1. Introduction. By Drozd’s Tame and Wild Theorem [8] the class of finite-dimensional algebras (associative, with identity) over an algebraically closed field may be divided into two disjoint classes. One class consists of the tame algebras, for which the indecomposable modules occur in each dimension $d$ in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory is as complicated as the study of finite-dimensional vector spaces together with two non-commuting endomorphisms, for which the classification of the indecomposable finite-dimensional modules is a well-known difficult problem. Hence we can hope to classify the modules only for tame algebras. Among the tame algebras we may distinguish the class of polynomial growth algebras $A$ for which there exists an integer $m$ (depending on $A$) such that, in each dimension $d$, the indecomposable $A$-modules occur in a finite number of discrete and at most $d^m$ one-parameter families.

Frequently, applying covering techniques, we may reduce the representation theory of a given tame (respectively, polynomial growth) algebra to that of the corresponding simply connected algebra. Recently, the class of polynomial growth simply connected algebras has been extensively investigated. In particular, a rather complete representation theory of polynomial growth strongly simply connected algebras has been established by the second author in [21]. One of the important open problems is to extend this theory to arbitrary simply connected algebras of polynomial growth. We are especially interested in criteria for a simply connected algebra to be of polynomial growth. This leads to the study of tame simply connected algebras which are minimal not of polynomial growth (they themselves are not of polynomial growth but every proper convex subcategory is).

The main aim of this article is to introduce and classify (by quivers and
relations) a class of tame minimal non-polynomial growth simply connected
algebras, which we call (generalized) polynomial growth critical algebras.
Moreover, we describe basic properties of polynomial growth critical alge-
bras and the structure of the category of indecomposable finite-dimensional
modules over such algebras. It is expected that the class of polynomial growth
critical algebras introduced and investigated here will play an important role
in the study of arbitrary tame non-polynomial growth simply connected al-
gebras.

The paper is organized as follows. In Section 2 we fix the notations
and recall the needed definitions. In Section 3 we introduce the polyno-
mial growth critical algebras and classify them by quivers and relations. In
particular, we prove that all such algebras are simply connected and their
opposite algebras are also polynomial growth critical. Moreover, applying
the main results of [21], we get a handy criterion for a strongly simply
connected algebra to be of polynomial growth. Section 4 is devoted to the
tilting classes of polynomial growth critical algebras. We prove that two
polynomial growth critical algebras with the same number of simple modules
belong to the same tilting class. Then we deduce that the Euler form of
any polynomial growth critical algebra is positive semi-definite with radical
of rank 2. In Section 5 we determine the Coxeter polynomial of any poly-
nomial growth critical algebra and show that the eigenvalues of its Coxeter
matrix are roots of unity. In the final Section 6 we investigate the module
category of polynomial growth critical algebras. We completely describe the
structure of all non-regular components of their Auslander–Reiten quivers
and discuss the behaviour of non-regular components in the category of in-
decomposable finite-dimensional modules. In particular, we show that the
Auslander–Reiten quiver of any polynomial growth critical algebra has ex-
actly one preprojective component, exactly one preinjective component, and
exactly one component containing both a projective module and an injective
module.

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2. Preliminaries. Throughout this article, $K$ will denote a fixed alge-
braically closed field. By an algebra is meant an associative finite-dimension-
al $K$-algebra with an identity, which we shall assume to be basic and con-
ected. An algebra $A$ can be written as a bound quiver algebra $A \cong KQ/I$,
where $Q = Q_A$ is the quiver of $A$ and $I$ is an admissible ideal in the path
algebra $KQ$ of $Q$. Equivalently, an algebra $A = KQ/I$ may be considered
as a $K$-category whose object class is the set of vertices of $Q$, and the set
of morphisms $A(x, y)$ from $x$ to $y$ is the quotient of the $K$-space $KQ(x, y)$
of all $K$-linear combinations of paths in $Q$ from $x$ to $y$ modulo the subspace $I(x,y) = I \cap KQ(x,y)$. An algebra $A$ with $Q_A$ having no oriented cycle is said to be \textit{triangular}. A full subcategory $C$ of $A$ is said to be \textit{convex} if any path in $Q_A$ with source and target in $Q_C$ lies entirely in $Q_C$. Following [1] a triangular algebra $A$ is called \textit{simply connected} if, for any presentation $A \cong KQ/I$ of $A$ as a bound quiver algebra, the fundamental group $\Pi_1(Q, I)$ of $(Q, I)$ is trivial. Moreover, following [20] an algebra $A$ is said to be \textit{strongly simply connected} if every convex subcategory of $A$ is simply connected. It was shown in [20] that a triangular algebra is strongly simply connected if and only if every convex subalgebra $C$ of $A$ satisfies the separation condition of Bautista, Larrión and Salmerón [3]. For example, if $Q_A$ is a tree, then $A$ is strongly simply connected.

For an algebra $A$, we denote by $\text{mod} A$ the category of finite-dimensional right $A$-modules and by $\text{ind} A$ its full subcategory consisting of the indecomposable modules. We shall denote by $\Gamma_A$ the Auslander–Reiten quiver of $A$ and by $\tau_A = \text{DTr}$ and $\tau_A^- = \text{TrD}$ the Auslander–Reiten translations. We shall agree to identify an indecomposable $A$-module with the vertex of $\Gamma_A$ corresponding to it. For each vertex $i$ of $Q_A$ we denote by $S_A(i)$ the simple $A$-module having $K$ at the vertex $i$, by $P_A(i)$ the projective cover of $S_A(i)$, and by $I_A(i)$ the injective envelope of $S_A(i)$. For a module $M$ in $\text{mod} A$ we shall denote by $\dim M$ the dimension vector $(\dim_K M(i))_{\text{vertex in } Q_A}$. The \textit{support} $\text{supp} M$ of a module $M$ in $\text{mod} A$ is the full subcategory of $A$ given by all vertices $i$ of $Q_A$ such that $M(i) \neq 0$.

Let $A$ be an algebra and $K[X]$ the polynomial algebra in one variable. Following [8], $A$ is said to be \textit{tame} if, for each dimension $d$, there exists a finite number of $K[X]$-$A$-bimodules $M_i$, $1 \leq i \leq n_d$, which are finitely generated and free as left $K[X]$-modules, and such that all but a finite number of isomorphism classes of indecomposable right $A$-modules of dimension $d$ are of the form $K[X]/(X - \lambda) \otimes_{K[X]} M_i$ for some $\lambda \in K$ and some $i$. Let $\mu_A(d)$ be the least number of $K[X]$-$A$-bimodules satisfying the above conditions. Then $A$ is said to be \textit{of polynomial growth} if there is a positive integer $m$ such that $\mu_A(d) \leq d^m$ for any $d \geq 1$ (cf. [19]). Examples of polynomial growth algebras are tilted algebras of Euclidean type and tubular algebras [15].

Let $A = KQ/I$ be a triangular algebra. Denote by $K_0(A)$ the Grothendieck group of $A$. Then $K_0(A) = \mathbb{Z}^n$, where $n$ is the number of vertices of $Q$. The \textit{Euler quadratic form} $\chi_A$ of $A$ is the integral quadratic form on $K_0(A)$ such that

$$\chi_A(\dim X) = \sum_{i=0}^{\infty} (-1)^i \dim_K \text{Ext}_A^i(X, X)$$

for any module $X$ in $\text{mod} A$ (see [15, (2.4)]). If $\text{gl.dim} A \leq 2$ then $\chi_A$ coin-
cides with the Tits form $q_A$ of $A$, defined for $x = (x_i)_{i \in Q_0} \in K_0(A)$ as follows:

$$q_A(x) = \sum_{i \in Q_0} x_i^2 - \sum_{(i \rightarrow j) \in Q_1} x_i x_j + \sum_{i, j \in Q_0} r_{i,j} x_i x_j,$$

where $Q_0$ and $Q_1$ are the sets of vertices and arrows of $Q$, respectively, and $r_{i,j}$ is the cardinality of $L \cap I(i,j)$ for a minimal set of generators $L \subseteq \bigcup_{i,j \in Q_0} I(i,j)$ of the ideal $I$ (see [4]).

For basic background on the representation theory of finite-dimensional algebras we refer to [15].

