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A VARIANT OF HÖRMANDER'S CONDITION FOR SINGULAR INTEGRALS

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Consider the operator T defined for $f \in L^2(\mathbb{R}) \cap L^p(\mathbb{R})$ by $\widehat{Tf}(\xi) = \chi_{[-1,1]}(\xi)\widehat{f}(\xi)$. Of course, it is well known that this operator is bounded from $L^p(\mathbb{R})$ to $L^p(\mathbb{R})$ if $1 < p < \infty$. In fact, T can be constructed from multiplication operators and the Hilbert transform, so the boundedness of T on L^p is just a consequence of the L^p boundedness of the Hilbert transform. It is curious that although the L^p boundedness of T follows from results on singular integrals, it does not follow directly, since the kernel of T, $(\sin x)/x$, has a derivative which does not decay quickly enough at infinity to apply the usual theory (see Davis and Chang [3]). Our aim in this paper is to show a result on singular integrals which in fact does include operators defined with kernels such as $(\sin x)/x$.

Our result will be a variant of a classical result of Calderón and Zygmund [1]. Actually, the statement will resemble that of a theorem found in Stein's [5] treatment of the Calderón and Zygmund theory. In order to state our results succinctly, we first introduce a little terminology.

We say that a function $p \geq 0$ on $\mathbb{R}^q$ satisfies a *reverse-L^∞ inequality* (abbreviated as "p satisfies RL^∞") if there is a constant C such that for every cube $Q \subseteq \mathbb{R}^q$ centered at the origin we have $0 < \|p|_Q\|_\infty \leq C p_Q$. Here and throughout, $p|_Q$ denotes the restriction of the function p to Q and p_Q denotes the average of the function p on Q. With these notations and conventions, we can now state our results.

THEOREM. *Let $K \in L^2(\mathbb{R}^n)$. Suppose that there exists a constant B such that*

(a) $\|\widehat{K}\|_\infty \leq B$,

(b) *There exist functions $A_1(x), \ldots, A_m(x)$ and $\varphi_1(y), \ldots, \varphi_m(y)$ such that each $\varphi_i(y)$ is bounded, $|\det[\varphi_j(y_i)]|^2$ satisfies RL^∞ on $\mathbb{R}^{nm}$, and*

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(c) *We have*

$$\int\limits_{|x|>2|y|} \Big|K(x-y) - \sum_{i=1}^{m} A_i(x)\varphi_i(y)\Big|\,dx \le B \quad \text{for all } |y|>0.$$

For $1<p<\infty$ *and* $f\in L^1\cap L^p$ *set*

$$Tf(x) = \int\limits_{\mathbb{R}^n} K(x-y)f(y)\,dy.$$

Then there exists a constant A_p *so that* $\|Tf\|_p \le A_p\|f\|_p$. *The constant* A_p *depends only on* p, B, n, m *and the constant in the* RL^∞ *condition for the* φ_i, *but not on the* L^2 *norm of* K.

The condition (b) is an alternation of what is known as Hörmander's condition. In fact, if we take $m=1$, $A_1(x)=K(x)$, $\varphi_1(y)\equiv 1$, then this is exactly Hörmander's condition and the theorem is Hörmander's [4] version of the result of Calderón and Zygmund. We will show in Section 3 how to apply this theorem to the kernel $(\sin x)/x$.

The proof of the theorem follows the strategy laid out by Calderón and Zygmund; L^2 boundedness of the operator T is a trivial consequence of hypothesis (a). Using (a) and (b) we will then show T is weak type $(1,1)$. Finally, interpolation and duality give the result for $1<p<\infty$. The difficult task is the weak type $(1,1)$ estimate. This is where the proof of our result differs from the proof of Calderón and Zygmund. To estimate the distribution of $Tf(x)$, that is, to estimate $|\{x : |Tf(x)|>\lambda\}|$ for $\lambda>0$, Calderón and Zygmund write $f=g+b$, where g is a function such that $|g|\le\lambda$, and b is a function which is supported on a union of cubes. On a typical such cube Q, $b=f-f_Q$. To motivate the present theorem, we think of $f_Q=g|_Q$ as the projection of f onto the space of constant functions. We, however, will consider a similar decomposition of f as $f=g+b$, but so that for each cube Q, $g|_Q$ is a projection of f onto the linear span of $\varphi_1,\ldots,\varphi_m$. With the proper estimates of these projections, we can proceed as in Calderón and Zygmund and obtain the necessary distribution estimates for Tg and Tb so that we can show T is weak $(1,1)$.

