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BORDISM OF SPIN MANIFOLDS WITH LOCAL ACTIONS OF TORI IN LOW DIMENSIONS

BY

PIOTR MIKRUT (WROCŁAW)

0. Introduction. The notion of a local action of tori on a manifold is a generalization of an action of a torus. Tori, possibly of different dimensions, act on open subsets of the manifold and these actions fit together in such a way that when two open sets meet the torus acting on one of them injects homomorphically into the torus acting on the second one. If we assume that each of these actions is without fixed points then the local action coincides with a T-structure introduced by Mikhael Gromov in the paper [G]. (See also Definition 1.5 in [CG]). Here we assume that a local action may admit fixed points.

Manifolds with such a structure can be classified up to a structure preserving bordism. Let us denote by Ω_n^S the bordism group of oriented n-manifolds with the structure S, where we put $S = t$ for T-structure, $S = pol$ for T-structure with polarization, $S = l.a.t.$ for local actions of tori and $S = l.a.t., spin$ for local actions of tori with a spin structure.

In low dimensions there are the following results:

- $\Omega_3^t \cong \mathbb{Z}_2$, $\mathrm{tor}\,\Omega_3^{pol} \cong \mathbb{Z}_{12} \times \mathbb{Z}_2$, $\Omega_3^{pol}/\mathrm{tor} \cong \mathbb{Z}^{(0,1/2)}$ ([HJ]);
- $\Omega_4^t \cong 2\mathbb{Z}$, $\Omega_4^{l.a.t.} \cong \mathbb{Z}$ ([Mi1]);
- $\Omega_4^{l.a.t.,spin} \cong 16\mathbb{Z}$ ([Mi2]).

In dimension 4 an isomorphism is given by signature.

In this paper we complete the list of bordism groups in low dimensions in the case of spin manifolds. We prove:

THEOREM. *The bordism groups of compact spin manifolds admitting local actions of tori in dimensions* 1, 2 *and* 3 *are the following*:

$$\Omega_1^{l.a.t.,spin} \cong \mathbb{Z} \quad \Omega_2^{l.a.t.,spin} \cong \mathbb{Z}_2 \quad \Omega_3^{l.a.t.,spin} \cong \mathbb{Z}_2.$$

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The calculation of the first two groups is just an exercise. Thus, the main result is the calculation of the third group. The methods we use are partly combinations of those for oriented manifolds with local actions of tori as in [HJ], [Mi1] and for spin manifolds with S^1 actions as in [B]. The important feature of spin manifolds is encoded in the property described in Lemma 3.7 and in the notion of an admissible isotropy subgroup (0.5, 0.6) which is related to notions of odd and even circle actions and is a simple consequence of the fact that $\Omega_1^{spin} = \mathbb{Z}_2$.

Definition 0.1. We say that a smooth manifold M admits a *local action of tori* if there is a covering $\{U_\alpha\}_{\alpha\in\Lambda}$ of M by open sets such that for each $\alpha \in \Lambda$ there is a torus T^{k_α} which acts smoothly and effectively on U_α:

$$\theta_\alpha : T^{k_\alpha} \times U_\alpha \to U_\alpha$$

and if $U_{\alpha\beta} := U_\alpha \cap U_\beta \neq \emptyset$ then up to changing the roles of α, β we have $k_\alpha \leq k_\beta$ and there exists a monomorphism

$$\xi_{\alpha\beta} : T^{k_\alpha} \to T^{k_\beta}$$

such that the following diagram is commutative:

$$\begin{array}{ccc} T^{k_\alpha} \times U_{\alpha\beta} & \xrightarrow{\theta_\alpha} & U_{\alpha\beta} \\ \Big\downarrow{\scriptstyle \xi_{\alpha\beta}\times \mathrm{id}_{U_{\alpha\beta}}} & & \Big\downarrow{\scriptstyle \mathrm{id}_{U_{\alpha\beta}}} \\ T^{k_\beta} \times U_{\alpha\beta} & \xrightarrow{\theta_\beta} & U_{\alpha\beta} \end{array}$$

Moreover, if $U_\alpha \cap U_\beta \cap U_\gamma \neq \emptyset$ then $\xi_{\beta\gamma}\xi_{\alpha\beta} = \xi_{\alpha\gamma}$ $(k_\alpha \leq k_\beta \leq k_\gamma)$.