3. Tame minimal non-polynomial growth algebras. In this section we give a complete description of a class of tame minimal non-polynomial growth algebras by quivers and relations. For an algebra $A = kQ/I$, generators of $I$ are usually called relations. In our bound quivers, a dashed line indicates a relation being the sum of all paths from the starting point to the end point. Moreover, a dotted line indicates a zero-relation along a path of length 2.

Recall that a concealed algebra is of concealed type $\Delta$ if it is an algebra $C$ of the form $C = \text{End}_H(T)$ where $H$ is a hereditary algebra of type $\Delta$ and $T$ is a preprojective tilting $H$-module. We know from [5], [10] that there is only one family of concealed algebras of type $\tilde{A}_n$, $n \geq 1$, given by the quivers

and four families of concealed algebras of type $\tilde{D}_n$, $n \geq 4$, given by the following quivers and relations:

(1) 

(2) 

(3) 

(4) 

where the number of vertices is equal to $n + 1$ and $\cdots$ means $\longrightarrow$ or $\longleftrightarrow$. It is known that if $C$ is a concealed algebra of Euclidean type then $\Gamma_C$ consists of a preprojective component $P$, a preinjective component $Q$ and a $\mathbb{P}_1(K)$-family $T = (T_\lambda)_{\lambda \in \mathbb{P}_1(K)}$ of stable tubes. It is shown in [15, (4.9)] that
an algebra $B$ is a tilted algebra $\text{End}_H(T)$, where $H$ is a hereditary algebra of type $\tilde{D}_n$ and $T$ a tilting $H$-module without preinjective (respectively, preprojective) direct summands, if and only if $B$ is a tubular extension (respectively, coextension) of tubular type $(2, 2, n-2)$ of a concealed algebra $C$ of type $\tilde{A}_m$ or $\tilde{D}_m$, $m \leq n$. In the case when $B$ is a tubular extension of $C$ (of type $(2, 2, n-2)$), $\Gamma_B$ consists of a preprojective component $P'$ (which is the preprojective component of $\Gamma_C$), a preinjective component $Q'$ having a complete slice of type $\tilde{D}_n$, and a $P_1(\mathbb{K})$-family $T' = (T'_{\lambda})_{\lambda \in P_1(\mathbb{K})}$ of ray tubes. Two tubes in $T'$ have 2 rays, one has $n-2$ rays, and the remaining are stable tubes of rank 1 (homogeneous tubes). We have the dual structure for $\Gamma_B$ in the case when $B$ is a tubular coextension of $C$ (of type $(2, 2, n-2)$). Finally, we note that any representation-infinite tilted algebra of type $\tilde{D}_n$ is of one of the above types.

The main objective of this article is to investigate the following class of algebras. By a polynomial growth critical algebra, briefly $pg$-critical algebra, we mean an algebra $A$ satisfying the following conditions:

(i) $A$ is of one of the forms:

$$B[M] = \begin{bmatrix} K & K & \cdots & K & K & N \\ K & \cdots & K & K & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & K & K & K & 0 \\ 0 & K & 0 & 0 \\ 0 & 0 & 0 & B \end{bmatrix}, \quad B[N,t] = \begin{bmatrix} K & K & \cdots & K & K & N \\ K & \cdots & K & K & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & K & K & K & 0 \\ 0 & K & 0 & 0 \\ 0 & 0 & 0 & B \end{bmatrix},$$

where $B$ is a representation-infinite tilted algebra of the form $\text{End}_H(T)$, for a hereditary algebra $H$ of type $\tilde{D}_n$ and a tilting $H$-module $T$ without non-zero preinjective direct summands, $M = \text{Hom}_H(T, R)$ (respectively, $N = \text{Hom}_H(T, S)$) for an indecomposable regular $H$-module $R$ of regular length 2 (respectively, indecomposable regular $H$-module $S$ of regular length 1) lying in a tube of $\Gamma_H$ with $n-2$ rays, and $t+1$ ($t \geq 2$) is the number of objects of $B[N,t]$ which are not in $B$.

(ii) Every proper convex subcategory of $A$ is of polynomial growth.

If $A = B[M]$ then the quiver $Q_A$ of $A$ consists of the quiver $Q_B$ of $B$ and an extension vertex $w$ (which is a source of $Q_A$) such that $M$ is the restriction of $P_A(w)$ to $B$. In the case when $A = B[N,t]$ the quiver $Q_A$ consists of $Q_B$ and the quiver

$$\begin{array}{c}
  w \\
  \vdots \\
  \vdots \\
  \vdots \\
  c \\
  b \\
\end{array}$$

and $N$ is the restriction of $P_A(w)$ to $B$. 
The following proposition motivates the name “pg-critical algebra”.

**Proposition 3.1.** Let $A$ be a pg-critical algebra. Then $A$ is tame but not of polynomial growth.

**Proof.** We may assume that $A$ is of the form $B[M]$ or $B[N,t]$. If $A = B[M]$ then the claim follows from [14] and [17]. In the case when $A = B[N,t]$, applying the APR-tilting module [2] induced by the simple projective $A$-module given by one of the vertices $a$ or $b$, we get an algebra of type $B'[M']$ for a tubular extension of a tilted algebra $B'$ of type $\tilde{D}_{n+t}$ and a regular indecomposable $B'$-module $M'$ of regular length 2 lying in the tube of $\Gamma_B$ having $n+t-2$ rays.

We note that the use of the term “pg-critical algebra” in the present paper slightly deviates from its use in an earlier publication by the authors [12]. Here, we consider a more general class of algebras which seems to be crucial for studying arbitrary tame simply connected algebras which are not of polynomial growth. Observe also that in the above definition of a pg-critical algebra both conditions (i) and (ii) are essential. Indeed, if $A$ is an algebra given by the following quiver and relations:

![Quiver](image1)

then $A$ satisfies (i) but not (ii), as the convex subcategory of $A$ formed by all vertices except $a$ is still not of polynomial growth. The algebra $\Gamma$ given by the following quiver and relations:

![Quiver](image2)

is tame (see [14, (3.9)]) with all proper convex subcategories representation-finite (hence of polynomial growth) but does not satisfy (i).

In order to save space in the theorem below and to make the list below more accessible we write down only the possible frames. Given such a frame, we allow the following admissible operations:

(i) Replacing each subgraph

![Subgraph](image3)
by

(ii) Choice of arbitrary orientations in non-oriented edges.
(iii) Constructing the opposite algebra.

**Theorem 3.2.** An algebra $A$ is pg-critical if and only if it is obtained from a frame in the following list by admissible operations:

1. 

2. 

3. 

4. 

5. 

Proof. Let $B$ be a representation-infinite tilted algebra of type $\tilde{D}_n$, $n \geq 4$, with a complete slice in its preinjective component. Then $B$ can be obtained from a concealed algebra $C$ (of type $\tilde{A}_m$ or type $\tilde{D}_m$, $m \leq n$) by adding branches $L_1, \ldots, L_r$ in the extension vertices $\omega_1, \ldots, \omega_r$ of a multiple one-point extension $C[E_1][E_2] \ldots [E_r]$ of $C$ by pairwise non-isomorphic simple regular $C$-modules $E_1, \ldots, E_r$. In this process we create two tubes with 2 rays, one tube with $n-2$ rays and the remaining tubes (of rank 1) are not changed (see [15, Section 4]). Moreover, if $C$ is of type $\tilde{A}_m$, then $B$ contains a convex, tilted subcategory $\overline{C}$ of type $\tilde{D}_s$, $m < s \leq n$, isomorphic to one of the following:

In this case, the structure of indecomposable regular $\overline{C}$-modules of regular length at most 2 is well known. In the case when $C$ is of type $\tilde{D}_m$, the indecomposable regular $C$-modules of regular length at most 2 are completely described in [12].