In Section 1 we show the necessary estimates on projections of functions. In Section 2 we prove the theorem by filling in the details of the strategy we have just outlined. Finally, in Section 3 we give some examples.

1. Projections. When we talk of the projection of an L^1-function f onto a finite-dimensional subspace, we mean that there is a function g in the subspace such that $\int f\overline{h}\,dx = \int g\overline{h}\,dx$ for every h in the subspace. If we fix a cube Q and consider the projection of $f|_Q$ onto the space of functions which are constant on Q, then this projection is just f_Q, which obviously satisfies $|f_Q| \le C|f|_Q$ with $C = 1$. We want to show this estimate persists in some situations when the space of constant functions is replaced by the linear span of a finite set of functions.

LEMMA. *Suppose $\varphi_1, \ldots, \varphi_m$ is a set of bounded functions on $\mathbb{R}^n$ such that $|\det[\varphi_j(x_i)]|^2$ satisfies RL^∞ on $\mathbb{R}^{nm}$. Then there exists a constant C with the property that whenever $Q \subseteq \mathbb{R}^n$ is a cube centered at the origin, $f \in L^1(Q)$ and $A_1\varphi_1(x) + \ldots + A_m\varphi_m(x)$ is the projection of $f|_Q$ onto the span of $\varphi_1, \ldots, \varphi_m$ in $L^1(Q)$, then we have $\sup_{x\in Q} |A_1\varphi_1(x) + \ldots + A_m\varphi_m(x)| \le C|f|_Q$. The constant C depends only on n, m and the constant in the RL^∞ condition for the φ_j.*

Proof. Fix Q. To simplify our notation we write

$$\langle p, q\rangle = \frac{1}{|Q|}\int_Q p(x)\overline{q(x)}\,dx$$

and let G be the $m \times m$ matrix defined by $G_{ij} = \langle \varphi_i, \varphi_j\rangle$. Then, with this notation, the $\overline{A}_i$ are solutions of the system $G[\overline{A}_1, \ldots, \overline{A}_m]^{\mathrm{T}} = [\langle \varphi_1, f\rangle \ldots \langle \varphi_m, f\rangle]^{\mathrm{T}}$. For $j = 1, \ldots, m$ let G^j be the matrix which is the same as G except that the elements in the jth column of G^j are $\langle \varphi_i, f\rangle$, $i = 1, \ldots, m$, instead of the $\langle \varphi_i, \varphi_j\rangle$ which occur in G. By a formula in Courant and Hilbert [2, p. 108],

$$\det G = \frac{1}{|Q|^m}\int_Q \ldots \int_Q |\det(\varphi_j(x_i))|^2\,dx_1 \ldots dx_m. \tag{1}$$

The RL^∞ condition implies that the last expression is non-zero, so by Cramer's rule,

$$\overline{A}_j = \frac{\det G^j}{\det G}.$$

Thus, it will suffice to show

$$\Big|\sum_{j=1}^m [\det G^j]\overline{\varphi_j(x)}\Big| \le C|f|_Q|\det G|, \qquad x \in Q. \tag{2}$$

For $j = 1, \ldots, m$, let $H^j(u, x)$ be the matrix which is the same as G except that the jth column consists of the elements $\varphi_i(u)\overline{\varphi_j(x)}$. Then

$$\begin{aligned}\Big|\sum_{j=1}^m [\det G^j]\overline{\varphi_j(x)}\Big| &= \Big|\sum_{j=1}^m \frac{1}{|Q|}\int_Q \overline{f(u)}[\det H^j(u, x)]\,du\Big| \\ &\le \frac{1}{|Q|}\int_Q |f(u)|\,\Big|\sum_{j=1}^m \det[H^j(u, x)]\Big|\,du.\end{aligned}$$

Thus, (2) follows if we show

$$\Big|\sum_{j=1}^{m}\det[H^j(u,x)]\Big| \le C|\det G|, \qquad x,u\in Q. \tag{3}$$

