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ON RINGS WHOSE FLAT MODULES FORM A GROTHENDIECK CATEGORY

BY

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1. Introduction. Throughout this paper, by a *ring* we shall mean "a ring with enough idempotents" in the sense of [4] and [26, p. 464], that is, an associative ring R containing a set $\{e_\lambda\}_{\lambda\in\Lambda}$ of pairwise orthogonal idempotent elements e_λ, $\lambda \in \Lambda$, such that

$$R = \bigoplus_{\lambda\in\Lambda} Re_\lambda = \bigoplus_{\lambda\in\Lambda} e_\lambda R. \tag{1.1}$$

We say that the ring R is *unitary* if it has an identity element 1. In this case the set Λ is finite.

By a right R-module we shall always mean a right R-module M which is *unitary*, that is, $MR = M$. We denote by $\mathrm{Mod}(R)$ the category of all unitary right R-modules, and thus $\mathrm{Mod}(R^{\mathrm{op}})$ will stand for the category of left R-modules. The full subcategory of $\mathrm{Mod}(R)$ formed by all finitely generated projective modules will be denoted by $\mathrm{proj}(R)$.

A right R-module M is *flat* if the tensor product functor

$$M \otimes_R (-) : \mathrm{Mod}(R^{\mathrm{op}}) \to \mathcal{A}b$$

is exact. The full subcategory of $\mathrm{Mod}(R)$ consisting of all flat right R-modules will be denoted by $\mathrm{Fl}(R)$. For convenience, we introduce the following definition.

DEFINITION 1.2. A ring $R = \bigoplus_{\lambda\in\Lambda} Re_\lambda = \bigoplus_{\lambda\in\Lambda} e_\lambda R$ as in (1.1) is *right panoramic* if the category $\mathrm{Fl}(R)$ of flat right R-modules is abelian, or, equivalently, if $\mathrm{Fl}(R)$ is a Grothendieck category.

Right panoramic rings appeared first in the Jøndrup–Simson paper [15, p. 29]. A characterization of unitary right panoramic rings was given later in [7].

One of our main aims in this paper is to study the relationship between right panoramic rings and locally finitely presented Grothendieck categories

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in connection with the following problem posed by Jøndrup and Simson in [15, p. 29]:

PROBLEM (JS). *Assume that R is a right panoramic unitary ring such that the right module R_R is a direct sum of indecomposable right ideals. Find a ring A and a right A-module U_A satisfying the following conditions:*

(i) *Every finitely presented right A-module is a direct sum of indecomposable modules.*

(ii) *The number of isomorphism classes of finitely presented indecomposable right A-modules is finite.*

(iii) *The module U_A is the direct sum of a complete set of representatives of isomorphism classes of indecomposable finitely presented right A-modules.*

(iv) *The ring R is Morita equivalent to the endomorphism ring* $\mathrm{End}(U_A)$ *of U_A.*

An affirmative solution of the problem (JS) would mean that right panoramic rings can be constructed in this specific way from module categories $\mathrm{Mod}(A)$ over rings A satisfying the conditions (i)–(iii) above.

We note that if A is an artin algebra of finite representation type then the ring $\mathrm{End}(U_A)$ is the Auslander algebra of A (see [3, Section VI.5] and [22, Section 11.2]).

Our results in this paper show that every right panoramic ring can be constructed in a similar way from a locally finitely presented Grothendieck category.

In Section 2 we collect preliminary results on panoramic rings. In particular, we show in Theorem 2.7 that a ring $R = \bigoplus_{\lambda\in\Lambda} e_\lambda R = \bigoplus_{\lambda\in\Lambda} Re_\lambda$ is right panoramic if and only if the following conditions are satisfied:

(i) For every flat right R-module F the injective hull $E_R(F)$ of F is a flat module.

(ii) The ring R is left locally coherent.

(iii) The ring R is right locally weakly $\boldsymbol{\delta}_R$-coherent (see (2.6)).

(iv) The weak global dimension $\mathrm{w.gl.dim}\, R$ of R is either 0 or 2.