Let now $A$ be a pg-critical algebra of one of the forms $B[M]$ or $B[N,t]$. Then a direct analysis shows that every proper convex subcategory of $B$ is of polynomial growth if and only if $A$ is a minimal non-polynomial growth algebra of one of the forms.
\[ D(U) = \begin{bmatrix} K & U \\ 0 & C \end{bmatrix}, \quad E(V) = \begin{bmatrix} K & 0 & V \\ 0 & K & V \\ 0 & 0 & C \end{bmatrix}, \quad F(V) = \begin{bmatrix} A & V^r \\ 0 & C \end{bmatrix}, \quad \text{or} \]

\[ G(V) = \begin{bmatrix} K & K & \ldots & K & W \\ K & K & \ldots & K & V \\ K & \ldots & K & V \\ \vdots & \vdots & \ddots & \vdots \\ 0 & K & V & C \end{bmatrix}, \]

where \( C = C \) if \( C \) is of type \( \tilde{D}_m \), \( U \) is an indecomposable regular \( C \)-module of regular length 2 lying in a tube with \( s - 2 \) rays, \( V \) is a simple regular \( C \)-module lying in a tube with \( s - 2 \) rays, \( W \) is the direct successor of \( V \) (in \( \Gamma_C \)), \( A \) is given by one of the following quivers:

\[
\begin{array}{c}
\begin{tikzpicture}
\node (w) at (0,0) {$w$};
\node (a) at (1,0) {$a$};
\draw (w) -- (a);
\end{tikzpicture}
\end{array}
\quad \begin{array}{c}
\begin{tikzpicture}
\node (w) at (0,0) {$w$};
\node (a) at (1,0) {$a$};
\draw (w) -- (a);
\end{tikzpicture}
\end{array}
\]

(possibly \( w = a \), but no loop) and such that \( F(N)(x,y) = N(y) \otimes_K A(x,w) \) for any object \( x \) in \( A \) and any object \( y \) in \( C \). Therefore it remains to describe the bound quivers of minimal non-polynomial growth algebras of the above types \( D(U), E(V), F(V), G(V) \) and their duals.

The strongly simply connected algebras of the form \( D(U), E(V), F(V), \) and their duals, have been described by the authors in [12]. All such algebras are minimal non-polynomial growth, hence pg-critical, and appear in the families (1)–(16). Further, a direct analysis of the remaining possibilities, using the known structure of the indecomposable regular \( C \)-modules of regular length at most 2, leads to \( A \) being given by one of the frames (1)–(31) or by one of the following forms:

\[
\begin{array}{c}
\begin{tikzpicture}
\end{tikzpicture}
\end{array}
\quad \begin{array}{c}
\begin{tikzpicture}
\end{tikzpicture}
\end{array}
\]

\( (r1) \) \quad \( (r2) \)
The above algebras (r1)–(r6) contain a proper convex subcategory of one of the forms

\[(s1)\]

\[(s2)\]

which are not of polynomial growth. Indeed, the universal Galois coverings of these algebras (with infinite cyclic group) admit convex subcategories given by pg-critical trees. Therefore, applying Proposition 3.1 and the properties of the associated push down functors [9], we deduce that (s1) and (s2) are not of polynomial growth. Hence the algebras (r1)–(r6) are not pg-critical. This finishes the proof.

**Corollary 3.3.** Let \( A \) be a pg-critical algebra. Then

(i) \( A \) is simply connected.

(ii) \( gl.\dim A = 2 \).

(iii) \( A^{op} \) is pg-critical.
Proof. This follows from the shape of the frames (1)–(31) and the fact that the algebras $B[M]$ and $B[N,t]$, defining the pg-critical algebras, have global dimension 2.

It is shown in [21] that a strongly simply connected algebra $A$ is of polynomial growth if and only if $A$ does not contain a convex subcategory which is hypercritical or pg-critical. The hypercritical algebras (which are the preprojective tilts of minimal wild hereditary tree algebras) have been classified by quivers and relations (cf. [22]). This together with the corollary below gives a handy criterion for a strongly simply connected algebra to be of polynomial growth.

**Corollary 3.4.** Let $A$ be an algebra. The following are equivalent:

(i) $A$ is tame minimal non-polynomial growth strongly simply connected.

(ii) $A$ is strongly simply connected pg-critical.

(iii) $A$ is obtained from one of the frames (1)–(16) by admissible operations.

**Proof.** This follows from Theorem 3.2, and Theorem 4.1 in [21].

Following [7] a triangular algebra $A$ is called completely separating if it is Schurian (that is, $\dim_K(P_A(x), P_A(y)) \leq 1$ for all vertices $x$ and $y$ of $Q_A$) and every convex subcategory of $A$ has the separation property. We then get the following consequence of the above corollary:

**Corollary 3.5.** Let $A$ be a completely separating algebra. Then $A$ is tame minimal of non-polynomial growth if and only if $A$ is obtained from one of the frames (1)–(11) by applying admissible operations.

4. The tilting classes of pg-critical algebras. Recall that the Euler form $\chi_A$ of an algebra $A$ is called positive semi-definite if $\chi_A(z) \geq 0$ for all $z \in K_0(A)$. In this case, the radical $\text{rad } \chi_A$ of $\chi_A$ is the set of all $z \in K_0(A)$ satisfying $\chi_A(z) = 0$. Moreover, $\text{rad } \chi_A$ is then a subgroup of $K_0(A)$ and its rank is said to be the radical rank of $\chi_A$.

For every $n \geq 4$ denote by $\Lambda_n$ the following pg-critical algebra of type (13):
which we call a canonical pg-critical algebra. Since \(\text{gl.dim} A_n = 2\), for \(x \in K_0(A_n) = \mathbb{Z}^{n+2}\), we have

\[
\chi_{A_n}(x) = \sum_{i=1}^{n+2} x_i^2 - \sum_{i=3}^{n+2} x_i x_{i+1}
- x_1 x_3 - x_2 x_3 - x_1 x_{n+2} - x_2 x_{n+2} - x_{n-1} x_n - x_{n-1} x_{n+1}
- x_n x_{n+2} - x_{n+1} x_{n+2} + x_3 x_{n+2} + x_{n-1} x_{n+2}
= (x_1 - \frac{1}{2} x_3 - \frac{1}{2} x_{n+2})^2 + (x_2 - \frac{1}{2} x_3 - \frac{1}{2} x_{n+2})^2
+ \frac{1}{2} \sum_{i=3}^{n+2} (x_i - x_{i+1})^2
+ \frac{1}{2} (x_{n-1} - x_n - x_{n+1} + x_{n+2})^2 + \frac{1}{2} (x_n - x_{n+1})^2.
\]

Hence \(\chi_{A_n}\) is positive semi-definite and \(\text{rad} \chi_{A_n}\) is the free abelian subgroup of \(K_0(A_n)\) generated by the vectors \(h_{\infty} = (1,1,0,0,\ldots,0,0,1,1,2)\) and \(h = (1,1,1,1,\ldots,1,1,1,1,1,1)\).

Observe that \(h_{\infty}\) is the positive generator of \(\text{rad} \chi_{H_{\infty}}\), where \(H_{\infty}\) is the tame hereditary convex subcategory of \(A_n\) given by the vertices \(1,2,3,\ldots, n, n+1, n+2\). Further, consider also the tame hereditary convex subcategory \(H_0\) of \(A_n\) given by the vertices \(1,2,3,\ldots, n-1, n, n+1\), and the positive generator \(h_0 = (1,1,2,2,\ldots,2,2,1,1,0)\) of \(\text{rad} \chi_{H_0}\). Then \(2h = h_0 + h_{\infty}\), and hence \(h_0\) and \(h_{\infty}\) generate a subgroup of \(\text{rad} \chi_{A_n}\) of index 2.

We say that an algebra \(A\) can be obtained from an algebra \(\Gamma\) by a sequence of tilts if there is a finite sequence of algebras \(\Gamma = \Gamma_0, \Gamma_1, \ldots, \Gamma_{r+1} = A\) and tilting \(\Gamma_i\)-modules \(T_i\), \(0 \leq i \leq r\), such that, for each \(i\), \(\Gamma_{i+1}\) is isomorphic to \(\text{End}_{\Gamma_i}(T_i)\).

The aim of this section is to prove the following theorem:

**Theorem 4.1.** Let \(A\) be a pg-critical algebra and \(n\) be the rank of \(K_0(A)\). Then

(i) \(A_{n-2}\) can be obtained from \(A\) by a sequence of tilts.

(ii) \(A\) can be obtained from \(A_{n-2}\) by a sequence of tilts.

(iii) \(\chi_A\) is positive semi-definite with radical rank 2.

In order to prove the theorem we need the following two lemmas.