Now let $M = M(x_1,\dots,x_{m-1})$ be the $m\times m-1$ matrix defined by $M_{ij} = \varphi_i(x_j)$, $j = 1,\dots,m-1$, and let $\Phi(u)$ be the $m\times 1$ matrix defined by $\Phi(u) = [\varphi_1(u)\dots\varphi_m(u)]^{\mathrm{T}}$. Then we claim

$$\frac{1}{|Q|^{m-1}}\int_Q\dots\int_Q \det[M|\Phi(u)]\det[\overline{M}|\overline{\Phi(x)}]\,dx_1\dots dx_{m-1} = (m-1)!\sum_{j=1}^{m}\det H^j(u,x). \tag{4}$$

Before proving (4), we show how (3) follows from it. The left hand side of (4) is dominated by

$$\begin{aligned}
&\frac{1}{2}\sup\{|\det[M|\Phi(u)]|^2 : x_1,\dots,x_{m-1},u\in Q\}\\
&\qquad + \frac{1}{2}\sup\{|\det[M|\Phi(v)]|^2 : x_1,\dots,x_{m-1},v\in Q\}\\
&\quad = \frac{C}{|Q|^m}\int_Q\dots\int_Q |\det[M|\Phi(u)]|^2\,dx_1\dots dx_{m-1}\,du\\
&\qquad + \frac{C}{|Q|^m}\int_Q\dots\int_Q |\det[M|\Phi(v)]|^2\,dx_1\dots d_{m-1}\,dv \le 2C|\det G|,
\end{aligned}$$

where the first inequality follows from our RL^∞ assumption, and the last inequality follows from (1). Thus, we are finished if we show (4).

The proof of (4) is essentially the same as the proof of the formula in Courant and Hilbert that we just used. The integrand on the right side of (4) can be rewritten as

$$\begin{aligned}
&\det[M|\Phi(u)]\det[\overline{M}|\overline{\Phi(x)}]\\
&\quad = \det([M|\Phi(u)][\overline{M}|\overline{\Phi(x)}]^{\mathrm{T}}) = \det\Big[\Big(\sum_{l=1}^{m-1}\varphi_i(x_l)\overline{\varphi_j(x_l)} + \varphi_i(u)\overline{\varphi_j(x)}\Big)_{ij}\Big]\\
&\quad = \sum_{j=1}^{m}\sum_{\sigma\in S_{m-1}}\det A_{\sigma j}
\end{aligned}$$

where

$$A_{\sigma j} = \begin{bmatrix} \varphi_1(x_{\sigma(1)})\overline{\varphi_1(x_{\sigma(1)})} & \ldots & \varphi_1(u)\overline{\varphi_j(x)} & \ldots & \varphi_1(x_{\sigma(m-1)})\overline{\varphi_m(x_{\sigma(m-1)})} \\ \vdots & & \vdots & & \vdots \\ \varphi_m(x_{\sigma(1)})\overline{\varphi_1(x_{\sigma(1)})} & \ldots & \varphi_m(u)\overline{\varphi_j(x)} & \ldots & \varphi_m(x_{\sigma(m-1)})\overline{\varphi_m(x_{\sigma(m-1)})} \end{bmatrix}.$$

The first two equalities above are clear; to obtain the last we expand the determinant to obtain m^m determinants. However, many are zero, and what remains is exactly the $m!$ determinants written out in the last term. Integrating both sides with respect to the variables $x_1, \ldots, x_{m-1}$ gives (4) and finishes the proof of the lemma. ∎

2. The proof of the theorem. As indicated previously, we will essentially follow Stein's treatment of the Calderón and Zygmund theory. At points when our proof is similar to that found in Stein, we will be rather brief but will be careful to elaborate at those points where our proof differs.

For $f \in L^2(\mathbb{R}^n)$ we have $\widehat{Tf}(\xi) = \widehat{K}(\xi)\widehat{f}(\zeta)$, so that by Plancherel's theorem $\|Tf\|_2 \le B\|f\|_2$.

To show that T is weak type $(1,1)$, fix $f \in L^1$ and $\lambda > 0$; we wish to show that $|\{x : Tf(x) > \lambda\}| \le C\lambda^{-1}\|f\|_1$ with C independent of f or λ. We can assume that f is real-valued.