An *atlas* of the local action of tori is the collection of the sets and maps

$$\langle \{U_\alpha\}_{\alpha\in\Lambda}; \{\theta_\alpha\}_{\alpha\in\Lambda}; \{\xi_{\alpha\beta}\}_{(\alpha,\beta)\in\Lambda_0}\rangle.$$

Here Λ_0 is the set of those pairs (α, β) for which $\xi_{\alpha\beta}$ is defined.

We say that a local action of tori is *pure* if k_α does not depend on α.

Compare with Definition 1.6 in [CG].

Definition 0.2. We say that two spin manifolds M_1, M_2 admitting local actions of tori are *cobordant* if there is a spin manifold W admitting local action of tori such that

$$\partial(W) = M_1 - M_2.$$

Moreover, the spin structure and local actions of tori on M_1, M_2 are restrictions of the spin structure and local actions of tori on W.

Notation 0.3. We use the notation T^1 if we think of the circle as a Lie group or a principal orbit of an action of T^1. The circle treated as a manifold is denoted by S^1.

NOTATIONS 0.4. Let us denote by $M^{(k)}$ the union of those U_α for which k_α is equal to an integer k. It is an open submanifold of M. If $m \in M^{(k)}$ then the *orbit* of m (denoted by $[m]_k$) for the local T^k action is the orbit of m for an action θ_α such that $m \in U_\alpha$ and $k_\alpha = k$. According to Definition 0.1, $[m]_k$ does not depend on α. Let $X^{(k)}$ denote the *orbit space* of $M^{(k)}$ with the quotient topology, which is determined by transversal slices. (See the slice theorem [Br], II.4.4).

Let us denote by ϖ_k the quotient map $M^{(k)} \to X^{(k)}$, and let a connected component of $M^{(k)}$ be denoted by $M_i^{(k)}$, where i is an index. Set $M^{(k,l)} = M^{(k)} \cap M^{(l)}$. If $m \in M^{(k,l)}, k \leq l$ then there is a map ϖ_l^k sending $[m]_k$ to $[m]_l$, i.e. $\varpi_l = \varpi_l^k \varpi_k$.

Let us denote by

- $\mathrm{Fix}_k^{(l)}$ the k-dimensional stratum of the fixed point set in $M^{(l)}$;
- $\mathrm{Fin}_k^{(l)}$ the k-dimensional stratum consisting of the orbits with finite isotropy subgroups;
- $\mathrm{Pr}^{(l)}$ the stratum consisting of the principal orbits.

DEFINITION 0.5. A T^1 action on a connected spin manifold M is called *even* if it lifts to a T^1 action on the spin bundle. Otherwise it is called *odd*. A T^k action on a connected spin manifold M is called *even* if for each subgroup $T^1 < T^k$ the action of T^1 is even. Otherwise it is called *odd*. A local T^k action on a connected spin manifold M is called *even* (*odd*) if for each local chart U_α the action θ_α is even (odd).

DEFINITION 0.6. Let i be the dimension of the acting torus. A subgroup $T^1 < T^i$, the action of which is odd, is called *admissible*. Let $\varepsilon : \mathbb{Z}_2^i \to \mathbb{Z}_2$ be a homomorphism defined as follows. A subgroup $T^1 < T^i$ represents an element of $H_1(T^i, \mathbb{Z}_2) \cong \mathbb{Z}_2^i$. The element of $\mathbb{Z}_2^i$ maps to zero if T^1 representing it is not admissible. Otherwise, the element maps to 1. The map is well defined, i.e. it is independent of the choice of a representative T^1. In the case where the action is odd the homomorphism ε is nontrivial.

Remark 0.7. A subgroup $T^1 < T^i$ is an isotropy subgroup with the corresponding stratum of codimension 2 iff its action is odd, i.e. it is admissible.

