & \cdots
\end{tikzcd}$$

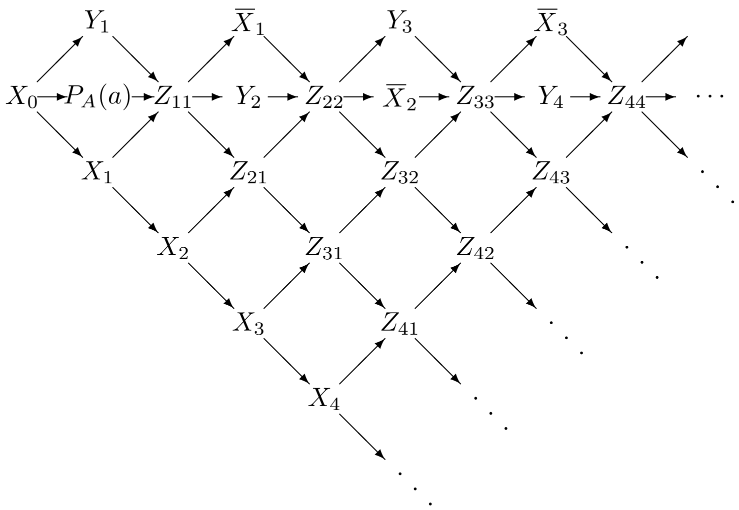
of $\mathcal{T}_0^{(\infty)}$ given by the corresponding two parallel rays of $\mathcal{T}_0^{(\infty)}$. We get the indecomposable A-modules

$$\overline{X}_i = (|X_i|, X_i, 1_{X_i}), \quad \overline{Y}_i = (|Y_i|, Y_i, 1_{Y_i}), \quad i \geq 1,$$

where $|X_i| = |Y_i| = K$. Clearly, $P_A(a) = (|X_0|, X_0, 1_{X_0})$. Moreover, since $\operatorname{Hom}_B(M, X_i)$ and $\operatorname{Hom}_B(M, Y_i)$ are, for $i \geq j$, orthogonal objects of $\operatorname{Hom}_B(M, \operatorname{mod} B)$, we get (see [16, (2.4)]) the indecomposable A-modules

$$Z_{i,j} = (K, X_i \oplus Y_j, \Delta_{i,j}), \quad i \geq j,$$

where $\Delta_{i,j} : \operatorname{Hom}_B(M, X_i \oplus Y_j) = K^2$ are the diagonal maps. Applying the above formula for Auslander–Reiten sequences in $\operatorname{mod} B[M]$ with the right terms being B-modules, and calculating the corresponding cokernels, we infer that $\mathcal{C}_0$ has a full translation subquiver $\mathcal{R}$ of the form



formed by the modules X_0, $P_A(a)$, $X_i, \overline{X}_i$ for $i \geq 1$, Y_j for $j \geq 1$, and $Z_{i,j}$ for $1 \leq j \leq i$. Obviously, $\mathcal{R} \cong (-\mathbb{N})\Delta$ and is closed under successors in $\mathcal{C}_0$. Consider also the modules $U_i = \tau_B^- Y_i$, $i \geq 1$. Observe that, if $\mathcal{T}_0^{(\infty)}$ has only two rays, then $U_{i+1} = X_i$ for $i \geq 0$. Calculating the Auslander–Reiten sequences with right terms U_i, $i \geq 1$, we conclude that $\mathcal{C}_0$ admits a full translation subquiver of the form

$$\begin{tikzcd}
\overline{Y}_1 \arrow[dr] & & U_1 \arrow[dr] & & & \\
& \overline{Y}_2 \arrow[ur] \arrow[dr] & & U_2 \arrow[dr] & & \\
& & \overline{Y}_3 \arrow[ur] \arrow[dr] & & U_3 \arrow[dr] & \\
& & & \overline{Y}_4 \arrow[ur] \arrow[dr] & & \ddots \\
& & & & \ddots &
\end{tikzcd}$$