We now consider a Calderón–Zygmund decomposition of $\mathbb{R}^n$ (see Stein [5, p. 17]): there exists a closed set F and an open set Ω such that

(i) $\mathbb{R}^n = F \cup \Omega$,

(ii) $|f(x)| \le \lambda$ a.e. on F,

(iii) Ω is the union of cubes, $\Omega = \bigcup_j Q_j$, whose interiors are disjoint, and so that for each Q_j, $\lambda \le |f|_{Q_j} \le 2^n\lambda$.

For each such Q_j we let y_j denote the center of Q_j, and then let $g_j(x)$ denote the projection of $f|_{Q_j}$ onto the span of $\varphi_1(\cdot - y_j), \ldots, \varphi_m(\cdot - y_j)$.

Now set

$$g(x) = \begin{cases} f(x) & \text{if } x \in F, \\ g_j(x) & \text{if } x \in \Omega \text{ and } x \in Q_j \end{cases}$$

and let $b(x) = f(x) - g(x)$. If $x \in F$, then by (ii), $|g(x)| \le \lambda$, while if $x \in Q_j$, $|g(x)| = |g_j(x)| \le C|f|_{Q_j} \le C\lambda$ by the lemma. Thus, $|g(x)| \le C\lambda$ for all x and therefore

$$(5)\qquad \left|\left\{x : |Tg(x)| > \frac{\lambda}{2}\right\}\right| \le \frac{C}{\lambda^2}\int_{\mathbb{R}^n} |g(x)|^2\,dx \le \frac{C}{\lambda}\|g\|_1$$

$$= \frac{C}{\lambda}\int_F |f(x)|\,dx + \frac{C}{\lambda}\sum_j \int_{Q_j} |g_j(x)|\,dx$$

$$\leq \frac{C}{\lambda}\int_F |f(x)|\,dx + \frac{C}{\lambda}\sum_j |Q_j|\,|f|_{Q_j} = \frac{C}{\lambda}\|f\|_1.$$

The function $b(x)$ is supported on $\bigcup_j Q_j$; on each Q_j we set $b_j(x) = f(x) - g_j(x)$, so that $b(x) = \sum_j b_j(x)$. For each j, let Q_j^* denote the cube which has the same center as Q_j but with sidelength $2\sqrt{n}$ times as long.

Then, since $\int_{Q_j} \varphi_i(y-y_j)b_j(y)\,dy = 0$,

$$\begin{aligned} Tb_j(x) &= \int_{Q_j} K(x-y)b_j(y)\,dy \\ &= \int_{Q_j} \Big[K(x-y) - \sum_{i=1}^m A_i(x-y_j)\varphi_i(y-y_j)\Big] b_j(y)\,dy \end{aligned}$$

and so

$$\begin{aligned} &\int_{\mathbb{R}^n\setminus\bigcup Q_j^*} |Tb(x)|\,dx \\ &\leq \sum_j \int_{x\notin Q_j^*}\int_{Q_j} \Big|K(x-y) - \sum_{i=1}^m A_i(x-y_j)\varphi_i(y-y_j)\Big|\,|b_j(y)|\,dy\,dx \\ &= \sum_j \int_{x\notin Q_j^*}\int_{Q_j-y_j} \Big|K(x-(y+y_j)) - \sum_{i=1}^m A_i(x-y_j)\varphi_i(y)\Big|\,|b_j(y+y_j)|\,dy\,dx \\ &= \sum_j \int_{Q_j-y_j}\int_{x\notin Q_j^*} \Big|K(x-(y+y_j)) - \sum_{i=1}^m A_i(x-y_j)\varphi_i(y)\Big|\,dx\,|b_j(y+y_j)|\,dy. \end{aligned}$$

Make the change of variables $x' = x - y_j$ in the inner integral in the last expression, and note that if $y \in Q_j - y_j$, $x' \notin Q_j^* - y_j$, then $|x'| > 2|y|$, so that the inner integral in the last expression is dominated by

$$\int_{|x'|>2|y|} \Big|K(x'-y) - \sum_{i=1}^m A_i(x')\varphi_i(y)\Big|\,dx' \leq B.$$