1. 1-manifolds

THEOREM 1.1. *The bordism group of compact spin* 1*-manifolds with local actions of tori is isomorphic to* $\mathbb{Z}$ *and is generated by* T^1 *with the nonbounding spin-structure.*

Proof. A compact 1-manifold with a local action of tori is equivariantly diffeomorphic to a disjoint sum of T^1's with the canonical T^1 action. T^1 with the bounding spin structure bounds D^2 and the T^1 action and spin structure obviously extend to D^2. We only have to check that T^1 with the non-bounding spin structure has infinite order. This follows from the fact that any compact 2-manifold with boundary admitting a local action of tori is diffeomorphic to a disjoint sum of manifolds equivariantly diffeomorphic to $[0,1]\times T^1$ or to D^2. Thus, if some non-zero multiple of T^1 with non-bounding spin structure bounds a 2-manifold then the 2-manifold is diffeomorphic to a multiple of $[0,1]\times T^1$. Looking at the orientations of the components of the boundary we arrive at a contradiction.

2. 2-manifolds

THEOREM 2.1. *The two-dimensional spin bordism group of manifolds with local actions of tori is isomorphic to* $\mathbb{Z}_2$ *and is generated by* $\overline{T}^2$, *i.e., a torus with the non-bounding spin structure and standard action of* T^2.

Proof. An oriented 2-manifold with local action of tori is equivariantly diffeomorphic to a disjoint sum of several copies of S^2 and T^2. Both of them are spin manifolds. It follows from the slice theorem ([Br]) that S^2 with local action of tori is equivariantly diffeomorphic to S^2 with the standard action of T^1. It bounds D^3 with the standard action of T^1. The unique spin structure on S^2 extends to a spin structure on D^3.

T^2 has essentially two spin structures: the bounding and non-bounding one. Up to a bordism, changing only the atlas on T^2, we can assume that T^2 is a single orbit of the T^2 action.

In case of a bounding spin structure we can choose the filling to be $D^2\times T^1$ and the action of T^2 on T^2 obviously extends to the action of T^2 on $D^2\times T^1$.

Thus, we have proved that a compact spin 2-manifold with a local action of tori is spin bordant with a disjoint sum of T^2's with non-bounding spin structures.

The disjoint sum of two copies of such T^2 is spin bordant with T^2 with the bounding spin structure. Let us first observe that by means of a bordism we can change the T^2 action into a T^1 action by choosing a $T^1 < T^2$. In this way we obtain a manifold M consisting of two copies of the trivial T^1 bundle over S^1. By Lemma 3.7, which we prove in the next section, we can construct a bordism between the manifold M and a trivial T^1 bundle over S^1 having the bounding spin structure. The bordism corresponds to a connected sum of the bases of the trivial T^1 bundles over S^1. The orientation and (non-bounding) spin structure on two copies of the trivial T^1 bundle over S^1 extends to the 3-manifold giving the orientation and (bounding) spin structure on the remaining component of the boundary.

A slightly different argument goes as follows. According to [B] the base of an even free T^1 action on a spin manifold inherits a spin structure. Then we use the result $\Omega_1^{Spin} = \mathbb{Z}_2$.

3. 3-manifolds. Some methods used in this section resemble methods used in the oriented case. See [HJ]. The spin case is less flexible since we have to take care of the spin structure on the trace of the bordism. One of the obstructions to a spin surgery operation is the necessity of preserving the admissibility of isotropy subgroups corresponding to codimension 2 strata.

Remark 3.1. Any 3-dimensional compact manifold M admits a spin structure. $M^{(3)}$ is a disjoint sum of tori T^3. We can assume that $M^{(3)} \cap (M^{(1)} \cup M^{(2)}) = \emptyset$. T^3 has eight spin structures. The standard $SL_3(\mathbb{Z})$ action on T^3 permutes these structures. There are two orbits for this action. They are distinguished by the Rokhlin invariant. The first one consists of the spin structures on T^3 which bound a spin structure on $T^2 \times D^2$. Thus, T^3 with one of these spin structures, as a manifold with a local action of tori, represents zero in the bordism group. The second orbit consists of the non-bounding spin structure on T^3. We change the atlas by replacing the action of T^3 on itself by an action of a subgroup $T^1 < T^3$. After the change T^3 is included in $M^{(1)}$. Thus, we can assume that $M^{(3)} = \emptyset$.