(v) For every $\lambda \in \Lambda$, the flat-dominant dimension of $e_\lambda R$ is ≥ 2.

We also show in Corollary 2.11 that a ring $R = \bigoplus_{\lambda\in\Lambda} Re_\lambda = \bigoplus_{\lambda\in\Lambda} e_\lambda R$ is both right and left panoramic if and only if R is left and right locally coherent, $\mathrm{w.gl.dim}\, R \leq 2$, and the flat-dominant dimension of $e_\lambda R$ is at least two for every $\lambda \in \Lambda$.

The main result of Section 3 is Theorem 3.3 asserting that the map $R \mapsto \mathrm{Fl}(R)$ induces a one-to-one correspondence between Morita equivalence classes of right panoramic rings $R = \bigoplus_{\lambda\in\Lambda} Re_\lambda = \bigoplus_{\lambda\in\Lambda} e_\lambda R$ and equivalence classes of locally finitely presented Grothendieck categories. The inverse correspondence is given by attaching to any family $\mathcal{G} = \{G_\lambda\}_{\lambda\in\Lambda}$ of finitely presented strong generators of the locally finitely presented Grothendieck category $\mathcal{C}$ the Gabriel functor ring $R_{\mathcal{G}}$ (see 3.2) of the family $\mathcal{G}$.

In Section 4 we discuss the problem (JS) stated above in connection with panoramic rings and the correspondence $R \mapsto \mathrm{Fl}(R)$ presented above. In particular, we show in Theorem 4.1 that the map $R \mapsto \mathrm{Fl}(R)$ defines a one-to-one correspondence between Morita equivalence classes of unitary right panoramic rings R such that R_R is a direct sum of indecomposable right ideals and equivalence classes of locally finitely presented Grothendieck categories $\mathcal{C}$ with a finite family of indecomposable finitely presented strong generators. We also show in Proposition 4.5 that there exists a one-to-one correspondence $R \mapsto A = eRe$ between:

(a) Morita equivalence classes of right panoramic unitary rings R such that:

- (a1) There is an idempotent $e \in R$ such that ReR is a minimal finitely generated left faithful right ideal.
- (a2) There exist primitive idempotents $e_1, \ldots, e_n$ such that every finitely generated projective right module is isomorphic to a direct sum of the modules $e_1R, \ldots, e_nR$; and

(b) Morita equivalence classes of unitary rings A such that:

- (b1) There are finitely many indecomposable finitely presented right A-modules.
- (b2) Every finitely presented right A-module is a direct sum of indecomposable modules.

By restricting this correspondence to semiperfect rings we get in Theorem 4.6 a one-to-one correspondence between Morita equivalence classes of right panoramic unitary semiperfect rings R and equivalence classes of locally finitely presented Grothendieck categories $\mathcal{C}$ such that the number of isomorphism classes of indecomposable finitely presented objects in $\mathcal{C}$ is finite, and every finitely presented object of $\mathcal{C}$ admits a direct sum decomposition that complements direct summands (see [1]).

Section 5 contains some concluding remarks and comments. We also present some open problems related to right panoramic rings and to the problem (JS).

2. Preliminary results on panoramic rings. We collect some known facts related to right panoramic rings. We also extend here several results given in [7] for unitary rings to arbitrary rings with enough idempotents.

First, we recall that the *dominant dimension* of a right R-module M_R (respectively, the *flat-dominant dimension* of M_R) is ≥ 2 in case there exists an exact sequence

$$0 \to M \to E_0 \to E_1$$

where E_0, E_1 are injective and projective (resp. injective and flat) R-modules.

Let us start with the following reformulation of a theorem of Tachikawa [25].

