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EXTENSIONLESS MODULES OVER TAME HEREDITARY ALGEBRAS

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Introduction. This paper is a sequel to [15]. However, the emphasis here is on modules over finite-dimensional tame hereditary algebras, R. Finite-dimensional R-modules M with the property that $\mathrm{Ext}^1_R(M,M)=0$ have established their importance in the theory of R-modules; see for instance [3]. Various adjectives have been attached to such modules; see [4], [9], and [11]. The last two references also deal with categories of sheaves. The neutral term *extensionless* is used here because our modules are not assumed to be finite-dimensional.

Let R be a finite-dimensional hereditary algebra. In studying R-modules there is no loss of generality in assuming that R is basic, i.e. $R/\mathrm{rad}\,R$ is a finite direct sum of division rings; see [2, Corollary 2.6]. In that case R gives rise to a quadratic form q. If q is semidefinite but not positive definite, R is said to be *tame*. See Chapter 8 of [2], Chapter 14 of [20], or [19] for a treatise on this class of algebras. The category of R-modules in this paper is encapsulated in Theorem 0.1.

THEOREM 0.1. *Let R be a two-sided indecomposable tame finite-dimensional hereditary algebra over an algebraically closed field K. Then the category of R-modules is equivalent to the category of K-representations of an oriented extended Coxeter–Dynkin diagram without oriented cycles.*

Proof. This follows from Section 4.3 of [8], Corollary 14.7 and Theorem 14.15 of [20]. ∎

Let R be the tame finite-dimensional hereditary algebra

$$\begin{pmatrix} K & K^2 \\ 0 & K \end{pmatrix},$$

where K^2 is the two-dimensional K-vector space and the multiplication is

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given by

$$\begin{pmatrix} d & u \\ 0 & c \end{pmatrix}\begin{pmatrix} f & v \\ 0 & e \end{pmatrix} = \begin{pmatrix} df & dv+ue \\ 0 & ce \end{pmatrix}.$$

We follow [2] and [20] and call such an algebra R a *Kronecker algebra*. Theorem 0.1 allows us to view an R-module, called a *Kronecker module*, as a quadruple $V = (V_1, V_2, \phi, \psi)$, where (V_1, V_2) is a pair of K-vector spaces and (ϕ, ψ) is a pair of K-linear maps from V_1 to V_2. All this and more can be found in Example 1.5 and on p. 282 of [20]. See also Section 8.7 of [2]. N. Aronszajn became interested in *systems* as a result of his work in perturbation theory. His joint work with Fixman has now been incorporated into the more general theory of R-modules. However, many results on Kronecker modules do not yet have analogues for R-modules. See for instance [12] and Section 3 of the present paper.

In Section 1 we define our terms and state preliminary results needed in the rest of the paper. The main result of Section 2 describes the countable-dimensional torsion-free extensionless R-modules of finite rank. In Section 3 we show that if K has the same cardinality as the field of complex numbers then any rank two torsion-free extensionless Kronecker module, V, may be considered a $K[X]$-module, where $K[X]$ is the polynomial ring in the variable X. It then follows from [16, Theorem 3.4] that V is a direct sum of rank one torsion-free Kronecker modules. At the end of Section 3 we briefly contrast the results in this paper with related results for modules over Dedekind domains.

NOTATION. The following notation prevails throughout the paper.

- R will stand for the algebra with all the properties mentioned in Theorem 0.1. When *module* is unappended we mean R-module, while *vector space* means K-vector space. Moreover, K is algebraically closed.
- $M^n = M \oplus \ldots \oplus M$ (n copies)
- Ext will stand for Ext_R^1 and the $\mathrm{Hom}_R(M, N)$ will often do without R.
- $\dim U$ is the dimension of the K-vector space U.
- $\mathrm{card}\, S$ is the cardinality of the set S.

1. Preliminary results. We begin by recalling a torsion theory on R-modules due to Aronszajn and Fixman in [1] when R is the Kronecker algebra and to Ringel in [19] for arbitrary R.

The non-zero finite-dimensional indecomposable R-modules are of three types: *pre-injective*, *preprojective*, and *regular*. A finite-dimensional module is *torsion* if it has no preprojective direct summand. Given an arbitrary R-module, M, the *torsion submodule* tM of M is the submodule of M generated by all finite-dimensional torsion submodules of M. We say that M is *torsion* if $tM = M$ and *torsion-free* if $tM = 0$. If M is neither torsion nor torsion-free, M is *mixed*. It is convenient to allow the zero module to float among all three types.

A torsion module is *regular* if every finite-dimensional indecomposable direct summand is regular. It is shown in [19] that a torsion regular module M decomposes as

$$M = \bigoplus_{p\in K\cup\{\infty\}} M(p), \tag{1}$$

where each $M(p)$ is analogous to the p-primary component of a torsion abelian group. Consequently, we refer to $M(p)$ as the p-primary component of M. There is also the following analogue of a Prüfer group.

Let M_1 be a non-zero torsion regular module with no non-zero proper regular submodule. There is a sequence of torsion regular modules M_n of regular length n with $M_n \subseteq M_{n+1}$. The module

$$M = \bigcup_{n=1}^{\infty} M_n \tag{2}$$

is called a *Prüfer module* in [1] and [19]. It is studied in Section 8 of [1] under the notation II_θ^∞, where II_θ^n corresponds to the $K[X]$-module $K[X]/(X-\theta)^n K[X]$. See also p. 10 of [20].