**Lemma 4.2.** Let \(H\) be a hereditary algebra of type \(\tilde{D}_n\), \(n \geq 4\), let \(T\) be a tilting \(H\)-module without non-zero preinjective direct summands, and \(B = \text{End}_H(T)\). Let \(M\) be an indecomposable regular \(B\)-module of regular length 2 lying in a tube of \(\Gamma_B\) with \(n-2\) rays, let \(R\) be the indecomposable regular \(H\)-module of regular length 2 such that \(M = \text{Hom}_H(T,R)\), and let \(\Lambda\) be the one-point extension \(\Lambda = H[R]\) of \(H\) by \(R\), say with the extension
we get

Moreover, Ext$_1(T', T') = 0$ since Ext$_1^H(T, T) = 0$. P$_A(w)$ is projective, and Ext$_1^H(T, P_A(w)) = Ext^1_H(T, R) = 0$, because R belongs to the torsion class of the tilting theory in mod $H$ determined by T. Finally, we get

\[
\text{End}_A(T') \cong \begin{pmatrix} K & \text{Hom}_H(T, R) \\ 0 & \text{Hom}_H(T, T) \end{pmatrix} = \begin{pmatrix} K & M \\ 0 & B \end{pmatrix} = B[M].
\]

**Lemma 4.3.** Let H be a hereditary algebra of Euclidean type and $R_1$, $R_2$ two indecomposable regular $H$-modules lying in the same $\tau_H$-orbit of $\Gamma_H$, say $R_2 = \tau_H^{-m}R_1$ for some $m \geq 0$. Consider the one-point extensions $\Lambda_1 = H[R_1]$ and $\Lambda_2 = H[R_2]$, say with the extension vertices $w_1$ and $w_2$, respectively. Then $T = \tau_H^{-m}H \oplus P_{\Lambda_2}(w_2)$ is a tilting $\Lambda_2$-module and End$_{\Lambda_2}(T) \cong \Lambda_1$.

**Proof.** Clearly, pd$_{\Lambda_2}(T) \leq 1$ and T has the correct number of pairwise non-isomorphic indecomposable direct summands. Moreover,

\[
\text{Ext}^1_{\Lambda_2}(\tau_H^{-m}H, \tau_H^{-m}H) \cong \text{Ext}^1_H(H, H) = 0,
\]

$P_{\Lambda_2}(w_2)$ is projective and

\[
\text{Ext}^1_{\Lambda_2}(\tau_H^{-m}H, P_{\Lambda_2}(w_2)) \cong \text{DHom}_{\Lambda_2}(P_{\Lambda_2}(w_2), \tau_{\Lambda_2}^{-m}H) 
\cong \text{DHom}_{\Lambda_2}(P_{\Lambda_2}(w_2), \tau_H(\tau_H^{-m}H)) = 0.
\]

Further,

\[
R_1 = \text{Hom}_H(H, R_1) \cong \text{Hom}_H(\tau_H^{-m}H, \tau_H^{-m}R_1) 
= \text{Hom}_H(\tau_H^{-m}H, R_2) = \text{Hom}_H(\tau_H^{-m}H, P_{\Lambda_2}(w_2)).
\]

Therefore,

\[
\text{End}_{\Lambda_2}(T) \cong \begin{pmatrix} K & R_1 \\ 0 & H \end{pmatrix} = \Lambda_1.
\]

**Proof of Theorem 4.1.** (i) First observe that if A is of type $B[N, t]$ then applying the APR-tilting module [2] associated with one of the two simple projective A-modules which are not B-modules, we get a $pg$-critical algebra of type $B'[M']$. Hence we may assume that A is of type $B[M]$. Now applying Lemmas 4.2 and 4.3 we infer that after one or two tilts we may pass to an algebra $H[R]$, where H is the hereditary algebra given by the quiver

\[
\begin{array}{c}
1 \\
\vdots \\
n+1
\end{array}
\]

\[
\begin{array}{c}
2 \\
4 \\
n
\end{array}
\]

Hence we may assume that A is of type $B[M]$. Now applying Lemmas 4.2 and 4.3 we infer that after one or two tilts we may pass to an algebra $H[R]$, where H is the hereditary algebra given by the quiver

\[
\begin{array}{c}
1 \\
\vdots \\
n+1
\end{array}
\]

\[
\begin{array}{c}
2 \\
4 \\
n
\end{array}
\]
and $R$ is an arbitrary indecomposable regular $H$-module of regular length 2 lying in a tube of $\Gamma_H$ of rank $n-2$. For $n \geq 5$ we have only one tube of rank $n-2$ in $\Gamma_H$ and this contains the following indecomposable module of regular length 2:

$$
\begin{array}{c}
K(\begin{smallmatrix} 1 \\
-1 \\
0 
\end{smallmatrix}) \\
K(\begin{smallmatrix} 1 \\
0 \\
0 
\end{smallmatrix}) \\
K(\begin{smallmatrix} 0 \\
0 \\
0 
\end{smallmatrix})
\end{array} \quad \begin{array}{c}
1 \\
\cdots \\
1
\end{array} \quad \begin{array}{c}
K \\
K \\
K
\end{array}

Taking this module as $R$ we find that $H[R]$ is isomorphic to $A_n$.

For $n = 4$, $\Gamma_H$ has three tubes of rank 2, which contain the following indecomposable modules of regular length 2:

$$
\begin{array}{c}
K(\begin{smallmatrix} 1 \\
-1 \\
0 
\end{smallmatrix})K \\
K(\begin{smallmatrix} 1 \\
0 \\
0 
\end{smallmatrix})K \\
K(\begin{smallmatrix} 0 \\
0 \\
0 
\end{smallmatrix})K
\end{array} \quad \begin{array}{c}
K(\begin{smallmatrix} 0 \\
-1 \\
0 
\end{smallmatrix})K \\
K(\begin{smallmatrix} 0 \\
0 \\
0 
\end{smallmatrix})K \\
K(\begin{smallmatrix} 1 \\
0 \\
-1 
\end{smallmatrix})K
\end{array}

Taking any of these modules as $R$ we deduce that $H[R]$ is isomorphic to $A_4$. This proves (i).

(ii) From Corollary 3.3 we know that $A^\text{op}$ is also a pg-critical algebra. Further, by (i), there exists a sequence of algebras $A^\text{op} = \Gamma_0, \Gamma_1, \ldots, \Gamma_{r+1} = A_{n-2}$ and tilting $\Gamma_i$-modules $T_i$ such that $\Gamma_{i+1} = \text{End}_{\Gamma_i}(T_i)$ for $0 \leq i \leq r$. Then each $T_i$ is also a tilting $\Gamma_{i+1}^\text{op}$-module and $\Gamma_{i+1}^\text{op} = \text{End}_{\Gamma_{i+1}}(T_i)$. Therefore we get a sequence of tilts $A_{n-2} = A_{n-2}^\text{op} = \Gamma_{r+1}^\text{op}, \Gamma_r^\text{op}, \ldots, \Gamma_1^\text{op}, \Gamma_0^\text{op} = (A^\text{op})^\text{op} = A$ leading from $A_{n-2}$ to $A$.

(iii) Since $A$ can be obtained from $A_{n-2}$ by a sequence of tilts, the forms $\chi_A$ and $\chi_{A_{n-2}}$ are $\mathbb{Z}$-congruent (see [15, (4.1)(7)]). Hence $\chi_A$ is positive semi-definite with radical rank 2, by the above description of the properties of $\chi_{A_{n-2}}$.

5. The Coxeter matrix. Let $A$ be a triangular algebra and $P_1, \ldots, P_n$ a complete set of pairwise non-isomorphic indecomposable $A$-modules. The Cartan matrix $C_A$ of $A$ is the $(n \times n)$-matrix whose $(i,j)$-entry is given by $\text{dim}_K \text{Hom}_A(P_i, P_j)$. Then $C_A$ is invertible over $\mathbb{Z}$ and we get a symmetric bilinear form $(\cdot, \cdot)_A$ on $K_0(A)$, given by $(x, y)_A = \frac{1}{2} x(C_A^{-1} y)^T$, such that $\chi_A(x) = (x, x)_A$. Further, the matrix $\Phi_A = -C_A^{-T} C_A$ is called the Coxeter polynomial of $A$. The characteristic polynomial of $\Phi_A$ is called the Coxeter polynomial of $A$. Note also that, for $\chi_A$ positive semi-definite, we have $\{ x \in K_0(A) \mid \chi_A(x) = 0 \} = \{ x \in K_0(A) \mid x \Phi_A = x \}$. 