Remark 3.2. Any Seifert orbit in $M^{(1)}$ has a tubular neighbourhood on which the T^1 action can be extended to an effective T^2 action. Thus after suitable change of the atlas the neighbourhood can be included in $M^{(2)}$. Thus, we can assume that the local action on $M^{(1)}$ is semi-free, i.e. its isotropy subgroups are all trivial or equal to T^1.

LEMMA 3.3. *A spin 3-manifold with a local action of tori satisfying the above assumptions is spin bordant to a spin 3-manifold M that satisfies the above assumptions and the additional condition*:

- *All connected components of $M^{(1)}$ have even local T^1 action.*

Proof. Let $M_j^{(1)}$ be a connected component of $M^{(1)}$ with an odd local T^1 action. In each principal orbit of $M_j^{(1)}$ (with T^1 invariant parallelization in a neighbourhood) the inherited spin structure corresponds to the non-zero element of $H^1(T^1, \mathbb{Z}_2)$. Since the local action is odd we can apply a method similar to that for a semi-free circle action. See [B]. Here we use a method better adapted to local actions. It can be generalized to the case where the dimension of the acting torus is greater than the dimension of the orbit space under suitable conditions on orbit types. The construction goes as follows.

Let us find a triangulation of $X_j^{(1)}$ such that the closure of each stratum consisting of points with non-trivial isotropy subgroups is a subcomplex. Such a triangulation exists since the orbit space has a natural structure of a smooth manifold with boundary ([Br]). The interior of each top dimensional simplex $\triangle^2$ corresponds to principal orbits. For each such simplex let us make a surgery over a point in the interior of $\triangle^2$:

$$M' = (M - (D_1^2 \times T^1)) \cup_\partial (S^1 \times D_2^2)$$

where T^1 corresponds to the acting torus and the action on the second set is given by the standard action on D^2.

D_1^2 is a smooth disk included in the interior of $\triangle^2$. The gluing map is chosen to preserve the actions.

If two simplices $\triangle_1^2$, $\triangle_2^2$ intersect along $\triangle^1$, their common face in the original triangulation, then the interior of $\triangle^1$ corresponds to principal orbits bounded from both sides by a 1-dimensional stratum of fixed points after the surgery. A similar situation occurs when $\triangle^1$ in the original triangulation lies in the boundary of the orbit space, i.e., corresponds to a 1-dimensional stratum of fixed points. We make a surgery over a point in the interior of $\triangle^1$ which "makes a corridor" between D_1^2 and D_2^2 in the first case or between D^2 and "the outside" of $X^{(1)}$ in the second:

$$M' = (M - (D^1 \times S^2)) \cup_\partial (S^0 \times D^3).$$

The action of T^1 on the first set is by the standard action on S^2. D^1 corresponds to $\triangle^1$ with a neighbourhood of the boundary deleted. The action on the second set is by the standard rotation of D^3. After the construction we have obtained a disjoint sum of spheres S^3, smooth suspensions of S^2, corresponding to vertices of the triangulation lying in the principal part or on $\mathrm{Fix}_1^{(1)}$. They are T^1 equivariantly spin null bordant. The remaining part of $M^{(1)}$ is a neighbourhood of $\partial(M^{(2)})$ diffeomorphic to a disjoint sum of trivial D^2 bundles over S^1 with the standard action of T^1 on fibers. $M^{(2)}$ intersects the boundary of the bundle. The local T^2 action can be extended to the whole D^2 bundle.

LEMMA 3.4. *A spin* 3*-manifold with a local action of tori satisfying the above assumptions is spin bordant with a spin* 3*-manifold M that satisfies the above assumptions and the additional condition*:

- *All connected components of $M^{(2)}$ have even local T^2 action.*

Proof. A connected component of $M^{(2)}$ with an odd T^2 local action can be changed by means of a bordism to a disjoint sum of at most two copies of $T^1 \times D^2$.