PROPOSITION 1.1 ([19]). *Submodules and direct products of torsion-free R-modules are torsion-free. Extensions of torsion-free R-modules by torsion-free modules are also torsion-free.*

For use in Section 3 we note that a Kronecker module $V = (V_1, V_2, \phi, \psi)$ is torsion-free if $a\phi + b\psi : V_1 \to V_2$ is one-to-one for every non-zero element (a, b) in K^2.

Let M be a torsion-free R-module. Let X be a subset of M. Let $\mathcal{F} = \{L \subseteq M : X \subseteq L$ and M/L is torsion-free$\}$. Then $N = \bigcap_{L\in\mathcal{F}} L$ is the smallest submodule of M containing X with M/N torsion-free. We call N the *torsion-closure* of X in M and denote it by $\operatorname{tc}_M X$ or $\operatorname{tc} X$ if M is understood.

A submodule M_1 of M is said to be *torsion-closed in* M if $\operatorname{tc}_M M_1 = M_1$. Let

$$\mathcal{C} = \{B \subseteq M : \operatorname{tc}_M B = M\}. \tag{3}$$

Since $M \in \mathcal{C}$, $\mathcal{C}$ is non-empty. In [6] Fixman calls $B_0 \in C$ a *basis of M with respect to generation* if no proper subset of B_0 is in $\mathcal{C}$. The *rank* of M, denoted by $\operatorname{rank} M$, is by definition $\operatorname{card} B_0$. This is well-defined by Theorem 1.2(a).

THEOREM 1.2. *Let M be a torsion-free R-module.*

(a) *If* B_1 *and* B_2 *are two bases of* M *with respect to generation then* $\operatorname{card} B_1 = \operatorname{card} B_2$.

(b) *If* $B \in \mathcal{C}$ *then* B *contains a basis of* M *with respect to generation.*

(c) *If* $S \subseteq M$ *and* $\operatorname{tc}_M S \neq \operatorname{tc}_M S'$ *for every proper subset* S' *of* S, *then* S *is contained in a basis of* M *with respect to generation.*

(d) *If* M_1 *and* M_2 *are torsion-free* R*-modules and* M *is an extension of* M_1 *by* M_2 *then* $\operatorname{rank} M = \operatorname{rank} M_1 + \operatorname{rank} M_2$.

Proof. When R is the Kronecker algebra, see [6]. It was noted in [17] that the results in Sections 4 and 5 of [19] can be used to adapt the proofs in [6] for arbitrary algebras in Theorem 0.1. ■

COROLLARY 1.3. *Let* L *and* M *be torsion-free* R*-modules. Then a homomorphism* $\phi : L \to M$ *is determined on any basis of* L *with respect to generation. In particular*, *if* L *is of rank one then* ϕ *is determined by its value on any non-zero element of* L.

We need more facts on rank one torsion-free R-modules. For this purpose we recall from [6] that a *height function* is a function

$$h : K \cup \{\infty\} \to \{\infty, 0, 1, 2, \ldots\}.$$

Two height functions h_1 and h_2 are *equivalent* if the following conditions are satisfied:

(i) $h_1(t) = \infty$ if and only if $h_2(t) = \infty$.

(ii) The set $\Delta = \{t \in K \cup \{\infty\} : h_1(t) \neq h_2(t)\}$ is finite.

(iii) If h_1, hence h_2, does not assume ∞, then $\sum_{t\in\Delta} h_1(t) = \sum_{t\in\Delta} h_2(t)$.

Equivalence classes of height functions characterize isomorphism classes of rank one torsion-free R-modules; see Theorem 3.3 of [6] and Section 6.4 of [19]. The principal ideal domain version of this result is due to R. Baer; see [7, Section 85].

Let h be a height function. We shall denote by $M(h)$ the corresponding rank one torsion-free R-module. So we have the correspondence

$$h \mapsto M(h). \tag{4}$$

Conversely, every torsion-free rank one R-module is isomorphic to $M(h)$ for some height function h. We illustrate (4) below in the case of Kronecker modules and $K[X]$-modules.

EXAMPLE 1.4 (Section 3 of [6] and Section 85 of [7]). (a) We want $M(h) = (V_1, V_2, \phi, \psi)$. Let $K(X)$ be the field of rational functions. Then V_2 is the K-subspace of $K(X)$ consisting of those functions that have a pole at each $\theta \in K \cup \{\infty\}$ of order $\leq h(\theta)$, while V_1 is the subspace of V_2 consisting of those functions in V_2 with a pole at ∞ of order $< h(\infty)$. So V_2 has as basis the set

$$\{(X-\theta)^{-k} : 0 \le k < h(\theta)+1,\ \theta \ne \infty\} \cup \{X^k : 0 < k < h(\infty)+1\}.$$

The map $\phi : V_1 \to V_2$ is the inclusion map and $\psi : V_1 \to V_2$ is given by $\psi(v_1) = Xv_1$.

(b) Since K is algebraically closed, there is a one-to-one correspondence between the monic irreducible polynomials in $K[X]$ and the elements of K. A *height function* h in this context is a function

$$h : K \to \{\infty, 0, 1, 2, \ldots\}.$$

Only (i) and (ii) are needed in the definition of *equivalence* of these height functions. The corresponding $K[X]$-module $M(h)$ is the $K[X]$-submodule of $K(X)$ generated by V_2 in (a).

DEFINITION 1.5. The module $M(h)$ is said to be *idempotent* if h is equivalent to a height function whose image is in $\{0, \infty\}$.