Theorem 5.1. Let $A$ be a pg-critical algebra and $n$ be the rank of $K_0(A)$. Then

(i) $\Phi_A^{2(n-5)}$ is the identity matrix.

(ii) The Coxeter polynomial of $A$ is of the form $(T^{n-5}+1)(T-1)^2(T+1)^3$.

In particular, the spectral radius of $\Phi_A$ is 1.

Proof. It follows from Section 4 that $A$ is in the same tilting class as an algebra $A = A_m$, $m \geq 4$. Then, by [15, (4.1)(7)], there exists an invertible matrix $\Psi$ such that $\Phi_A = \Psi \Phi_{A_m} \Psi^{-1}$. Therefore it is sufficient to prove the claim for $A_m$. This can be done by elementary calculations. In fact, the Cartan matrix $C_{A_m}$ is by definition given by

$$
C_{A_m} = \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 2 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 1 & 1 & 1 & 1 & & & & \\
1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 1 & 0 & 1 & & & & \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\end{bmatrix}
$$

Its inverse $C_{A_m}^{-1}$, depending on $m$, is, for $m = 4$,

$$
C_{A_4}^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & 0 & -1 \\
-1 & -1 & 1 & -1 & -1 & 2 \\
1 & 0 & -1 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix},
$$

and for $m > 4$,

$$
C_{A_m}^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \\
0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & -1 \\
-1 & -1 & 1 & -1 & 0 & \cdots & 0 & 0 & 0 & 1 \\
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & -1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
$$

$$
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & -1 \\
0 & 0 & 1 \\
\end{bmatrix}
$$
Thus the Coxeter matrix $\Phi = -C^{-T}C$ is, for $m = 4$,

$$\Phi_{A_4} = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ -1 & -1 & -1 & -1 & -2 \\ 1 & 1 & 1 & 0 & 1 \\ -1 & -1 & -2 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 \end{bmatrix},$$

and for $m > 4$,

$$\Phi_{A_m} = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & -2 \\ 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & \cdots & -1 & -2 & -1 & -1 & 0 \end{bmatrix}.$$  

Now (i) is a matter of direct verification, while (ii) follows by induction and expansion, using the $(m - 1)$st row of the corresponding determinant.

6. The Auslander–Reiten quiver. The main aim of this section is to describe the structure of non-regular components of the Auslander–Reiten quiver of a pg-critical algebra. Moreover, we give a view on the structure of the category of indecomposable modules over such an algebra.

Let $A$ be a pg-critical algebra. If $A = B[M]$ we put $B_0 = B$ and denote by $w$ the extension vertex of $A = B_0[M]$. Assume now $A = B[N,t]$. Then $Q_A$ consists of $Q_B$ and the quiver

$$w \rightarrow \cdots \rightarrow e \rightarrow b$$
and $\mathcal{N}$ is the restriction of $P_A(w)$ to $B$. Denote by $B_0$ the convex subcategory of $A$ given by all objects of $A$ except $a$. Then $B_0$ is a tilted algebra of type $\tilde{D}_{n+1}$ having a complete slice in the preinjective component, containing $B$ as a convex subcategory, and $A$ is the one-point coextension of $B_0$ by the injective module $I_{B_0}(c)$. In both cases, $B_0$ is a tubular extension of tubular type $(2,2,r)$, with $r = n$ or $r + n + t$, of its unique tame concealed convex subcategory $C_0$ (of type $\tilde{A}_m$ or $\tilde{D}_m$). It follows from [15, (4.9)] that $\Gamma_{B_0}$ consists of a preprojective component $P_0$, formed by the indecomposable preprojective $C_0$-modules, a $\mathbb{P}_1(K)$-family $T_0(\lambda), \lambda \in \mathbb{P}_1(K)$, of pairwise orthogonal ray tubes, and a preinjective component $I_0$, containing all indecomposable injective $B_0$-modules. In $(T_0(\lambda))_{\lambda \in \mathbb{P}_1(K)}$ two tubes have 2 rays, one has $r - 2$ rays, and the remaining ones are homogeneous (stable tubes of rank 1). Without loss of generality, we may assume that $T_0(b)$ and $T_0(c)$ are tubes with 2 rays and $T_0^{(\infty)}$ is the tube with $r - 2$ rays containing a module which is a direct predecessor (if $A = B[N]$) or direct successor (if $A = B[N,l]$) of $P_A(a)$ in $\Gamma_A$.

Since $A^{\text{op}}$ is also pg-critical, by Corollary 3.3, we conclude that $A$ is also of one of the forms

$$[M'B'] = \begin{bmatrix} B' & 0 & 0 & \cdots & 0 & D(N') \\ 0 & D(M') & K & K & \cdots & K \\ K & K & \cdots & K & K \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & K & K & K \end{bmatrix}$$

where $B'$ is a representation-infinite tilted algebra of type $\tilde{D}_{n'}$ with a complete slice in the preprojective component, $M'$ (respectively, $N'$) is an indecomposable regular $B'$-module of regular length 2 (respectively, 1) lying in a tube of $\Gamma_{B'}$ with $n' - 2$ corays, and $t' + 1 (t \geq 2)$ is the number of objects of $[t',N']B'$ which are not in $B'$.

If $A = [M']B'$ we put $B'_\infty = B'$ and denote by $a'$ the coextension vertex of $A = [M']B_\infty$. Assume now that $A = [t',N']B'$. Then the quiver $Q_A$ of $A$ consists of $Q_{B'}$ and the quiver,

```
  w' \rightarrow a' \rightarrow b'
```

and $N'$ is the restriction of $J_A(w')$ to $B'$. Denote by $B_\infty$ the convex subcategory of $A$ given by all objects of $A$ except $a'$. Then $B_\infty$ is a tilted algebra
of type $\tilde{D}_{n'+t'}$ having a complete slice in the preprojective component, containing $B'$ as a convex subcategory, and $A'$ is the one-point extension of $B_\infty$ by the projective module $P_{B_\infty}(c')$. In both cases, $B_\infty$ is a tubular coextension of tubular type $(2,2,r)$, with $r = n'$ or $r = n'+t'$, of its unique tame concealed convex subcategory $C_\infty$ (of type $\tilde{A}_{m'}$ or $\tilde{D}_{m'}$).

It follows from [15, (4.9)] that $\Gamma_{B_\infty}$ consists of a preinjective component $Q_\infty$, formed by the indecomposable preinjective $C_\infty$-modules, a $\mathbb{P}_1(K)$-family $T^{(\lambda)}_\infty$, $\lambda \in \mathbb{P}_1(K)$, of pairwise orthogonal coray tubes, and a preprojective component $P'_\infty$ containing all indecomposable projective $B_\infty$-modules. In $(T^{(\lambda)}_\infty)_{\lambda \in \mathbb{P}_1(K)}$ two tubes have 2 corays, one has $r-2$ corays and the remaining ones are homogeneous. We may assume that $T^{(0)}_\infty$, $T^{(1)}_\infty$ are tubes with 2 corays and $T^{(\infty)}_\infty$ is the tube with $r-2$ corays containing a module which is a direct successor (if $A = [M'B']$) or a direct predecessor (if $A = [t',N']B'$) of $I_A(a')$ in $\Gamma_A$.

Further, denote by $T_0$ the tubular $K$-family $T^{(\lambda)}_0$, $\lambda \in K$, by $T_\infty$ the tubular $K$-family $T^{(\lambda)}_\infty$, $\lambda \in K$, by $Q_0$ the class of indecomposable $A$-modules whose restrictions to $B_0$ have no non-zero direct summands from $P_0$ and $T_0$, and by $P_\infty$ the class of indecomposable $A$-modules whose restrictions to $B_\infty$ have no non-zero direct summands from $T_\infty$ and $Q_\infty$. Finally, denote by $\Delta$ the following quiver of type $D_\infty$:

\[
\xymatrix{ & \cdot \ar[rd] & \cdot \\
\cdot \ar[ru] & \cdot \ar[ld] & \cdot \ar[ld]
}
\]

**Theorem 6.1.** Let $A$ be a pg-critical algebra. Then

(i) $\Gamma_A = T_0 \cup T_\infty \cup Q_0 \cap P_\infty \cup T_\infty \cup Q_\infty$, where $Q_0 \cap P_\infty$ is a disjoint union of regular components and one non-regular component $C$, and the ordering from left to right indicates that there are non-zero maps (in mod $A$) only from any of these families to itself or to the families to its right.