Moreover, the Auslander–Reiten sequences in $\operatorname{mod} B$ with right terms in $\mathcal{T}_0^{(\infty)}$ but different from the modules Y_i, U_i, $i \geq 1$, are Auslander–Reiten sequences in $\operatorname{mod} B[M] = \operatorname{mod} A$. Hence all rays of $\mathcal{T}_0^{(\infty)}$ except the one containing the modules Y_i, $i \geq 1$, form infinite sectional paths also in $\mathcal{C}_0$. This also shows that $\mathcal{C}_0$ contains all indecomposable projective A-modules which do not belong to $\mathcal{P}_0 \vee \mathcal{T}_0$. Further, the stable part of $\mathcal{C}_0$ is then isomorphic to $\mathbb{Z}\mathbf{A}_\infty$. Finally, observe that, if L is an indecomposable A-module whose restriction to B_0 is a direct sum of modules from $\mathcal{T}_0^{(\infty)}$, then L belongs to $\mathcal{C}_0$. This is clear if L is a B-module. Suppose L is not a B-module. Then $L = (V, Z, \varphi)$ for some vector space V, $Z \in \operatorname{mod} B$, and $\varphi : V \to \operatorname{Hom}_B(M, Z)$. By our assumption, $V \neq 0$, $\varphi \neq 0$ and Z is a direct sum of modules of the form X_i, $i \geq 0$, and Y_j, $j \geq 1$. Then the structure of indecomposable representations of partially ordered sets of width 2 (see [16, (2.4)]) implies that L is one of the modules $P_A(a) = \overline{X}_0$, $\overline{X}_i$, $\overline{Y}_i$, $i \geq 1$, or $Z_{i,j}$, $1 \leq j \leq i$, and we are done.

Applying dual arguments for the one-point coextension (extension) leading from B_∞ to A, we infer that $\Gamma_A = \mathcal{P}_\infty \vee \mathcal{T}_\infty \vee \mathcal{Q}_\infty$. Therefore

$$(*) \qquad \Gamma_A = \mathcal{P}_0 \vee \mathcal{T}_0 \vee \mathcal{Q}_0 \cap \mathcal{P}_\infty \vee \mathcal{T}_\infty \vee \mathcal{Q}_\infty.$$

Further, the connected component, say $\mathcal{C}_\infty$, of Γ_A containing the module $I_A(a')$ admits a full translation subquiver $\mathcal{L}$ which is isomorphic to $\mathbb{N}\Delta^{\mathrm{op}}$ and is closed under predecessors in $\mathcal{C}_\infty$. Moreover, $\mathcal{C}_\infty$ contains all indecomposable A-modules whose restrictions to B_∞ are direct sums of modules from $\mathcal{T}_\infty^{(\infty)}$. All corays of $\mathcal{T}_\infty^{(\infty)}$ except one (whose modules are distributed in $\mathcal{L}$) are infinite sectional paths in $\mathcal{C}_\infty$. Finally, $\mathcal{C}_\infty$ contains all indecomposable injective A-modules which do not belong to $\mathcal{T}_\infty \vee \mathcal{Q}_\infty$, and the stable part of $\mathcal{C}_\infty$ is isomorphic to $\mathbb{Z}\mathbf{A}_\infty$.

It follows from the above considerations and the decomposition $(*)$ that $\mathcal{R}$ consists of modules whose restrictions to B_∞ are direct sums of preprojective B_∞-modules and the restrictions to B_0 are direct sums of modules from $\mathcal{T}_0^{(\infty)}$. Similarly, $\mathcal{L}$ consists of modules whose restrictions to B_0 are direct sums of preinjective B_0-modules and the restrictions to B_∞ are direct sums of modules from $\mathcal{T}_\infty^{(\infty)}$. In particular, $\mathrm{Hom}_A(\mathcal{L},\mathcal{R}) = 0$ and $\mathrm{Hom}_A(\mathcal{R},\mathcal{L}) \neq 0$. Moreover, the regular components of $\mathcal{Q}_0 \cap \mathcal{P}_\infty$ consist entirely of modules whose restrictions to B_0 have non-zero preinjective direct summands and the restrictions to B_∞ have non-zero preprojective direct summands. Consequently, in order to complete the proof, it is enough to show that $\mathcal{C}_0 = \mathcal{C}_\infty$. For this we have several cases to consider, depending on the shape of the quiver of A. We will not go through all these cases in detail, but rather discuss one example, which essentially describes all situations that occur in a complete analysis.