The following notation will be used for any height function h:

$$F(h) = \{p \in K \cup \{\infty\} : h(p) \ne \infty\},$$
$$P(h) = \{p \in K \cup \{\infty\} : h(p) \ne 0\},$$
$$I(h) = \{p \in K \cup \{\infty\} : h(p) = \infty\}.$$

We have the following lemma on rank one torsion-free R-modules.

LEMMA 1.6. (a) *Suppose that $M(h)$ is infinite-dimensional. Then it is idempotent if and only if $I(h)$ is non-empty and $F(h) \cap P(h)$ is finite.*

(b) *$M(h)$ is infinite-dimensional if and only if $I(h)$ is non-empty or $P(h)$ is infinite. In that case* $\dim M(h) = \aleph_0 \operatorname{card} P(h)$.

(c) *$M(h)$ is countably infinite-dimensional if and only if $P(h)$ is countably infinite or $I(h)$ is non-empty and countable.*

Proof. (a) follows from the definitions, while (b) and (c) follow from Theorem 6.5 of [19]. ∎

PROPOSITION 1.7 ([15, Proposition 3.4]). *Let h_1 and h_2 be height functions. Suppose* $\dim M(h_1) + \dim M(h_2)$ *is infinite*, $F(h_1) \subseteq F(h_2)$, *and* $F(h_1) \cap P(h_2)$ *is finite. Then* $\operatorname{Ext}^1_R(M(h_2), M(h_1)) = 0$.

Lemma 1.6(a) and Proposition 1.7 imply that idempotent rank one torsion-free R-modules are extensionless. Proposition 1.7 also implies that if $M(h_2)$ is finite-dimensional and $M(h_1)$ is infinite-dimensional, then $\operatorname{Ext}(M(h_2), M(h_1)) = 0$.

PROPOSITION 1.8 ([19, Section 2.2]). *Let M be a torsion-free R-module.*

(a) *If M has no non-zero finite-dimensional direct summand then* $\operatorname{Ext}^1_R(N, M) = 0$ *for every torsion-free finite-dimensional R-module N.*

(b) *If* N *is a non-zero finite-dimensional torsion-free* R*-module and* $\operatorname{Hom}_R(M,N) \neq 0$, *then* M *has a non-zero finite-dimensional direct summand.*

PROPOSITION 1.9 ([15, Lemma 3.1]). *Let* M *be a non-zero torsion module with only one non-zero primary part at* $t \in K \cup \{\infty\}$ *and let* h *be a height function. Then* $\operatorname{Ext}(M, M(h)) = 0$ *if and only if* $h(t) = \infty$.

We shall also need results on the dimensions of some vector spaces of R-homomorphisms.

LEMMA 1.10. *Let* M *and* N *be torsion-free modules with* M *of finite rank.*

(a) *If* N *is infinite-dimensional*, *then* $\dim \operatorname{Hom}_R(M,N) \le \dim N$.

(b) *If* N *is finite-dimensional*, *then* $\operatorname{Hom}_R(M,N)$ *is finite-dimensional.*

Proof. (a) Let C and H be respective bases of the K-vector spaces N and $\operatorname{Hom}_R(M,N)$. Let $\mathcal{P}_f(C)$ be the set of finite subsets of C. Let $B = \{b_1, \ldots, b_r\}$ be a basis of M with respect to generation. The proof is by induction on r.

Suppose $r = 1$. Let $\alpha \in H$ and let $\alpha(b_1) = k_1c_1 + \ldots + k_nc_n$, where $k_i \in K$ and $c_i \in C$. We now define a map $\psi : H \to \mathcal{P}_f(C)$ by setting $\psi(\alpha) = \{c_1, \ldots, c_n\}$. By Corollary 1.3, α is determined by $\alpha(b_1)$. So ψ is a finite-to-one map. Hence $\operatorname{card} H \le \operatorname{card} C = \dim N$.

We now assume the result for all torsion-free modules of rank $< r$. Let $M_1 = \operatorname{tc}_M\{b_1, \ldots, b_{r-1}\}$ and $M_2 = M/M_1$. From the exact sequence

$$0 \to M_1 \to M \to M_2 \to 0 \tag{5}$$

we get the exact sequence

$$0 \to \operatorname{Hom}(M_2,N) \to \operatorname{Hom}(M,N) \to \operatorname{Hom}(M_1,N). \tag{6}$$

That $\dim \operatorname{Hom}(M,N) \le \dim N$ follows from (6), the induction hypothesis, and the fact that N is infinite-dimensional.

(b) Since M is of finite rank we can write it as $M = L_1 \oplus L_2$, where L_1 is finite-dimensional and L_2 has no non-zero finite-dimensional direct summand. By Proposition 1.8(b) we obtain $\operatorname{Hom}(L_2,N) = 0$. So $\operatorname{Hom}(M,N) \cong \operatorname{Hom}(L_1,N)$. ■

LEMMA 1.11. *The endomorphism algebra of a Prüfer* R*-module* M *is an uncountable-dimensional vector space over* K.