(ii) The regular components in $Q_0 \cap P_\infty$ consist entirely of modules whose restrictions to $B_0$ have non-zero preinjective direct summands and whose restrictions to $B_\infty$ have non-zero preprojective direct summands.

(iii) The component $C$ has the following properties:

(a) $C$ contains all modules of $T^{(\infty)}_0$ and $T^{(\infty)}_\infty$.

(b) The stable part of $C$ is of the form $\mathbb{Z}A_\infty$.

(c) $C$ admits a full translation subquiver $R = (-N)\Delta$ which is closed under successors in $C$ and consists of modules whose restrictions to $B_0$ are direct sums of modules from $T^{(\infty)}_0$ and whose restrictions to $B_\infty$ are direct sums of preprojective modules.
(d) \( \mathcal{C} \) admits a full translation subquiver \( \mathcal{L} = \mathbb{N} \Delta^\text{op} \) which is closed under predecessors in \( \mathcal{C} \) and consists of modules whose restrictions to \( B_0 \) are direct sums of preinjective modules and whose restrictions to \( B_\infty \) are direct sums of modules from \( T^{(\infty)} \).

(e) \( \text{Hom}_A(\mathcal{L}, \mathcal{R}) = 0 \) and \( \text{Hom}_A(\mathcal{R}, \mathcal{L}) \neq 0 \).

Proof. Assume first that \( A = B[N, t] \). Then, in the above notation, \( A_A = P' \oplus S_A(a) \). Consider the APR-tilting \( A \)-module \( T = P' \oplus \tau_A S_A(a) \) and the algebra \( A' \) obtained from \( A \) by reversing the arrow \( c \rightarrow a \) to \( c \leftarrow a \). Then \( A' = \text{End}_A(T) \) is a pg-critical algebra of the form \( B_0[P_{B_0}(c)] \). Moreover, by [2], the functor \( \text{Hom}_A(T, -) : \text{mod} A \rightarrow \text{mod} A' \) induces an equivalence between the full subcategory of \( \text{mod} A \) formed by all modules without direct summands isomorphic to \( S_A(a) \) and the full subcategory of \( \text{mod} A' \) formed by all modules without direct summands isomorphic to \( S_A'(a) \). Moreover, \( \Gamma_A' \) is obtained from \( \Gamma_A \) by replacing

\[
\begin{array}{c}
S_A(b) \\
S_A(a) \rightarrow P_A(c) \rightarrow I_A(b) \\
\end{array}
\quad \text{and} \quad 
\begin{array}{c}
I_A(b) \\
I_A(a) \rightarrow I_A(c) \\
\end{array}
\]

by

\[
\begin{array}{c}
S_A'(b) \\
P_A'(c) \rightarrow P_A'(a) \rightarrow I_A'(b) \\
\end{array}
\quad \text{and} \quad 
\begin{array}{c}
I_A'(b) \\
I_A'(a) \rightarrow I_A'(c) \\
\end{array}
\]

respectively. Therefore, in order to prove the theorem, we may assume that \( A \) is of the form \( B[M] \). We identify the objects of \( \text{mod} B[M] = \text{mod} A \) with the triples \((V, X, \varphi)\), where \( V \) is a (finite-dimensional) vector space over \( K \), \( X \) an object of \( \text{mod} B \) and \( \varphi : V \rightarrow \text{Hom}_B(M, X) \) is a \( K \)-linear map. Then the \( B \)-modules \( X \) are the triples \((0, X, 0)\). Moreover, a \( B[M] \)-homomorphism \((V, X, \varphi) \rightarrow (W, Y, \psi)\) consists of a pair \((\alpha, f)\), where \( \alpha : V \rightarrow W \) is a \( K \)-linear map, \( f : X \rightarrow Y \) a \( B \)-homomorphism, and \( \psi = \text{Hom}_B(M, f) \varphi \). It is known that if \( 0 \rightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \rightarrow 0 \) is an Auslander–Reiten sequence in \( \text{mod} B \) then in \( \text{mod} B[M] \) we have an Auslander–Reiten sequence

\[
0 \rightarrow (|X|, X, 1_{|X|}) \xrightarrow{(|1_{|X|}, f)} (|X|, Y, |f|) \rightarrow Z \rightarrow 0,
\]

where \(|X| = \text{Hom}_B(M, X)\) and \(|f| = \text{Hom}_B(M, f)\) (see [15, (2.5)(6)]). Since in our situation \( \text{Hom}_B(M, P_0 \vee T_0) = 0 \) we conclude that \( \Gamma_A = P_0 \vee T_0 \vee Q_0 \).
Denote by $C_0$ the connected component of $\Gamma_A$ containing $P_A(a)$. Observe that the restriction of the vector space category $\text{Hom}_B(M, \text{mod } B)$ to the tube $T_0^{(\infty)}$ is the $K$-linear category of the partially ordered set given by the following full translation subquiver:

\[
\begin{align*}
Y_1 & \rightarrow Y_2 \rightarrow Y_3 \rightarrow \cdots \\
\downarrow & \quad \downarrow \quad \downarrow \\
M = X_0 & \rightarrow X_1 \rightarrow X_2 \rightarrow \cdots
\end{align*}
\]

of $T_0^{(\infty)}$ given by the corresponding two parallel rays of $T_0^{(\infty)}$. We get the indecomposable $A$-modules $X_i = (|X_i|, X_i, 1_{X_i})$, $Y_i = (|Y_i|, Y_i, 1_{Y_i})$, $i \geq 1$, where $|X_i| = |Y_i| = K$. Clearly, $P_A(a) = (|X_0|, X_0, 1_{X_0})$. Moreover, since $\text{Hom}_B(M, X_i)$ and $\text{Hom}_B(M, Y_i)$ are, for $i \geq j$, orthogonal objects of $\text{Hom}_B(M, \text{mod } B)$, we get (see [16, (2.4)]) the indecomposable $A$-modules $Z_{i,j} = (K, X_i \oplus Y_j, \Delta_{i,j})$, $i \geq j$, where $\Delta_{i,j} : \text{Hom}_B(M, X_i \oplus Y_j) = K^2$ are the diagonal maps. Applying the above formula for Auslander–Reiten sequences in $\text{mod } B[M]$ with the right terms being $B$-modules, and calculating the corresponding cokernels, we infer that $C_0$ has a full translation subquiver $R$ of the form

\[
\begin{align*}
Y_1 & \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow \cdots \\
\downarrow & \quad \downarrow \quad \downarrow \\
X_0 & \rightarrow P_A(a) \rightarrow Z_{12} \rightarrow Z_{22} \rightarrow \cdots \\
\downarrow & \quad \downarrow \quad \downarrow \\
X_1 & \rightarrow Z_{11} \rightarrow Z_{21} \rightarrow \cdots \\
\downarrow & \quad \downarrow \\
X_2 & \rightarrow Z_{12} \rightarrow Z_{22} \rightarrow \cdots \\
\downarrow & \\
X_3 & \rightarrow Z_{13} \rightarrow Z_{23} \rightarrow \cdots
\end{align*}
\]

formed by the modules $X_0$, $P_A(a)$, $X_i$, $X_i$ for $i \geq 1$, $Y_j$ for $j \geq 1$, and $Z_{i,j}$ for $1 \leq j \leq i$. Obviously, $R \cong (-N)\Delta$ and is closed under successors in $C_0$. Consider also the modules $U_i = \tau_B Y_i$, $i \geq 1$. Observe that, if $T_0^{(\infty)}$ has only two rays, then $U_{i+1} = X_i$ for $i \geq 0$. Calculating the Auslander–Reiten
sequences with right terms \( U_i, i \geq 1 \), we conclude that \( C_0 \) admits a full translation subquiver of the form

\[
\begin{array}{cccccccc}
Y_1 & \rightarrow & U_1 & \rightarrow & Y_2 & \rightarrow & U_2 & \rightarrow & Y_3 & \rightarrow & U_3 & \rightarrow & Y_4 & \rightarrow & \cdots \\
\end{array}
\]