Consider the following algebra A of type (16):

$$\begin{tikzcd}
 & & x_1 \arrow[r] \arrow[dll] \arrow[dl] \arrow[dd, dash, dashed] & x_2 \arrow[r] & \cdots \arrow[r] & x_{k-1} \arrow[r] & x_k \arrow[d] & w \arrow[l] \arrow[ddl, dash, dashed] \\
x_0 \arrow[drr] & x_{-1} \arrow[dr] & & & & & x_{k+1} \arrow[d] & \\
 & & w' & & \cdots \arrow[ll] & & x_{k+2} \arrow[ll] &
\end{tikzcd}$$

Then A is of the form $A = B_0[M]$, where B_0 is the full subcategory created by all objects except w, and $M = P_{B_0}(x_k)/P_{B_0}(x_{k+2})$. Further, in the above notation, $Y_0 = P_{B_0}(x_k)/P_{B_0}(x_{k+1}) = P_A(x_k)/P_A(x_{k+1})$. Also, A is of the form $A = [M']B_\infty$, where B_∞ is the full subcategory created by all objects except w'. There are essentially three cases, depending on k.

In case $k > 2$, any of the simple modules $S_A(x_j)$, $i < j < 2$, lies in $\mathcal{T}_0^{(\infty)}$ as well as in $\mathcal{T}_\infty^{(\infty)}$, and thus in $\mathcal{C}_0 \cap \mathcal{C}_\infty$.

For $k \le 2$, consider the module

$$L = \tau_B^- Y_0 = P_{B_0}(x_{k-1})/P_{B_0}(x_k) = P_A(x_{k-1})/P_A(x_k).$$

Then L lies in $\mathcal{T}_0^{(\infty)}$, thus in $\mathcal{C}_0$, and $\tau_A L = P_A(w)/P_A(x_{k+1})$.

In case $k = 2$, $\tau_A L = P_A(w)/P_A(x_3)$ lies in $\mathcal{T}_\infty^{(\infty)}$, thus in $\mathcal{C}_\infty$. So $\mathcal{C}_0$ and $\mathcal{C}_\infty$ are connected to each other, thus coincide.

In case $k = 1$, $\tau_A L = P_A(w)/P_A(x_2)$ is neither a module over B_0 nor over B_∞. However, $\tau_A^2 L = P_A(x_1)/(P_A(x_0) \amalg P_A(x_{-1}))$ again lies in $\mathcal{T}_0^{(\infty)}$ and $\mathcal{C}_\infty$, so that $\mathcal{C}_0$ and $\mathcal{C}_\infty$ coincide. This completes the proof.

Recall that the *component quiver* Σ_Λ of an algebra Λ is defined as follows [18]: the vertices of Σ_Λ are the connected components of Γ_Λ, and two components $\mathcal{D}$ and $\mathcal{E}$ are connected in Σ_Λ by an arrow $\mathcal{D} \to \mathcal{E}$ if $\mathrm{rad}^\infty(X,Y) \neq 0$ for some modules $X \in \mathcal{D}$ and $Y \in \mathcal{E}$. Here, $\mathrm{rad}^\infty(X,Y)$ denotes the intersection of all finite powers $\mathrm{rad}^i(X,Y)$ of the radical $\mathrm{rad}(X,Y)$.

From Theorem 6.1 we get the following information on the component quiver of a pg-critical algebra.

COROLLARY 6.2. *Let A be a pg-critical algebra. Then, in the above notation, the following statements hold*:

(i) $\mathcal{P}_0$ *is a unique source of* Σ_A *and there are arrows from* $\mathcal{P}_0$ *to all remaining vertices of* Σ_A.

(ii) $\mathcal{Q}_\infty$ *is a unique sink of* Σ_A *and there are arrows from all remaining vertices of* Σ_A *to* $\mathcal{Q}_\infty$.

(iii) *For each* $\lambda \in K$, *there are arrows in* Σ_A *from* $\mathcal{T}_0^{(\lambda)}$ *to all components of* $(\mathcal{Q}_0 \cap \mathcal{P}_\infty) \vee \mathcal{T}_\infty \vee \mathcal{Q}_\infty$, *and* $\mathcal{P}_0 \to \mathcal{T}_0^{(\lambda)}$ *is a unique arrow with target* $\mathcal{T}_0^{(\lambda)}$.