THEOREM 2.1. *Let R be a unitary right perfect ring. Then the following conditions are equivalent.*

(a) *The ring R is right panoramic.*

(b) *R is a semiprimary QF-3 ring, the global dimension* $\mathrm{gl.dim}\, R$ *of R is ≤ 2, and the dominant dimension of R_R is ≥ 2.*

(c) *There exist a ring A of finite representation type and a finitely generated A-module U_A such that every indecomposable right A-module is isomorphic to a direct summand of U_A and the ring R is isomorphic to the endomorphism ring* $\mathrm{End}(U_A)$.

There is a direct connection between the condition (c) of Theorem 2.1 and the problem (JS) of Jøndrup and Simson stated above. Namely, unitary right perfect right panoramic rings appear in the way foretold in the problem (JS).

The next two results we want to mention are proved in [7] for the particular case of unitary rings. But we are going now to state and prove them in the more general setting of rings with enough idempotents. It will be necessary to recall some definitions and notations.

For torsion theories and noncommutative localization, we refer to [24], while for general properties of rings and modules, we shall use mainly the terminology from [1] or [26].

DEFINITION 2.2 ([11, p. 531] in the unitary case). A ring $R = \bigoplus_{\lambda\in\Lambda} e_\lambda R = \bigoplus_{\lambda\in\Lambda} Re_\lambda$ (1.1) is called a *right* FTF-*ring* if the class of right R-modules which are (isomorphic to) submodules of flat modules is the torsionfree class for a hereditary torsion theory of $\mathrm{Mod}(R)$. In this case, the associated torsion radical will be denoted by $\boldsymbol{\tau}_R$.

DEFINITION 2.3 (see [26, p. 214]). Let $R = \bigoplus_{\lambda\in\Lambda} e_\lambda R = \bigoplus_{\lambda\in\Lambda} Re_\lambda$ be a ring. R is *right locally coherent* if each $e_\lambda R$ is coherent; or, equivalently, if every finitely presented right R-module is coherent.

PROPOSITION 2.4. *A ring $R = \bigoplus_{\lambda\in\Lambda} e_\lambda R = \bigoplus_{\lambda\in\Lambda} Re_\lambda$ is right locally coherent if and only if the direct product of any family of flat left R-modules is a flat module.*

Proof. The idea is similar to the usual one for unitary rings. The only problem here comes from the fact that the product in the category $\mathrm{Mod}(R)$ of unitary modules differs from the usual one, because the product $\prod_i L_i$ of unitary modules L_i is computed by taking first the abelian group product

L of the modules L_i and then, setting $\prod_i L_i = LR$. But it is not hard to prove that if M_R is a unitary right module and ${}_RL$ is a non-unitary left module, then the canonical homomorphism $M \otimes_R \ RL \to M \otimes_R \ L$ is a surjection. Taking this into account, one sees that a unitary module M_R is finitely generated (respectively, finitely presented) in $\mathrm{Mod}(R)$ if and only if the canonical homomorphism $M \otimes_R \prod_I L_i \to \prod_I (M \otimes_R \ L_i)$ is an epimorphism (resp. an isomorphism) for any set I and any family $\{L_i\}_{i \in I}$ of unitary left R-modules, where the product $\prod_I L_i$ is understood as the product in the category $\mathrm{Mod}(R^{\mathrm{op}})$. Then the proposition follows in a usual way (for example, as in [26, 12.16 and 26.6]). ∎

PROPOSITION 2.5. *A ring $R = \bigoplus_{\lambda \in \Lambda} e_\lambda R = \bigoplus_{\lambda \in \Lambda} Re_\lambda$ is a right FTF ring if and only if the following two conditions are satisfied.*

(i) *For every flat right R-module F the injective hull $E_R(F)$ is a flat module.*

(ii) *The direct product of any family of unitary injective and flat right R-modules is a flat R-module.*

Proof. Apply the arguments used in the proof of [11, Proposition 2.1] for unitary rings. ∎