Moreover, the Auslander–Reiten sequences in \( \text{mod}\, B \) with right terms in \( \mathcal{T}_0^{(\infty)} \) but different from the modules \( Y_i, U_i, i \geq 1 \), are Auslander–Reiten sequences in \( \text{mod}\, B[M] = \text{mod}\, A \). Hence all rays of \( \mathcal{T}_0^{(\infty)} \) except the one containing the modules \( Y_i, i \geq 1 \), form infinite sectional paths also in \( C_0 \). This also shows that \( C_0 \) contains all indecomposable projective \( A \)-modules which do not belong to \( P_0 \lor T_0 \). Further, the stable part of \( C_0 \) is then isomorphic to \( \mathbb{Z}A_{\infty} \). Finally, observe that, if \( L \) is an indecomposable \( A \)-module whose restriction to \( B_0 \) is a direct sum of modules from \( \mathcal{T}_0^{(\infty)} \), then \( L \) belongs to \( C_0 \). This is clear if \( L \) is a \( B \)-module. Suppose \( L \) is not a \( B \)-module. Then \( L = (V, Z, \varphi) \) for some vector space \( V \), \( Z \in \text{mod}\, B \), and \( \varphi : V \to \text{Hom}_B(M, Z) \). By our assumption, \( V \neq 0 \), \( \varphi \neq 0 \) and \( Z \) is a direct sum of modules of the form \( X_i, i \geq 0 \), and \( Y_j, j \geq 1 \). Then the structure of indecomposable representations of partially ordered sets of width 2 (see [16, (2.4)]) implies that \( L \) is one of the modules \( P_A(a) = X_0, X_i, Y_i, i \geq 1 \), or \( Z_{i,j}, 1 \leq j \leq i \), and we are done.

Applying dual arguments for the one-point coextension (extension) leading from \( B_\infty \) to \( A \), we infer that \( \Gamma_A = P_\infty \lor T_\infty \lor Q_\infty \). Therefore

\[
(\ast) \quad \Gamma_A = P_\infty \lor T_\infty \lor Q_\infty \cap P_\infty \lor T_\infty \lor Q_\infty.
\]

Further, the connected component, say \( C_\infty \), of \( \Gamma_A \) containing the module \( I_A(a') \) admits a full translation subquiver \( \mathcal{L} \) which is isomorphic to \( \mathbb{N}\Delta^{pp} \) and is closed under predecessors in \( C_\infty \). Moreover, \( C_\infty \) contains all indecomposable \( A \)-modules whose restrictions to \( B_\infty \) are direct sums of modules from \( \mathcal{T}_\infty^{(\infty)} \). All corays of \( \mathcal{T}_\infty^{(\infty)} \) except one (whose modules are distributed in \( \mathcal{L} \)) are infinite sectional paths in \( C_\infty \). Finally, \( C_\infty \) contains all indecomposable injective \( A \)-modules which do not belong to \( T_\infty \lor Q_\infty \), and the stable part of \( C_\infty \) is isomorphic to \( \mathbb{Z}A_{\infty} \).

It follows from the above considerations and the decomposition (\( \ast \)) that \( \mathcal{R} \) consists of modules whose restrictions to \( B_\infty \) are direct sums of preprojec-
tive $B_\infty$-modules and the restrictions to $B_0$ are direct sums of modules from $T_0^{(\infty)}$. Similarly, $L$ consists of modules whose restrictions to $B_0$ are direct sums of preinjective $B_0$-modules and the restrictions to $B_\infty$ are direct sums of modules from $T_\infty^{(\infty)}$. In particular, $\text{Hom}_A(L, R) = 0$ and $\text{Hom}_A(R, L) \neq 0$. Moreover, the regular components of $Q_0 \cap P_\infty$ consist entirely of modules whose restrictions to $B_0$ have non-zero preinjective direct summands and the restrictions to $B_\infty$ have non-zero preprojective direct summands. Consequently, in order to complete the proof, it is enough to show that $C_0 = C_\infty$. For this we have several cases to consider, depending on the shape of the quiver of $A$. We will not go through all these cases in detail, but rather discuss one example, which essentially describes all situations that occur in a complete analysis.

Consider the following algebra $A$ of type (16):

\[
\begin{array}{ccccccc}
 & x_1 & \rightarrow & x_2 & \rightarrow & \cdots & x_{k-1} & \rightarrow & x_k & \rightarrow & w \\
| & & & & & & & & & & \\
\downarrow & & & & & & & & & & \\
x_0 & \rightarrow & x_1 & \rightarrow & \cdots & \rightarrow & x_{k+1} & \rightarrow & x_{k+2} & \\
| & & & & & & & & & & \\
w' & & & & & & & & & & \\
\end{array}
\]

Then $A$ is of the form $A = B_0[M]$, where $B_0$ is the full subcategory created by all objects except $w$, and $M = P_{B_0}(x_k)/P_{B_0}(x_{k+2})$. Further, in the above notation, $Y_0 = P_{B_0}(x_k)/P_{B_0}(x_{k+1}) = P_A(x_k)/P_A(x_{k+1})$. Also, $A$ is of the form $A = [M']B_\infty$, where $B_\infty$ is the full subcategory created by all objects except $w'$. There are essentially three cases, depending on $k$.

In case $k > 2$, any of the simple modules $S_A(x_i)$, $i < j < 2$, lies in $T_0^{(\infty)}$ as well as in $T_\infty^{(\infty)}$, and thus in $C_0 \cap C_\infty$.

For $k \leq 2$, consider the module

\[ L = \tau_B Y_0 = P_{B_0}(x_{k-1})/P_{B_0}(x_k) = P_A(x_{k-1})/P_A(x_k). \]

Then $L$ lies in $T_0^{(\infty)}$, thus in $C_0$, and $\tau_A L = P_A(w)/P_A(x_{k+1})$.

In case $k = 2$, $\tau_A L = P_A(w)/P_A(x_3)$ lies in $T_\infty^{(\infty)}$, thus in $C_\infty$. So $C_0$ and $C_\infty$ are connected to each other, thus coincide.

In case $k = 1$, $\tau_A L = P_A(w)/P_A(x_2)$ is neither a module over $B_0$ nor over $B_\infty$. However, $\tau_A^2 L = P_A(x_1)/(P_A(x_0) \oplus P_A(x_{-1}))$ again lies in $T_0^{(\infty)}$ and $C_\infty$, so that $C_0$ and $C_\infty$ coincide. This completes the proof.

Recall that the component quiver $\Sigma_A$ of an algebra $A$ is defined as follows [18]: the vertices of $\Sigma_A$ are the connected components of $\Gamma_A$, and two components $D$ and $E$ are connected in $\Sigma_A$ by an arrow $D \rightarrow E$ if $\text{rad}^\infty(X, Y) \neq 0$ for some modules $X \in D$ and $Y \in E$. Here, $\text{rad}^\infty(X, Y)$ denotes the intersection of all finite powers $\text{rad}^j(X, Y)$ of the radical $\text{rad}(X, Y)$. 


From Theorem 6.1 we get the following information on the component quiver of a pg-critical algebra.

**Corollary 6.2.** Let \( A \) be a pg-critical algebra. Then, in the above notation, the following statements hold:

(i) \( P_0 \) is a unique source of \( \Sigma_A \) and there are arrows from \( P_0 \) to all remaining vertices of \( \Sigma_A \).

(ii) \( Q_\infty \) is a unique sink of \( \Sigma_A \) and there are arrows from all remaining vertices of \( \Sigma_A \) to \( Q_\infty \).

(iii) For each \( \lambda \in K \), there are arrows in \( \Sigma_A \) from \( T(\lambda)_0 \) to all components of \( (Q_0 \cap P_\infty) \vee T_\infty \vee Q_\infty \), and \( P_0 \to T_0^{(\lambda)} \) is a unique arrow with target \( T_0^{(\lambda)} \).

(iv) For each \( \lambda \in K \), there are arrows in \( \Sigma_A \) from all components of \( P_0 \vee T_0 \vee (Q_0 \cap P_\infty) \) to \( T_\infty^{(\lambda)} \), and \( T_\infty^{(\lambda)} \to Q_\infty \) is a unique arrow with source in \( T_\infty^{(\lambda)} \).