Proof. Let $E = \operatorname{End} M$. The modules M_n in (2) play the role of $\mathbb{Z}/p^n\mathbb{Z}$ in the proof that $\operatorname{End} \mathbb{Z}_p^\infty$ is a complete discrete valuation ring; see Section 43.3 of [7] or Remark 3, Section 4.4 of [19]. So E is a complete discrete valuation ring. Let I be its maximal ideal. Since $K \subseteq E$ and $K \subseteq E/I$, both E and E/I have the same characteristic. Therefore, by [21] for instance,

$E \cong (E/I)[[X]]$, the power series ring over E/I in the variable X, which is an uncountable-dimensional vector space over K. ∎

2. Countable-dimensional extensionless modules of finite rank. Throughout this section R stands for the algebra in Theorem 0.1. Theorem 2.3, Proposition 2.5, and Theorem 2.8 are the main results of the section. However, the other results are needed to prove them. The notation in (4) will be used.

If $M(h)$ is infinite-dimensional, then by Section 6.3 of [19], it contains a simple projective module P and

$$M(h)/P \cong \bigoplus_{p \in K \cup \{\infty\}} M(p), \tag{7}$$

where each $M(p)$ is a p-primary module of regular length $h(p)$. In particular, $M(p)$ is a Prüfer module when $h(p) = \infty$.

PROPOSITION 2.1. *Let $M(h_1)$ and $M(h_2)$ be countable-dimensional torsion-free rank one R-modules. Suppose that for some t in $K \cup \{\infty\}$, $h_2(t) = \infty$ while $h_1(t) < \infty$. Then $\mathrm{Ext}^1_R(M(h_2), M(h_1))$ is an uncountable-dimensional vector space over K.*

Proof. Let $M(h_3)$ be the rank one submodule of $M(h_2)$, where $h_3(t) = \infty$ and $h_3(p) = 0$ if $p \neq t$. Since R is hereditary it is enough to show that $\mathrm{Ext}(M(h_3), M(h_1))$ is uncountable-dimensional.

Let $M(h_4)$ be a rank one module that contains $M(h_1)$, where $h_4(t) = \infty$ and $h_4(p) = h_1(p)$ when $p \neq t$. For brevity, we write M_i for $M(h_i)$. From the exact sequence

$$0 \to M_1 \to M_4 \to M_4/M_1 \to 0$$

we get the exact sequence

$$\mathrm{Hom}(M_3, M_4) \to \mathrm{Hom}(M_3, M_4/M_1) \to \mathrm{Ext}(M_3, M_1) \to \mathrm{Ext}(M_3, M_4). \tag{8}$$

From the definitions of h_3 and h_4 we see that M_3 is isomorphic to a submodule of M_4. Since $F(h_4) \subseteq F(h_3)$ and $F(h_4) \cap P(h_3)$ is empty, $\mathrm{Ext}(M_3, M_4) = 0$, by Proposition 1.7. From $h_1(t) < \infty$, $h_4(t) = \infty$ and (7) we deduce that M_4/M_1 contains a Prüfer module, $M(t)$, as a direct summand. Hence from Lemma 1.11 we infer that $\mathrm{Hom}(M_3, M_4/M_1)$ is uncountable-dimensional. Since M_1 is countable-dimensional it follows from Lemma 1.6(c) that M_4 is also countable-dimensional. By Lemma 1.10, $\mathrm{Hom}(M_3, M_4)$ is countable-dimensional. When we take all these to (8) we deduce that $\mathrm{Ext}(M_3, M_1)$ is uncountable-dimensional. ∎

PROPOSITION 2.2. *Suppose that $M(h_1)$ and $M(h_2)$ are countable-dimensional torsion-free rank one R-modules and $F(h_1) \cap P(h_2)$ is an infinite* *set. Then* $\mathrm{Ext}^1_R(M(h_2), M(h_1))$ *is an uncountable-dimensional vector space over* K.

Proof. Denote $M(h_i)$ by M_i. With P as in (7) we have

$$0 \to P \to M_2 \to M_2/P \to 0. \tag{9}$$

From (9) we get the exact sequence

$$\mathrm{Hom}(P, M_1) \to \mathrm{Ext}(M_2/P, M_1) \to \mathrm{Ext}(M_2, M_1) \to \mathrm{Ext}(P, M_1). \tag{10}$$

The last entry is zero because P is a projective module. By Lemma 1.10(a), $\mathrm{Hom}(P, M_1)$ is countable-dimensional. From (7) we deduce that M_2/P contains $\bigoplus_{p\in F(h_1)\cap P(h_2)} M(p)$, where $M(p)$ is a nonzero p-primary module. Since $F(h_1) \cap P(h_2)$ is infinite, it follows from (7) and Proposition 1.9 that $\mathrm{Ext}(M_2/P, M_1)$ is uncountable-dimensional. So (10) implies that the module $\mathrm{Ext}(M_2, M_1)$ is isomorphic to the uncountable-dimensional vector space $\mathrm{Ext}(M_2, M_1)/\mathrm{Hom}(P, M_1)$. ∎

Propositions 1.7, 2.1, and 2.2 together give the next theorem.

THEOREM 2.3. *Let* $M(h_1)$ *and* $M(h_2)$ *be countable-dimensional torsion-free rank one* R*-modules with at least one of them infinite-dimensional. Then* $\mathrm{Ext}^1_R(M(h_2), M(h_1))$ *is either zero or an uncountable-dimensional* K*-vector space. It is zero if and only if* $F(h_1) \subseteq F(h_2)$ *and* $F(h_1) \cap P(h_2)$ *is finite.*

COROLLARY 2.4. *Suppose that* M *is a non-zero finite-dimensional torsion-free* R*-module and* N *is a countably infinite-dimensional torsion-free* R*-module of finite rank. Then* $\mathrm{Ext}^1_R(N, M)$ *is uncountable-dimensional. If* N *is also extensionless then* N *does not have a non-zero finite-dimensional direct summand.*

Proof. Since M is finite-dimensional we can write $M = \bigoplus_{i=1}^n M(h_i)$ for some positive integer n, and h_i height functions with $F(h_i) = K \cup \{\infty\}$, for $i = 1, \ldots, n$. So if $\mathrm{rank}\, N = 1$, the corollary follows from Theorem 2.3.