For every ring $R = \bigoplus_{\lambda \in \Lambda} e_\lambda R = \bigoplus_{\lambda \in \Lambda} Re_\lambda$, there is a biggest hereditary torsion class in $\mathrm{Mod}(R)$ with respect to the condition that R_R is torsionfree (the Lambek torsion class). The corresponding radical will be denoted by $\boldsymbol{\delta}_R$, and a right R-module X_R is $\boldsymbol{\delta}_R$-torsion if and only if $\mathrm{Hom}_R(X, E(e_\lambda R)) = 0$, for each $\lambda \in \Lambda$. A module M_R is said to be $\boldsymbol{\delta}_R$-*finitely generated* if there exists a finitely generated submodule N of M such that M/N is $\boldsymbol{\delta}_R$-torsion. Moreover, M_R is said to be $\boldsymbol{\delta}_R$-*finitely presented* if it is finitely generated and for any exact sequence

$$0 \to K \to P \to M \to 0$$

such that P is finitely generated, the module K is $\boldsymbol{\delta}_R$-finitely generated. The same definitions are in use for any other torsion theory (such as $\boldsymbol{\tau}_R$ for a right FTF ring R).

DEFINITION 2.6. A ring $R = \bigoplus_{\lambda \in \Lambda} e_\lambda R = \bigoplus_{\lambda \in \Lambda} Re_\lambda$ is called *right locally* (resp. *weakly*) $\boldsymbol{\delta}_R$-*coherent* if for each $\lambda \in \Lambda$, every finitely generated right ideal $I \subseteq e_\lambda R$ (resp. such that $e_\lambda R/I$ is $\boldsymbol{\delta}_R$-torsion) is $\boldsymbol{\delta}_R$-finitely presented.

We start with the following characterization of arbitrary right panoramic rings (compare with [7, Theorem 3] for the unitary case, and with [21, Proposition 1.1], [14, Proposition 1.4]).

THEOREM 2.7. *A ring $R = \bigoplus_{\lambda \in \Lambda} e_\lambda R = \bigoplus_{\lambda \in \Lambda} Re_\lambda$ is right panoramic if and only if the following conditions are satisfied*:

(i) *For every flat right R-module F the injective hull $E_R(F)$ is a flat module.*

(ii) *The ring R is left locally coherent.*

(iii) *The ring R is right locally weakly $\boldsymbol{\delta}_R$-coherent.*

(iii) *The weak global dimension* w.gl.dim R *of R is either* 0 *or* 2.

(iv) *For every $\lambda \in \Lambda$, the flat-dominant dimension of $e_\lambda R$ is ≥ 2.*

Further, if $R = \bigoplus_{\lambda\in\Lambda} Re_\lambda = \bigoplus_{\lambda\in\Lambda} e_\lambda R$ is right panoramic then $\{e_\lambda R\}_{\lambda\in\Lambda}$ is a family of finitely presented generators of the category $\mathrm{Fl}(R)$ *and* $\mathrm{proj}(R) = \mathrm{fp}(\mathrm{Fl}(R))$, *where* $\mathrm{fp}(\mathrm{Fl}(R))$ *is the full subcategory of* $\mathrm{Fl}(R)$ *formed by all finitely presented objects.*

Proof. Suppose that R is right panoramic. Then $\{e_\lambda R\}_{\lambda\in\Lambda}$ is a family of generators for the category $\mathrm{Fl}(R)$. We want to prove first that each $e_\lambda R$ is, indeed, a finitely presented object of $\mathrm{Fl}(R)$. To this end, consider a direct system $\{F_i\}$ in the category $\mathrm{Fl}(R)$. It is easy to see that $\{F_i\}$ is a direct system of flat right R-modules, and the direct limit $\varinjlim F_i = F$ is again flat (see [26, 36.1]). Therefore, F is also the direct limit of the system $\{F_i\}$ in the category $\mathrm{Fl}(R)$. Now, by applying the functor $\mathrm{Hom}_R(e_\lambda R, -)$ in the category $\mathrm{Mod}(R)$, we get an isomorphism

$$\varinjlim \mathrm{Hom}_R(e_\lambda R, F_i) \cong \mathrm{Hom}_R(e_\lambda R, F)$$

of abelian groups, because $e_\lambda R$ is finitely presented. This implies that, in the category $\mathrm{Fl}(R)$, there is an isomorphism

$$\varinjlim \mathrm{Hom}_{\mathrm{Fl}(R)}(e_\lambda R, F_i) \cong \mathrm{Hom}_{\mathrm{Fl}(R)}(e_\lambda R, \varinjlim F_i).$$

It follows that $e_\lambda R$ is a finitely presented object in $\mathrm{Fl}(R)$ (see [26, 25.2]) and therefore $\{e_\lambda R\}$ is a family of finitely presented generators of the category $\mathrm{Fl}(R)$.