(iv) *For each* $\lambda \in K$, *there are arrows in* Σ_A *from all components of* $\mathcal{P}_0 \vee \mathcal{T}_0 \vee (\mathcal{Q}_0 \cap \mathcal{P}_\infty)$ *to* $\mathcal{T}_\infty^{(\lambda)}$, *and* $\mathcal{T}_\infty^{(\lambda)} \to \mathcal{Q}_\infty$ *is a unique arrow with source in* $\mathcal{T}_\infty^{(\lambda)}$.

(v) *For each regular component* $\mathcal{D}$ *in* $\mathcal{Q}_0 \cap \mathcal{P}_\infty$, *there are arrows* $\mathcal{C} \to \mathcal{D}$ *and* $\mathcal{D} \to \mathcal{C}$ *in* Σ_A.

(vi) Σ_A *admits a loop* $\mathcal{C} \to \mathcal{C}$.

Proof. We know that there are only finitely many indecomposable B_0-modules (respectively, B_∞-modules) whose restrictions to $\mathcal{C}_0$ (respectively, $\mathcal{C}_\infty$) are zero. The statements (i)–(iv) follow from Theorem 4.1 and the facts that $(\mathcal{T}_0^{(\lambda)})_{\lambda \in \mathbb{P}_1 K}$ separates $\mathcal{P}_0$ from $\mathcal{I}_0$ in $\operatorname{ind} B_0$, and $(\mathcal{T}_\infty^{(\lambda)})_{\lambda \in \mathbb{P}_1 K}$ separates $\mathcal{P}'_\infty$ from $\mathcal{Q}_\infty$ in $\operatorname{ind} B_\infty$. Moreover, (vi) follows from $\operatorname{Hom}_A(\mathcal{R}, \mathcal{L}) \neq 0$ and the fact that $\mathcal{C}$ has no oriented cycles. Take now an arbitrary regular component $\mathcal{D}$ in $\mathcal{Q}_0 \cap \mathcal{P}_\infty$. For (v) it is enough to show that $\operatorname{Hom}_A(P_A(a), X) \neq 0$ and $\operatorname{Hom}_A(Y, I_A(a)) \neq 0$ for some modules X and Y in $\mathcal{D}$. Suppose that $\operatorname{Hom}_A(P_A(a), X) = 0$ for all X in $\mathcal{D}$. Then $\mathcal{D}$ consists of B_0-modules which, by Theorem 6.1, must be preinjective. But then $\mathcal{D}$ is not regular, a contradiction. We get a similar contradiction assuming $\operatorname{Hom}_A(Y, I_A(a)) = 0$ for all $y \in \mathcal{D}$. This finishes the proof.

It follows from the decomposition $\Gamma_A = \mathcal{P}_0 \vee \mathcal{T}_0 \vee \mathcal{Q}_0 \cap \mathcal{P}_\infty \vee \mathcal{T}_\infty \vee \mathcal{Q}_\infty$ that $\operatorname{Hom}_A(\mathrm{D}(A), \mathcal{P}_0 \vee \mathcal{T}_0) = 0$ and $\operatorname{Hom}_A(\mathcal{T}_\infty \vee \mathcal{Q}_\infty, A) = 0$. Hence $\operatorname{pd}_A X \leq 1$ for all modules X in $\mathcal{P}_0 \vee \mathcal{T}_0$ and $\operatorname{id}_A Y \leq 1$ for all modules Y in $\mathcal{T}_\infty \vee \mathcal{Q}_\infty$ (see [15, (2.4)]). We also have the following information on the homological behaviour of the non-regular component $\mathcal{C}$ of $\mathcal{Q}_0 \cap \mathcal{P}_\infty$.

COROLLARY 6.3. *Let A be a pg-critical algebra. Then*

(i) $\operatorname{pd}_A X \leq 1$ *for all modules* X *in* $\mathcal{R}$.

(ii) $\operatorname{id}_A Y \leq 1$ *for all modules* Y *in* $\mathcal{L}$.

(iii) *There are infinitely many indecomposable modules* Z *in* $\mathcal{C}$ *with* $\operatorname{pd}_A Z = 2$ *and* $\operatorname{id}_A Z = 2$.

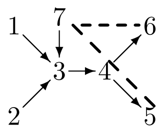
P r o o f. (i) From the structure of $\mathcal{C}$ described in Theorem 6.1 we know that if X belongs to $\mathcal{R}$ then the restriction of $\tau_A X$ to B_0 is a direct sum of modules from $\mathcal{T}_0^{(\infty)}$. On the other hand, the restriction of any indecomposable injective A-module to B_0 is a direct sum of indecomposable preinjective B_0-modules. Hence $\mathrm{Hom}_A(\mathrm{D}(A), \tau_A X) = 0$, and so $\mathrm{pd}_A X \le 1$.