Suppose $\mathrm{rank}\, N > 1$. Then we have an exact sequence

$$0 \to N_1 \to N \to N_2 \to 0, \tag{11}$$

where both N_1 and N_2 are non-zero and torsion-free with at least one of them infinite-dimensional. From (11) we get the exact sequence

$$\mathrm{Hom}(N_1, M) \to \mathrm{Ext}(N_2, M) \to \mathrm{Ext}(N, M) \to \mathrm{Ext}(N_1, M) \to 0. \tag{12}$$

By induction either $\mathrm{Ext}(N_2, M)$ or $\mathrm{Ext}(N_1, M)$ is uncountable-dimensional. Since $\mathrm{Hom}(N_1, M)$ is finite-dimensional by Lemma 1.10(b), it follows from (12) that $\mathrm{Ext}(N, M)$ is uncountable-dimensional.

Suppose $N = M \oplus N_1$, $M \neq 0$ and finite-dimensional. Then N_1 is infinite-dimensional. So $\mathrm{Ext}(N_1, M) \neq 0$. This contradicts $\mathrm{Ext}(N, N) = 0$. ∎

In light of Corollary 2.4 we point out that the investigation of extensionless R-modules with a non-zero finite-dimensional direct summand requires different techniques from those in this paper; see [18].

PROPOSITION 2.5. *Let M and N be countable-dimensional torsion-free modules of finite rank with N infinite-dimensional. Then* $\mathrm{Ext}^1_R(N,M)$ *is either zero or an uncountable-dimensional K-vector space.*

Proof. Suppose that $\mathrm{Ext}(N,M)\neq 0$. We show by induction on $r=\operatorname{rank} M+\operatorname{rank} N$ that $\mathrm{Ext}(N,M)$ is uncountable-dimensional. Theorem 2.3 starts the induction. We may, therefore, assume that $r\geq 3$. Either N or M has rank ≥ 2.

Suppose that $\operatorname{rank} M=1$. Then $\operatorname{rank} N\geq 2$ and we have the exact sequences (11) and (12). If M is finite-dimensional then Corollary 2.4 gives us the required conclusion. So we assume that M is infinite-dimensional. Since $\mathrm{Ext}(N,M)\neq 0$ it follows from (12) that at least one of $\mathrm{Ext}(N_1,M)$ and $\mathrm{Ext}(N_2,M)$ is non-zero. If $\mathrm{Ext}(N_i,M)\neq 0$, $i=1$ or 2, then by Proposition 1.8(a), N_i is infinite-dimensional. Induction then shows that $\mathrm{Ext}(N_i,M)$ is uncountable-dimensional. Since $\mathrm{Hom}(N_1,M)$ is countable-dimensional by Lemma 1.10, (12) implies that $\mathrm{Ext}(N,M)$ is uncountable-dimensional when $\operatorname{rank} M=1$.

Suppose $\operatorname{rank} M\geq 2$. Then we get from (5) the exact sequence

$$\mathrm{Hom}(N,M_2)\to\mathrm{Ext}(N,M_1)\to\mathrm{Ext}(N,M)\to\mathrm{Ext}(N,M_2)\to 0. \tag{13}$$

Since $\operatorname{rank} M_i+\operatorname{rank} N<\operatorname{rank} M+\operatorname{rank} N$, $i=1,2$, induction and (13) imply that $\mathrm{Ext}(N,M)$ is uncountable-dimensional. ■

Remark 2.6. If N is finite-dimensional, Proposition 2.5 may fail when M is infinite-dimensional and has a non-zero finite-dimensional direct summand. For instance, let N and L be finite-dimensional torsion-free R-modules with $\mathrm{Ext}^1_R(N,L)\neq 0$. Let $M(h)$ be an infinite-dimensional torsion-free rank one R-module. Then by Proposition 1.8(a), $\mathrm{Ext}^1_R(N,L\oplus M(h))\cong\mathrm{Ext}^1_R(N,L)$, which is a non-zero finite-dimensional vector space.

LEMMA 2.7. *Let M be a countably infinite-dimensional torsion-free extensionless R-module of finite* rank ≥ 2. *Then every submodule M_1 of M with $M_2=M/M_1$ torsion-free is an infinite-dimensional extensionless direct summand of M.*

Proof. We have the exact sequence

$$0\to M_1\to M\to M_2\to 0. \tag{14}$$

Proposition 1.8(b) and Corollary 2.4 imply that M_2 is infinite-dimensional. The sequence (14) gives the exact sequence

$$\mathrm{Hom}(M,M_2)\to\mathrm{Ext}(M,M_1)\to\mathrm{Ext}(M,M)\to\mathrm{Ext}(M,M_2)\to 0. \tag{15}$$

Since $\mathrm{Hom}(M, M_2)$ is countable-dimensional and $\mathrm{Ext}(M, M) = 0$ we deduce from Proposition 2.5 that $\mathrm{Ext}(M, M_1) = 0$. Hence $\mathrm{Ext}(M_1, M_1) = 0$.