Combining this estimate with the previous ones yields

$$\begin{aligned} \int_{\mathbb{R}^n\setminus\bigcup Q_j^*} |Tb(x)|\,dx &\leq B\sum_j \|b_j\|_1 \\ &= B\sum_j |Q_j|\,|b_j|_{Q_j} \leq C\sum_j |Q_j|\,|f|_{Q_j} \leq C\|f\|_1. \end{aligned}$$

Thus

$$\Big|\Big\{x\in\mathbb{R}^n\setminus\bigcup Q_j^* : |Tb(x)| > \frac{\lambda}{2}\Big\}\Big| \leq \frac{C}{\lambda}\|f\|_1. \tag{6}$$

But also, using (iii),

$$\Big|\bigcup_j Q_j^*\Big| \le C\sum_j |Q_j| \le C\sum_j \frac{1}{\lambda}|Q_j|\,|f|_{Q_j} = \frac{C}{\lambda}\|f\|_1. \tag{7}$$

Combining estimates (6) and (7) gives the estimate

$$\left|\left\{x \in \mathbb{R}^n : |Tb(x)| > \frac{\lambda}{2}\right\}\right| \le \frac{C}{\lambda}\|f\|_1. \tag{8}$$

Combining (5) and (8) gives the desired weak type $(1,1)$ inequality.

Since T is of weak type $(1,1)$ and bounded on L^2, we can apply the Marcinkiewicz interpolation theorem (see Stein [5, p. 21]) to conclude that T is bounded on L^p if $1 < p < 2$.

Now note that if $K(x)$ satisfies the hypothesis of the theorem, then so does $K(-x)$; thus, the duality argument found in Stein [5, p. 33] applies to show that T is bounded on L^p for $2 < p < \infty$. ■

3. Examples

Example 1. As noted earlier, if we take $m = 1$, $A_1(x) = K(x)$, and $\varphi_1(x) \equiv 1$, we get Hörmander's version of the Calderón–Zygmund Theorem [4].

Example 2. Let T be the operator defined by $\widehat{Tf}(\xi) = s(\xi)\widehat{f}(\xi)$, where $s(\xi)$ is a step function. The kernel corresponding to this operator is in $L^2(\mathbb{R})$ and has the form

$$K(x) = \sum_{j=1}^{m} \frac{c_j e^{i\lambda_j x}}{x},$$

where λ_j is real for every j and $\sum_{j=1}^m c_j = 0$. By construction, condition (a) of the theorem is satisfied. Set $A_j(x) = (c_j e^{i\lambda_j x})/x$ and $\varphi_j(y) = e^{-i\lambda_j y}$ to get the integral estimate in (c). We need to show that $D(y_1, \ldots, y_m) = |\det[\exp(-i\lambda_j y_k)]|^2$ satisfies the RL^∞ condition. To see this, use Taylor's theorem to write $D = p + r$, where $p \not\equiv 0$ is a homogeneous polynomial of degree l and $r(y_1, \ldots, y_m) = O(|(y_1, \ldots, y_m)|^{l+1})$. For $a > 0$, set $Q_a = [-a, a]^m$; then the homogeneity of p implies that $p_{Q_a} = a^l p_{Q_1}$. Let M be a constant such that $\|r|_{Q_a}\|_\infty \le Ma^{l+1}$, and $\|p|_{Q_a}\|_\infty \le Ma^l$ for all $0 < a \le 1$. Since $D > 0$ and $D_{Q_a} > 0$ for all $a > 0$, we see that for small enough a, $p_{Q_a} > 0$, hence $p_{Q_1} > 0$. Then if $a < (2M)^{-1}p_{Q_1}$, we have $p_{Q_a} \le 2D_{Q_q}$, so

$$\|D|_{Q_a}\|_\infty \le 2Ma^l \le \frac{2M}{p_{Q_1}} p_{Q_a} \le \frac{2M}{p_{Q_1}} D_{Q_a}.$$

The fact that the RL^∞ condition holds for larger a is an easy consequence of the following properties of D_{Q_a}:

(i) $$D_{Q_a} > 0 \quad \text{for all } a > 0,$$

(ii) $$\|D\|_\infty < \infty,$$

(iii) $$\lim_{a\to\infty} D_{Q_a} = m! \lim_{a\to\infty} \det[(\exp(i(\lambda_j - \lambda_k)x))_{Q_a}] = m! \det I_m = m!,$$

where formula (1) is used again.

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