(v) For each regular component \( D \) in \( Q_0 \cap P_\infty \), there are arrows \( C \to D \) and \( D \to C \) in \( \Sigma_A \).

(vi) \( \Sigma_A \) admits a loop \( C \to C \).

**Proof.** We know that there are only finitely many indecomposable \( B_0 \)-modules (respectively, \( B_\infty \)-modules) whose restrictions to \( C_0 \) (respectively, \( C_\infty \)) are zero. The statements (i)–(iv) follow from Theorem 4.1 and the facts that \( (T_0^{(\lambda)})_{\lambda \in P_+ K} \) separates \( P_0 \) from \( T_0 \) in \( \text{ind} B_0 \), and \( (T_\infty^{(\lambda)})_{\lambda \in P_+ K} \) separates \( P_\infty \) from \( Q_\infty \) in \( \text{ind} B_\infty \). Moreover, (vi) follows from \( \text{Hom}_A(R, L) \neq 0 \) and the fact that \( C \) has no oriented cycles. Take now an arbitrary regular component \( D \) in \( Q_0 \cap P_\infty \). For (v) it is enough to show that \( \text{Hom}_A(P_\lambda(a), X) \neq 0 \) and \( \text{Hom}_A(Y, I_\lambda(a)) \neq 0 \) for some modules \( X \) and \( Y \) in \( D \). Suppose that \( \text{Hom}_A(P_\lambda(a), X) = 0 \) for all \( X \) in \( D \). Then \( D \) consists of \( B_0 \)-modules which, by Theorem 6.1, must be preinjective. But then \( D \) is not regular, a contradiction. We get a similar contradiction assuming \( \text{Hom}_A(Y, I_\lambda(a)) = 0 \) for all \( y \in D \). This finishes the proof.

It follows from the decomposition \( \Gamma_A = P_0 \vee T_0 \vee Q_0 \cap P_\infty \vee T_\infty \vee Q_\infty \) that \( \text{Hom}_A(D(a), P_0 \vee T_0) = 0 \) and \( \text{Hom}_A(T_\infty \vee Q_\infty, A) = 0 \). Hence \( \text{pd}_A X \leq 1 \) for all modules \( X \) in \( P_0 \vee T_0 \) and \( \text{id}_A Y \leq 1 \) for all modules \( Y \) in \( T_\infty \vee Q_\infty \) (see [15, (2.4)]). We also have the following information on the homological behaviour of the non-regular component \( C \) of \( Q_0 \cap P_\infty \).

**Corollary 6.3.** Let \( A \) be a pg-critical algebra. Then

(i) \( \text{pd}_A X \leq 1 \) for all modules \( X \) in \( R \).

(ii) \( \text{id}_A Y \leq 1 \) for all modules \( Y \) in \( L \).

(iii) There are infinitely many indecomposable modules \( Z \) in \( C \) with \( \text{pd}_A Z = 2 \) and \( \text{id}_A Z = 2 \).
**Proof.** (i) From the structure of $\mathcal{C}$ described in Theorem 6.1 we know that if $X$ belongs to $\mathcal{R}$ then the restriction of $\tau_A X$ to $B_0$ is a direct sum of modules from $\mathcal{T}_0^{(\infty)}$. On the other hand, the restriction of any indecomposable injective $A$-module to $B_0$ is a direct sum of indecomposable preinjective $B_0$-modules. Hence $\text{Hom}_A(D(A), \tau_A X) = 0$, and so $\text{pd}_A X \leq 1$.

The proof of (ii) is dual.

(iii) It follows from the proof of Theorem 6.1 that $\text{Hom}_A(D(A), \tau_A Y_i) \neq 0$ for all $i \geq 1$. Hence $\text{pd}_A Y_i = 2$ for all $i \geq 1$. On the other hand, there is an infinite sequence $1 < i_1 < i_2 < \ldots$ such that, for each $s \geq 1$, there is a sectional path $Y_{i_s} \rightarrow L_{i_{s-1}} \rightarrow \ldots \rightarrow R$, with $R$ an indecomposable projective $B_0$-module (if $A = B[M]$ and $M$ is directing) or the radical of $P_A(a)$. Moreover, $L_{i_s} = \tau_A^{-1} Y_{i_{s-1}}$ for any $s \geq 1$. Hence $\text{Hom}(\tau_A^{-1} Y_{i_{s-1}}, A) \neq 0$ for $s \geq 1$, as the composition of irreducible maps forming a sectional path is non-zero. Consequently, $\text{id}_A Y_{i_{s-1}} = 2$. We then have an infinite family $Y_{i_{s-1}}$, $s \geq 1$, of modules in $\mathcal{C}$ with both projective and injective dimension equal to 2.

We shall now present an example illustrating the above considerations.

**Example 6.4.** Let $A$ be the algebra given by the bound quiver

\[
\begin{array}{c}
1 \rightarrow 7 \\
\downarrow \\
3 \rightarrow 6
\end{array}
\]

Let $B_0 = C_0$ be the convex subcategory of $A$ given by all objects of $A$ except 7. Then $B_0$ is a concealed algebra of type $\tilde{D}_5$ and $A$ is a one-point extension $B_0[M]$ of $B_0$ by an indecomposable regular $B_0$-module lying in the following stable tube $\mathcal{T}_0^{(\infty)}$ of rank 3 in $\Gamma_{B_0}$:

\[
\begin{array}{ccccccc}
0 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]

where we represent a module by its dimension vector in $K_0(B_0)$ and the vertical lines have to be identified in order to obtain a tube. Hence $A$ is a pg-critical algebra. Observe also that $A = [M']B_\infty$, where $B_\infty$ is the
convex subcategory of $A$ given by all vertices except 6, and $M'$ is given by the dimension vector $\begin{bmatrix} 10 \\ 11 \\ 10 \end{bmatrix}$ in $K_0(B_\infty)$. Further, $B_\infty$ is a tubular one-point coextension of the concealed convex subcategory given by the objects 1, 2, 3, 4, and 7, and its unique tube $T_\infty$ with 3 corays is of the form

$$
\begin{bmatrix}
0 & 1 & 0 \\
0 & 0 & 0 \\
2 & 1 & 3
\end{bmatrix} = I_{B_\infty}(5)
$$

which again the modules are represented by their dimension vectors in $K_0(B_\infty)$, and the vertical lines have to be identified in order to obtain a (coray) tube. Then

$$
\Gamma_A = P_0 \vee T_0 \vee Q_0 \vee P_\infty \vee T_\infty \vee Q_\infty,$$

where $P_0$ is the preprojective component of type $\tilde{D}_5$, $T_0$ (respectively, $T_\infty$) consists of the two stable tubes of rank 2 and the homogeneous tubes, $Q_\infty$ is the preinjective component of type $\tilde{D}_4$, and the non-regular component $C$ of $Q_0 \vee P_\infty$ is of the form
We end the paper with an example showing the possible shapes of regular components of the Auslander–Reiten quiver of a pg-critical algebra.

**Example 6.5.** Let $A$ be the pg-critical algebra given by the bound quiver

```
1 -- 4 -- 7
|       |
|       |
2 -- 3 -- 5 -- 6 -- 8 -- 9
```

Then $A$ admits an automorphism group $G$ of order 2 generated by the twist $g$ such that $g(1) = 1$, $g(2) = 3$, $g(3) = 2$, $g(4) = 4$, $g(5) = 6$, $g(6) = 5$, $g(7) = 7$, $g(8) = 9$, and $g(9) = 8$. Assume that the characteristic of $K$ is not equal to 2. Then the twisted group algebra $B = A[G]$ (in the sense of [13]) is given by the following bound quiver:

```
\[ \begin{array}{c}
\vdots \\
. \\
. \\
. \\
. \\
. \\
. \\
\end{array} \]
```

Hence, $B$ is a string (special biserial) algebra and the regular components in $\Gamma_B$ are of the form $ZA_\infty^\infty$ and $ZA_\infty^\infty/\tau$ (see [6]). Then, applying [11] or [13], we find that the regular components of $\Gamma_A$ are of the form $ZA_\infty^\infty$, $ZD_\infty$, $ZA_\infty^\infty/\tau$ and $ZA_\infty^\infty/\tau^2$. In fact, there are infinitely many regular components in $\Gamma_A$ of each of these types.

**REFERENCES**


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