We now want to show that (14) splits. From (14) we get the exact sequence

$$\mathrm{Hom}(M_1, M_1) \to \mathrm{Ext}(M_2, M_1) \to \mathrm{Ext}(M, M_1). \tag{16}$$

Since $\mathrm{Hom}(M_1, M_1)$ is countable-dimensional, $\mathrm{Ext}(M, M_1) = 0$, and M_2 is infinite-dimensional, Proposition 2.5 and (16) imply that $\mathrm{Ext}(M_2, M_1) = 0$ as required. ∎

THEOREM 2.8. *A countably infinite-dimensional torsion-free R-module M, of finite rank, n, is extensionless if and only if there is a non-zero idempotent height function h such that M is isomorphic to $M(h)^n$.*

Proof. Proposition 1.7 shows that if h is idempotent then $M(h)^n$ is extensionless.

Suppose M is extensionless. Using (5), Lemma 2.7, and induction on the rank of M we deduce that there are height functions h_j, $j = 1, \ldots, n$ with

$$M \cong M(h_1) \oplus \ldots \oplus M(h_n). \tag{17}$$

Since $\mathrm{Ext}(M, M) = 0$, we have $\mathrm{Ext}(M(h_j), M(h_i)) = 0$ for every pair (i, j). Using the notation in Lemma 1.6, Theorem 2.3 implies that, for every pair (i, j), $F(h_i) = F(h_j)$, $F(h_i) \cap P(h_j)$ is a finite set, and $I(h_i) = I(h_j)$.

Since each $M(h_i)$ is infinite-dimensional, by Corollary 2.4, it also follows from Theorem 2.3 that each $I(h_j)$ is non-empty. So by Lemma 1.6(a), the height functions in (17) are non-zero, idempotent, and equivalent. ∎

3. Uncountable-dimensional torsion-free extensionless rank two Kronecker modules.

In this section R denotes the Kronecker algebra and R-modules will be written as $V = (V_1, V_2, \phi, \psi)$ (see the introduction). We shall need the results in [14]. The only properties of the field of complex numbers used in [14] are that it is algebraically closed and has the cardinality of the continuum. So in this section we shall assume that not only is K algebraically closed but also that K, hence R, has the cardinality of the continuum. In that case we have Theorem 3.1, which is a more precise version of Theorem 2.3 for Kronecker modules. The notation in (4) is still in force.

THEOREM 3.1 ([14]). *Suppose the Kronecker algebra R has the cardinality of the continuum. Let $M(h_1)$ and $M(h_2)$ be torsion-free rank one R-modules, at least one of which is infinite-dimensional. Then*

$$\dim \mathrm{Ext}(M(h_2), M(h_1)) = 2^{\mathrm{card}\, F(h_1) \cap P(h_2)}$$

if $\mathrm{card}\, F(h_1) \cap P(h_2)$ is infinite. Otherwise, $\dim \mathrm{Ext}(M(h_2), M(h_1))$ is either $2^{\aleph_0}$ or 0. It is zero if and only if, in addition, $F(h_1) \subseteq F(h_2)$.

Next comes a refinement of Corollary 2.4.

PROPOSITION 3.2. *Suppose the Kronecker algebra R has the cardinality of the continuum. Let M be a non-zero finite-dimensional torsion-free R-module and let N be an infinite-dimensional*, *torsion-free R-module of finite rank. Then* $\dim \operatorname{Ext}(N, M) = 2^{\dim N}$. *If N is also extensionless then N does not have a non-zero finite-dimensional direct summand.*

Proof. If N is of rank one and infinite-dimensional then the result follows from Theorem 3.1 and Lemma 1.6(b). Suppose $\operatorname{rank} N > 1$. In that case we proceed as in the proof of Corollary 2.4 beginning with (11). ∎

COROLLARY 3.3. *Suppose the Kronecker algebra R has the cardinality of the continuum. Let M be an infinite-dimensional extensionless torsion-free R-module of finite rank. Then every non-zero torsion-closed submodule of M is infinite-dimensional.*

Proof. Let M_1 be a non-zero torsion-closed submodule of M and let $M_2 = M/M_1$. So we have the sequences (14) and (15). If M_1 were finite-dimensional, then $\dim \operatorname{Ext}(M, M_1)$ would be $2^{\dim M}$ by Proposition 3.2. Since $\operatorname{Ext}(M, M) = 0$, and $\dim \operatorname{Hom}(M, M_2) \le \dim M_2 < 2^{\dim M}$, (15) implies that $\dim M_2 \ge 2^{\dim M}$, a contradiction. ∎

Before we continue, we have to recall some relationships between $K[X]$-modules and some Kronecker modules.

Let $V = (V_1, V_2, \phi, \psi)$. Let $e = (e_1, e_2)$ be a non-zero element of K^2. We say that V is *non-singular via* e if $e_1\phi + e_2\psi : V_1 \to V_2$ is an isomorphism. As shown in [1, p. 281] we may in that case consider V as a $K[X]$-module. Conversely, any $K[X]$-module M gives rise to a Kronecker module $(M, M, \mathrm{id}, \beta)$, where id is the identity map on M and $\beta(m) = Xm$. For a fixed $e \in K^2$, the subcategory of Kronecker modules non-singular via e is an exact full subcategory equivalent to the category of $K[X]$-modules. We shall use the following facts on non-singular Kronecker modules.