The proof of (ii) is dual.

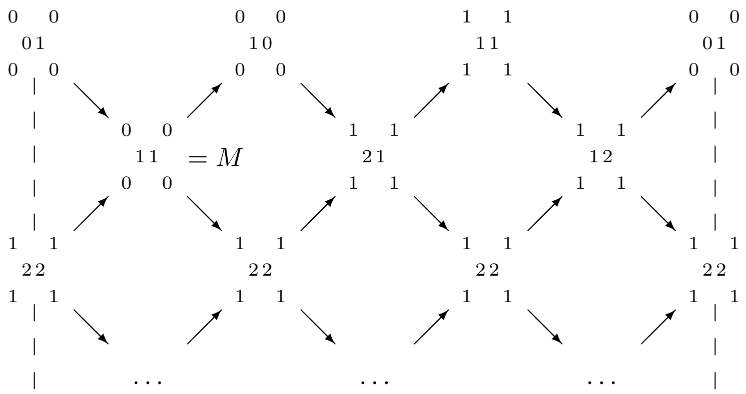
(iii) It follows from the proof of Theorem 6.1 that $\mathrm{Hom}_A(\mathrm{D}(A), \tau_A \overline{Y}_i) \neq 0$ for all $i \ge 1$. Hence $\mathrm{pd}_A \overline{Y}_i = 2$ for all $i \ge 1$. On the other hand, there is an infinite sequence $1 < i_1 < i_2 < \ldots$ such that, for each $s \ge 1$, there is a sectional path $Y_{i_s} \to L_{i_s} \to \ldots \to R$, with R an indecomposable projective B_0-module (if $A = B[M]$ and M is directing) or the radical of $P_A(a)$. Moreover, $L_{i_s} = \tau_A^{-1} \overline{Y}_{i_{s-1}}$ for any $s \ge 1$. Hence $\mathrm{Hom}(\tau_A^{-1} \overline{Y}_{i_{s-1}}, A) \neq 0$ for $s \ge 1$, as the composition of irreducible maps forming a sectional path is non-zero. Consequently, $\mathrm{id}_A \overline{Y}_{i_{s-1}} = 2$. We then have an infinite family $\overline{Y}_{i_{s-1}}$, $s \ge 1$, of modules in $\mathcal{C}$ with both projective and injective dimension equal to 2.

We shall now present an example illustrating the above considerations.

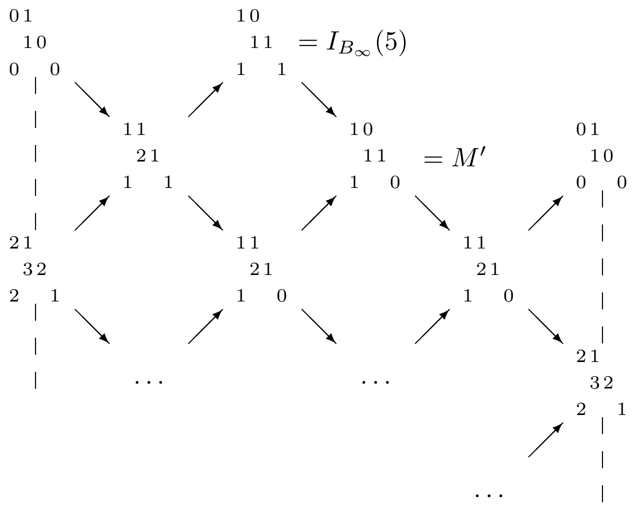
EXAMPLE 6.4. Let A be the algebra given by the bound quiver



Let $B_0 = C_0$ be the convex subcategory of A given by all objects of A except 7. Then B_0 is a concealed algebra of type $\widetilde{\mathbf{D}}_5$ and A is a one-point extension $B_0[M]$ of B_0 by an indecomposable regular B_0-module lying in the following stable tube $\mathcal{T}_0^{(\infty)}$ of rank 3 in Γ_{B_0}:



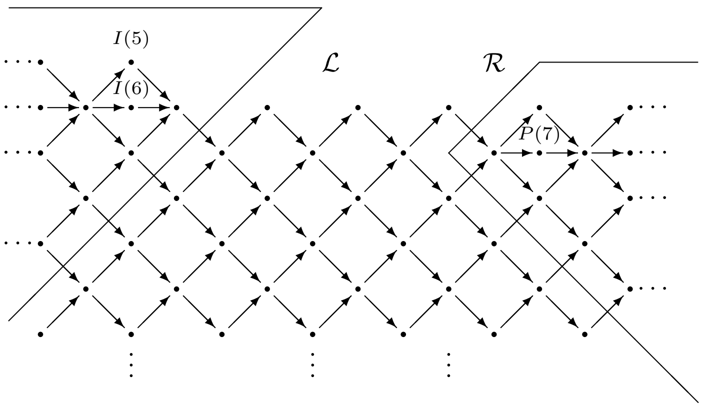
where we represent a module by its dimension vector in $K_0(B_0)$ and the vertical lines have to be identified in order to obtain a tube. Hence A is a pg-critical algebra. Observe also that $A = [M']B_\infty$, where B_∞ is the convex subcategory of A given by all vertices except 6, and M' is given by the dimension vector $\begin{smallmatrix} 1\,0 & \\ 1\,1 & \\ 1 \quad 0 \end{smallmatrix}$ in $K_0(B_\infty)$. Further, B_∞ is a tubular one-point coextension of the concealed convex convex subcategory given by the objects 1, 2, 3, 4, and 7, and its unique tube $\mathcal{T}_\infty^{(\infty)}$ with 3 corays is of the form



where again the modules are represented by their dimension vectors in $K_0(B_\infty)$, and the vertical lines have to be identified in order to obtain a (coray) tube. Then

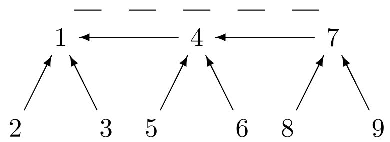
$$\Gamma_A = \mathcal{P}_0 \vee \mathcal{T}_0 \vee \mathcal{Q}_0 \vee \mathcal{P}_\infty \vee \mathcal{T}_\infty \vee \mathcal{Q}_\infty,$$

where $\mathcal{P}_0$ is the preprojective component of type $\widetilde{\mathbf{D}}_5$, $\mathcal{T}_0$ (respectively, $\mathcal{T}_\infty$) consists of the two stable tubes of rank 2 and the homogeneous tubes, $\mathcal{Q}_\infty$ is the preinjective component of type $\widetilde{\mathbf{D}}_4$, and the non-regular component $\mathcal{C}$ of $\mathcal{Q}_0 \vee \mathcal{P}_\infty$ is of the form

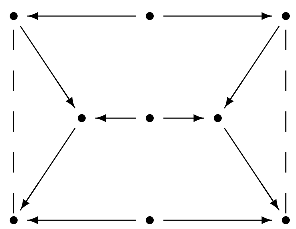


We end the paper with an example showing the possible shapes of regular components of the Auslander–Reiten quiver of a pg-critical algebra.

EXAMPLE 6.5. Let A be the pg-critical algebra given by the bound quiver



Then A admits an automorphism group G of order 2 generated by the twist g such that $g(1) = 1$, $g(2) = 3$, $g(3) = 2$, $g(4) = 4$, $g(5) = 6$, $g(6) = 5$, $g(7) = 7$, $g(8) = 9$, and $g(9) = 8$. Assume that the characteristic of K is not equal to 2. Then the twisted group algebra $B = A[G]$ (in the sense of [13]) is given by the following bound quiver:



Hence, B is a string (special biserial) algebra and the regular components in Γ_B are of the form $\mathbb{Z}\mathbf{A}_\infty^\infty$ and $\mathbb{Z}\mathbf{A}_\infty/(\tau)$ (see [6]). Then, applying [11] or [13], we find that the regular components of Γ_A are of the form $\mathbb{Z}\mathbf{A}_\infty^\infty$, $\mathbb{Z}\mathbf{D}_\infty$, $\mathbb{Z}\mathbf{A}_\infty/(\tau)$ and $\mathbb{Z}\mathbf{A}_\infty/(\tau^2)$. In fact, there are infinitely many regular components in Γ_A of each of these types.

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