Such a component is diffeomorphic to one of the following:

- A T^2 bundle over S^1,

• A T^2 bundle over $(0,1)$,

• $(0,1]\times T^2/\sim$, where $\sim$ corresponds to collapsing a subgroup $T^1<T^2$ over $\{1\}$,

• $[0,1]\times T^2/\sim$, where $\sim$ corresponds to collapsing a subgroup $T^1_1<T^2$ over $\{0\}$ and a subgroup $T^1_2<T^2$ over $\{1\}$.

Except for the first case, the local T^2 action is an action. In the first two cases let us choose an admissible subgroup $T^1_1<T^2$ and a splitting $T^2=T^1_1\times T^1_2$ and perform a T^2 equivariant surgery killing the free homotopy class of the loop given by an orbit of T^1_1:

$$(M^{(2)}-T^2\times D^1)\cup_\partial D^2\times T^1_2\times S^0.$$

Here $D^1\subset S^1$ in the first case or $\overline{D}^1\subset(0,1)$ in the second. After the surgery in the first case we obtain the fourth case and in the second the third one.

In the third case let us denote by T^1_1 the subgroup acting on the nearby component of $M^{(1)}$ and let T^1_2 be the isotropy subgroup. T^1_2 is admissible by 0.7.

In the fourth case let T^1_1 and T^1_2 denote the isotropy subgroups. They are both admissible.

There is a chain of admissible subgroups $T^1_{\alpha_1},\ldots,T^1_{\alpha_k}$ such that in the third case $T^1_{\alpha_1}\cap T^1_1=\{1\}$ and $T^1_{\alpha_k}=T^1_2$ or $T^1_{\alpha_k}\cap T^1_2=\{1\}$ and in the fourth case $T^1_{\alpha_1}=T^1_1$ and $T^1_{\alpha_2}=T^1_2$. Moreover, we can assume that in both cases $T^1_{\alpha_i}\cap T^1_{\alpha_{i+1}}=\{1\}$. Now by a surgery similar to that in [HJ], Section 4, we can decompose by means of a spin bordism the component of $M^{(2)}$ into a disjoint sum of several copies of S^3 and $D^2\times S^1$ in the third case and disjoint sum of spheres S^3 in the fourth case. The subgroups $T^1_{\alpha_i}$ $(i=1,\ldots,k)$ are the corresponding isotropy subgroups. For each component $D^2\times S^1$ of $M^{(2)}$ obtained, the T^1 locally acting on the nearby component of $M^{(1)}$ acts on the boundary of $D^2\times S^1$. The T^1 action extends to the whole of $D^2\times S^1$. Moreover, since we have assumed that $T^1_{\alpha_1}\cap T^1_1=\{1\}$, we will not obtain any orbit with a non-trivial isotropy subgroup.

S^3 with the standard action of T^2, which is odd, is spin null bordant: $S^3=\partial D^4$, where D^4 is a smooth cone over S^3.

Remark 3.5. We have proved so far that $M=M^{(1)}\cup M^{(2)}$ consists only of even parts. Moreover, $M^{(1)}$ and $M^{(2)}$ consist only of principal orbits.

By means of an equivariant spin surgery corresponding to connected sum on $X^{(1)}$, $M^{(1)}$ can be made connected.

FACT 3.6. If we trivialize a part of T^1 fibering over some simple closed curve S^1 on the base as $(-1,1)\times S^1\times T^1$ with the tangent frame corresponding to the splitting then the curve on the base space can be killed by spin surgery (an inverse of the local connected sum on the base):

$$M' = (M - D^1 \times S^1 \times T^1) \cup_\partial S^0 \times D^2 \times T^1$$

if the element of $H^1((-1,1) \times S^1 \times T^1; \mathbb{Z}_2)$ representing the induced spin structure acts non-trivially on the homology class of S^1.

Using an argument from [B], Proposition 2.2, we know that $(-1,1) \times S^1$ inherits a spin structure from M. The core $\{0\} \times S^1$ inherits a spin structure from $(-1,1) \times S^1$. The surgery above can be performed if the spin structure on the core is the bounding one.