PROPOSITION 3.4. (a) ([6, Corollary 3.5]). *A rank one torsion-free Kronecker module is non-singular if and only if its height function assumes the value* ∞.

(b) *If* $h_1(\theta) = h_2(\theta) = \infty$ *then any extension of* $M(h_1)$ *by* $M(h_2)$ *is non-singular.*

(c) *A $K[X]$-module M is torsion-free if and only if the corresponding Kronecker module (M, M) is torsion-free. Moreover*, $\operatorname{rank} M = \operatorname{rank}(M, M)$.

The main result of this section ultimately rests on Theorem 3.5. It was proved in [16] for Dedekind domains that are not local rings.

THEOREM 3.5 ([16, Theorem 3.4). *Let M be a torsion-free $F[X]$-module of arbitrary finite rank n*, *and F a field of arbitrary cardinality and not* *necessarily algebraically closed. Then M is extensionless if and only if there is an idempotent height function h such that M is isomorphic to $M(h)^n$.*

We recall that a height function h has domain K or $K \cup \{\infty\}$ depending on whether we are dealing with $K[X]$-modules or R-modules. When h is the zero function the corresponding R-module, $M(h)$, is a finite-dimensional projective R-module, while the corresponding $K[X]$-module is $K[X]$. This explains why, in order to get an infinite-dimensional R-module from an idempotent height function, *non-zero idempotent height function* is stipulated in Theorems 2.8 and 3.6. No non-zero assumption is needed in Theorem 3.5. However, no problem arises in applying Theorem 3.5 in the proof of Theorem 3.6 because when $K[X]$ is considered as a Kronecker module, the way described before Proposition 3.4, its corresponding height function h assumes 0 on K and $h(\infty) = \infty$. So what was a zero height function when dealing with $K[X]$-modules becomes a non-zero height function when ∞ is included in its domain.

THEOREM 3.6. *Suppose the Kronecker algebra R has the cardinality of the continuum. Let $V = (V_1, V_2, \phi, \psi)$ be a torsion-free rank two infinite-dimensional R-module. Then V is extensionless if and only if there is a non-zero idempotent height function h such that V is isomorphic to $M(h)^2$.*

Proof. Sufficiency follows from Proposition 1.7.

Suppose that V is extensionless. Since $\operatorname{rank} V = 2$ we have an exact sequence

$$0 \to M(h_1) \to V \to M(h_2) \to 0, \tag{18}$$

where $M(h_1)$ and $M(h_2)$ are torsion-free rank one modules.

Since $\operatorname{Ext}(V, V) = 0$ and R is hereditary we have $\operatorname{Ext}(M(h_1), M(h_2)) = 0$.

Suppose that for some $\theta \in K \cup \{\infty\}$ we have $h_1(\theta) = \infty$.

Then by Theorem 3.1, we get $h_2(\theta) = \infty$. So by Proposition 3.4(b), V may be considered a $K[X]$-module. Hence Theorem 3.5 implies that V is isomorphic to $M(h)^2$ for some idempotent height function h. If h is the zero height function in the category of $K[X]$-modules, then $M(h)$ is isomorphic to $K[X]$ as a $K[X]$-module. However, when $M(h)$ is viewed as an R-module, $h(\infty) = \infty$. Therefore h is a non-zero idempotent height function as required.

Suppose that

$$F(h_1) = K \cup \{\infty\}. \tag{19}$$

We shall show that (19) leads to a contradiction. Using $\operatorname{Ext}(M(h_1), M(h_2)) = 0$, it follows from (19), Lemma 1.6(b) and Theorem 3.1 that $\dim M(h_2) \geq \dim M(h_1)$. So (18) gives $\dim M(h_2) = \dim V$. With $M_i = M(h_i)$, (18) gives the exact sequence

(20) $\mathrm{Hom}(M_2, M_2) \to \mathrm{Ext}(M_2, M_1) \to \mathrm{Ext}(M_2, V) \to \mathrm{Ext}(M_2, M_2)$.

From (19), Lemma 1.6(b), Corollary 3.3 and Theorem 3.1 we get

$$\dim \mathrm{Ext}(M(h_2), M(h_1)) = 2^{\dim M(h_2)} = 2^{\dim V}.$$

By Lemma 1.10(a), $\dim \mathrm{Hom}(M_2, M_2) \le \dim M_2$. Therefore, (20) yields that $\dim \mathrm{Ext}(M_2, V) = 2^{\dim M_2} = 2^{\dim V}$. The exact sequence

(21) $\mathrm{Hom}(V, V) \to \mathrm{Hom}(M_1, V) \to \mathrm{Ext}(M_2, V) \to \mathrm{Ext}(V, V)$

is obtained from (18). Feeding (21) with $\dim \mathrm{Hom}(M_1, V) \le \dim V$ and $\mathrm{Ext}(V, V) = 0$ leads to the contradiction $2^{\dim V} \le \dim V$. Therefore (19) is untenable. So for some $\theta \in K \cup \{\infty\}$, $h_1(\theta) = \infty$. We quote the paragraph before (19) to complete the proof. ■

We conclude the paper with some remarks on the cardinality assumptions in the paper.

REMARKS 3.7. (a) At first sight it seems that *two* in Theorem 3.6 can be replaced by n as in Theorem 3.5. However, the reduction to $K[X]$-modules in the proof of Theorem 3.6 required Theorem 3.1, which in turn required the explicit structure of rank one torsion-free Kronecker modules.

(b) If M is an idempotent torsion-free rank one R-module then any finite direct sum of copies of M is extensionless—even when M is not countable-dimensional. This follows from Proposition 1.7. So only one direction in Theorems 2.3 and 2.8 requires a countability hypothesis.

In [15] results analogous to those in this paper were obtained for countable Dedekind domains and countable R. Here we have shifted the countability hypothesis from the algebra to the module. The format of the proofs is the same. However, the example below shows that the analogue of Theorem 2.8 is false for Dedekind domains.

EXAMPLE 3.8. Let D be an incomplete discrete valuation ring whose completion $\widehat{D}$ is a module of finite rank $r > 1$ over D, for instance the *bad* noetherian ring in [13, p. 207]. In [16] it was observed that $\widehat{D}$ is an indecomposable extensionless D-module of finite rank. We now show that $\widehat{D}$ is a countably generated D-module: First, F, the quotient field of D, is generated over D by $\{1/p^n : n = 1, 2, \ldots\}$, where p is a generator of the maximal ideal of D. Let $\widehat{F}$ be the quotient field of $\widehat{D}$. Since it is finitely generated over F, it follows that $\widehat{F}$ is countably generated over D. Since D is noetherian and $\widehat{D} \subseteq \widehat{F}$, it follows that $\widehat{D}$ is also countably generated over D. Therefore we have found a countably generated extensionless D-module that is not a direct sum of rank one D-modules.

The problem of describing extensionless modules is a special case of the problem of classifying pairs of modules (M_2, M_1) with $\mathrm{Ext}(M_2, M_1) = 0$. The latter problem is solved in [22] for torsion-free abelian groups of finite rank. As observed in Theorem 2.16 of [5], Pontryagin's criterion, [7, Theorem 19.1], extends the result in [22] to countable rank. As noted in [5], $\mathbb{Z}$ can be replaced by any countable principal ideal domain. Theorem 3.5 is a sample of the results in [16] on extensionless modules that do not require the Dedekind domain to be countable. We refer to [10] for the structure of modules of countable rank over complete discrete valuation rings.

Let G be a countable torsion-free abelian group and let A be an uncountable torsion-free abelian group. There is a characterization in [5], under various assumptions each consistent with ZFC, of pairs (A, G) with $\mathrm{Ext}(A, G) = 0$. Such a theorem is still inaccessible for the algebras in Theorem 0.1 even if we restrict to Kronecker algebras.

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REFERENCES

[1] N. Aronszajn and U. Fixman, *Algebraic spectral problems*, Studia Math. 30 (1968), 273–338.
[2] M. Auslander, I. Reiten and S. Smalø, *Representation Theory of Artin Algebras*, Cambridge Univ. Press, 1995.
[3] K. Bongartz, *Tilted algebras*, in: Lecture Notes in Math. 903, Springer, Berlin, 1982, 26–38.
[4] F. V. Colhoe, *A note on preinjective partial tilting modules*, in: CMS Conf. Proc. 14, Amer. Math. Soc., 1993, 109–115.
[5] P. Eklof and M. Huber, *On the rank of Ext*, Math. Z. 174 (1980), 159–185.
[6] U. Fixman, *On algebraic equivalence between pairs of linear transformations*, Trans. Amer. Math. Soc. 113 (1964), 424–453.
[7] L. Fuchs, *Infinite Abelian Groups*, Vols. I, II, Academic Press, New York, 1970, 1973.
[8] P. Gabriel, *Auslander–Reiten sequences and representation-finite algebras*, in: Lecture Notes in Math. 831, Springer, Berlin, 1980, 1–71.
[9] W. Geigle and H. Lenzing, *Perpendicular categories with applications to representations and sheaves*, J. Algebra 144 (1991), 273–343.
[10] I. Kaplansky, *Modules over Dedekind rings and valuation rings*, Trans. Amer. Math. Soc. 72 (1952), 327–340.
[11] H. Lenzing and H. Meltzer, *Sheaves on a weighted projective line of genus one*, in: CMS Conf. Proc. 14, Amer. Math. Soc., 1993, 313–337.
[12] H. Lenzing and F. Okoh, *The separability of direct products of modules over a canonical algebra*, ibid., 339–352.
[13] M. Nagata, *Local Rings*, Interscience, New York, 1962.
[14] F. Okoh, *Indecomposable rank two modules over some artinian ring*, J. London Math. Soc. 22 (1980), 411–422.

[15] F. Okoh, *Extensionless modules of finite rank over countable Dedekind domains*, Arch. Math. (Basel) 59 (1992), 232–238.

[16] —, *The rank of a completion of a Dedekind domain*, Comm. Algebra 21 (1993), 4561–4574.

[17] —, *Bouquets of Baer modules*, J. Pure Appl. Algebra 93 (1994), 297–310.

[18] —, *Infinite-dimensional partial tilting modules*, in preparation.

[19] C. M. Ringel, *Infinite-dimensional representations of finite-dimensional hereditary algebras*, Sympos. Mat. Inst. Alta Mat. 23 (1979), 321–412.

[20] D. Simson, *Linear Representations of Partially Ordered Sets and Vector Space Categories*, Gordon and Breach, 1992.

[21] O. Teichmüller, *Diskret bewertete perfekte Körper mit unvollkommenen Restklassenkörper*, J. Reine Angew. Math. 176 (1936), 141–152.

[22] R. B. Warfield, Jr., *Extensions of torsion-free Abelian groups of finite rank*, Arch. Math. (Basel) 23 (1972), 145–150.

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