LEMMA 3.7. *Assume that a* 2*-manifold* Y *diffeomorphic to* S^2 *with three disjoint open disks deleted is included in* $X^{(1)}$. *Then at least one boundary loop inherits a bounding spin structure.*

Proof. If all three boundary loops inherit non-bounding spin structures then we obtain a relation

$$3\overline{T}^1 = 0 \quad \text{in} \quad \Omega_1^{spin}$$

where $\overline{T}^1$ is a generator of $\Omega_1^{spin} = \mathbb{Z}_2$. This is a contradiction.

COROLLARY 3.8. *With the same assumptions made*, *the* 3*-manifold is spin bordant to a manifold satisfying the assumptions and having the following property*:

A connected component of $M^{(1)}$ *is diffeomorphic to one of the following*:

- $(-1,1) \times S^1 \times T^1$;
- *A* T^1 *bundle over* S^2;
- *A* T^1 *bundle over* RP^2 *with a section.*

In each case the local action of T^1 *is even.*

Proof. By 3.6 and 3.7 we can decompose $X^{(1)}$ into minimal pieces. We obtain $(-1,1) \times S^1$ and X, where X has no boundary. According to 3.5 the boundary tori in each component of $\partial(M^{(2)})$ with boundary inherit a non-bounding spin structure $\overline{T}^2$. Thus, the two pieces $D^2, RP^2 - D^2$ do not occur.

If a component of $M^{(1)}$ has no boundary then it is a T^1 bundle over X with an even T^1 action on the fibers. There is a section of the bundle outside a small disk D^2. $\partial(D^2)$ inherits a bounding spin structure and the surgery described in 3.6 can be performed. We obtain a T^1 bundle over X with a section and a T^1 bundle over S^2.

Similarly as in [B], Proposition 2.2, we can show that X inherits a pin$^-$ structure. A pin$^-$ structure on X determines a spin structure on a T^1 bundle over X associated with the orientation sheaf. A pin$^-$ bordism relation between the orbit spaces corresponds to a spin bordism relation of the total spaces. By [KT], Ω_2^{pin-} is generated by RP^2 with one of its pin$^-$ structures.

Thus, we can assume that each connected component of $X^{(1)}$ having no boundary and different from S^2 is equal to RP^2.

R e m a r k 3.9. A connected component of $M^{(2)}$ admits an even T^2 action. Thus, it is a T^2 bundle over $(-1,1)$ or over S^1. In the second case the component of $M^{(2)}$ can be assumed to be disjoint from $M^{(1)}$. In the first case we have two possibilities:

• The component intersects one component of $M^{(1)}$ with the orbit space $X^{(1)}$ diffeomorphic to $(-1,1) \times S^1$. We obtain a connected component of M, which is diffeomorphic to a T^2 bundle over S^1.

• The component intersects two components of $M^{(1)}$ with orbit spaces diffeomorphic to $(-1,1) \times S^1$. In this case the corresponding connected component of M is diffeomorphic to a T^2 bundle over S^1. It is a sum of a chain of components of $M^{(1)}$, each with an orbit space diffeomorphic to $(-1,1) \times S^1$, and of components of $M^{(2)}$, each with an orbit space diffeomorphic to $(-1,1)$.

In both cases the canonical T^2 local action on the T^2 bundle is compatible and thus bordant to the original local action on the component of M.

3.10. *Generators and relations.* After suitably changing the atlas we obtain the following set of generators:

(1) A T^1 bundle over S^2;
(2) A T^2 bundle over S^1 with a monodromy A from $SL_2(\mathbb{Z})$;
(3) A T^1 bundle over RP^2 with a section.

All of them are assumed to have even local actions.

(1) *A T^1 bundle over S^2.* By means of the surgery described in 3.6 we can show that it is bordant to a disjoint sum of products $T^1 \times S^2$, several copies of the T^1 bundle over S^2 with Euler number ± 2 and several copies of the T^1 bundle over S^2 with Euler number ± 1. We have $T^1 \times S^2 = \partial(T^1 \times D^3)$ and this filling is spin.

The T^1 bundle over S^2 with Euler number ± 2 can be pictured as $T^2 \times [0,1]/\sim$ where $\sim$ corresponds to collapsing orbits over $\{0\}$ by the subgroup $(1,0)$ and over $\{1\}$ by the subgroup $(1,\pm 2)$. Both subgroups are admissible if we think of the T^2 action. The action of T^1 corresponds to the subgroup $(1,1)$. The total space is diffeomorphic to RP^3 and thus admits two spin structures, one with an odd and one with an even action of $T^1 \cong (1,1)$. The manifold is the boundary of the D^2 bundle over S^2 with Euler number ± 2 given as $T^2 \times [0,1] \times [0,1/2]/\sim$, where $\sim$ corresponds to collapsing orbits over $\{0\} \times [0,1/2]$ by the subgroup $(1,0)$ and over $\{1\} \times [0,1/2]$ by the subgroup $(1,\pm 2)$ and over $[0,1] \times \{0\}$ by the subgroup $(0,1)$. See [OR] for the convention of this notation. All 1-dimensional isotropy subgroups are admissible for the action of T^2. Thus the filling is spin.

The T^1 bundle over S^2 with Euler number ± 1 is S^3 and thus admits a unique spin structure. It bounds D^4, the smooth cone over S^3. The action and the spin structure obviously extend.

(2) *The T^2 bundle over S^1 with a monodromy A from $SL_2(\mathbb{Z})$.* Let N denote the manifold, and let A be expressed as a product of parabolic matrices having trace 2: $A = A_1 \cdot \ldots \cdot A_k$. Let X be S^2 with $k+1$ disjoint open disks deleted. Construct a T^2 bundle over X with monodromy group included in $SL_2(\mathbb{Z})$ such that the corresponding monodromy matrices along boundary loops are A, $A_k^{-1}, \ldots, A_1^{-1}$. Let M be the manifold thus obtained. The boundary of M consists of the disjoint sum of the manifolds: N and $N_1, \ldots, N_k$. Each N_i, as the total space of a T^2 bundle over S^1 with parabolic monodromy matrix having trace 2, is diffeomorphic to the total space of a T^1 bundle over T^2, where T^1 corresponds to the eigenspace (with eigenvalue 1) of the monodromy matrix. Let us note that the matrix $A \in SL_2(\mathbb{Z})$ having trace 2 has eigenvalue 1 since it is conjugate in $SL_2(\mathbb{Z})$ to $\left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$ for some $k \in \mathbb{Z}$.

Let us choose a T^2 invariant framing (e_1, e_2, e_3) on N such that e_1 is a horizontal vector field and (e_2, e_3) is the frame tangent to fibers.

The bundle tangent to M has a splitting into the Whitney sum of the bundle tangent to the fibers and the horizontal bundle, i.e., the lift of the bundle tangent to the base. Let us choose a horizontal vector field e_4 on N tangent to M and transversal to N. The frame (e_1, e_2, e_3, e_4), as a section over N of the tangent frame bundle to M compatible with the splitting, can be extended over the whole manifold M to a section (e_1, e_2, e_3, e_4) of the tangent frame bundle to M compatible with the splitting. This is due to the fact that the orbit space is homotopy equivalent to $S^1 \vee \ldots \vee S^1$. In particular, M as a parallelizable manifold admits spin structures parametrized by $H^1(M; \mathbb{Z}_2)$. The T^2 action on N is even. Thus, the element s of $H^1(N; \mathbb{Z}_2)$ corresponding to the spin structure on N according to the framing (e_1, e_2, e_3) acts trivially on elements of $\operatorname{im}(H_1(T^2) \hookrightarrow H_1(N))$. The element s is in the image of the map $H^1(M; \mathbb{Z}_2) \to H^1(N; \mathbb{Z}_2)$ induced by the inclusion $N \subset M$. Thus, the spin structure on N is induced from a spin structure on M.

We have just proved that N is spin bordant as a manifold with a local action of tori to a disjoint sum of $N_1, \ldots, N_k$ with spin structures induced from M. Each of the latter is bordant to a T^1 bundle over T^2 with an even T^1 local action, which we can see by changing the atlas of the local action. This case was treated above.

From (1) and (2) above we see that the set of generators can be reduced to the following single generator:

(3) *A T^1 bundle over RP^2 admitting a section, with an orientable total space*. From [KT] we know that RP^2 with the pin$^-$ structure has order 8 in Ω_2^{pin-}. Thus the bordism group of spin 3-manifolds with local actions of tori is a cyclic group $\mathbb{Z}_s$, where s divides 8.

The T^1 bundle over $\mathbb{R}P^2$ admitting a section with orientable total space cannot bound a spin 4-manifold with a local action of tori. The argument is similar to that in [HJ] in the proof of Theorem 7.2. Assume that there is a filling W. Let W_0 be a connected component of $W^{(1)}$ which meets the boundary of W. The local T^1 action on W_0 extends the local T^1 action on $\partial(W)$ and thus is even. Thus, the set of fixed points consists only of isolated points and the orbit space of W_0 is a topological 3-manifold. All boundary components of W_0 except $\partial(W)$ admit local T^2 actions and compatible local T^1 actions. They are either T^2 bundles over S^1 with parabolic monodromy or lens spaces. This follows from the slice theorem and the description of 3-manifolds with T^2 actions in [OR]. In each case the orbit space of the local T^1 action is a surface which bounds. Since RP^2 does not bound we obtain a contradiction.

The double of the T^1 bundle over RP^2 is spin bordant to a T^1 bundle over $RP^2 \# RP^2$ with a section. The bordism can be obtained by performing a connected sum of orbit spaces, which corresponds to a T^1 equivariant surgery on the total spaces. $RP^2 \# RP^2$ admits a non-orientable T^1 fibration over S^1 and the homotopy class of a fibre cannot be killed by pin$^-$ surgery since the double of RP^2 is not zero in Ω_2^{pin-}. The T^1 bundle over $RP^2 \# RP^2$ with a section is diffeomorphic to a T^2 bundle over S^1 with monodromy $-I$. The local T^2 action is compatible with the local T^1 action. The local action of T^2 is even. From (2) we know that the manifold is spin bordant to a disjoint sum of T^2 bundles over S^1 with parabolic monodromy of trace 2 with even local actions of T^2. Each such bundle as a T^1 bundle over T^2 is spin bordant to the trivial T^1 bundle over T^2. If the induced spin structure on the base bounds, the manifold bounds. Thus, we have to consider only the trivial T^1 bundle over T^2, i.e., T^3 with a spin structure such that the canonical T^3 action is even. Let us denote the spin manifold $\overline{T}^3$. By [KT] the induced pin$^-$ structure on the base torus represents 4 in $\mathbb{Z}_8 \cong \Omega_2^{pin-}$. Thus we obtain a relation in $\mathbb{Z}_s$:

$$2 \equiv 4k$$

for some integer k. If k is even then $2 \equiv 0$ and $s = 2$. If k is odd then $2 \equiv 4$ and $s = 2$.

We finally obtain

THEOREM 3.11. $\Omega_3^{l.a.t.,spin} \cong \mathbb{Z}_2$, *i.e., the bordism group of compact spin 3-manifolds with local action of tori is isomorphic to $\mathbb{Z}_2$ and is generated by the non-orientable T^1 bundle over RP^2 with a section.*

Let M be a compact spin 3-manifold with a local action of tori. Let $\overline{X}$ denote a surface which is constructed from components of $X^{(1)}$ corresponding to even local action by collapsing each boundary component to a point. Then from the method of the proof that $\Omega_3^{l.a.t.,spin} \cong \mathbb{Z}_2$ we can deduce the following:

THEOREM 3.12. *If M is a compact spin* 3*-manifold with a local action of tori then its bordism class in $\Omega_3^{l.a.t.,spin} \cong \mathbb{Z}_2$ coincides with $\chi(\overline{X}) \bmod 2$, where χ denotes the Euler characteristic.*

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Mathematical Institute
University of Wrocław
pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: mikrut@math.uni.wroc.pl

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