The Gabriel functor ring (see [5] and [6, p. 138])

$$\bigoplus_{\lambda\in\Lambda}\bigoplus_{\mu\in\Lambda} \mathrm{Hom}_{\mathrm{Fl}(R)}(e_\lambda R, e_\mu R) = \bigoplus_{\lambda\in\Lambda}\bigoplus_{\mu\in\Lambda} \mathrm{Hom}_R(e_\lambda R, e_\mu R)$$

of the family $\{e_\lambda R\}$ is naturally isomorphic to $\bigoplus_{\lambda\in\Lambda}\bigoplus_{\mu\in\Lambda} e_\mu R e_\lambda \cong R$. Then, the construction of [7, Sec.1] shows that the functor

$$\mathbf{H} = \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}(e_\lambda R, -) : \mathrm{Fl}(R) \to \mathrm{Mod}(R)$$

induces an equivalence between $\mathrm{Fl}(R)$ and the quotient category of $\mathrm{Mod}(R)$ corresponding to a certain hereditary torsion theory $\boldsymbol{\sigma}$ of $\mathrm{Mod}(R)$. Since the functor $\mathbf{H}$ is naturally isomorphic to the inclusion functor $\mathrm{Fl}(R) \to \mathrm{Mod}(R)$ then $\boldsymbol{\sigma}$ is a hereditary torsion theory of $\mathrm{Mod}(R)$ such that the $\boldsymbol{\sigma}$-closed modules are precisely the flat right R-modules. As a consequence, a right R-module M_R is $\boldsymbol{\sigma}$-torsionfree if and only if there exists an embedding of

M_R into a flat right R-module. This shows that R is a right FTF ring with respect to the torsion theory $\boldsymbol{\tau}_R = \boldsymbol{\sigma}$ and (i) follows from Proposition 2.5. Moreover, since flat right R-modules are the $\boldsymbol{\tau}_R$-closed modules, we infer that the product of any family of unitary flat right R-modules is flat, so that R is left locally coherent (by Proposition 2.4) and (ii) follows.

Next we prove (iv). Assume that

$$0 \to K \to F_1 \to F_0 \to X \to 0$$

is an exact sequence in $\mathrm{Mod}(R)$ with F_1 and F_0 flat. By the foregoing remarks, K is $\boldsymbol{\tau}_R$-torsionfree and embeds in a $\boldsymbol{\tau}_R$-closed object with $\boldsymbol{\tau}_R$-torsionfree cokernel. Consequently, K is also $\boldsymbol{\tau}_R$-closed, by [24, Proposition IX.4.2], and hence flat. This proves that $\mathrm{w.gl.dim}\, R \leq 2$. Moreover, if we had $\mathrm{w.gl.dim}\, R = 1$, then, for each X_R we would have an exact sequence $0 \to F_1 \to F_0 \to X \to 0$, where each F_i is flat. By [24, Proposition IX.4.2], X is $\boldsymbol{\tau}_R$-torsionfree. But then the torsion theory $\boldsymbol{\tau}_R$ is trivial and every right R-module is flat, because every right R-module is $\boldsymbol{\tau}_R$-closed, so that $\mathrm{w.gl.dim}\, R = 0$, which is a contradiction. This proves that $\mathrm{w.gl.dim}\, R$ is either 0 or 2.

Now we prove (iii). Fix $\mu \in \Lambda$ and set $e = e_\mu$. If $I \subseteq eR$ is a finitely generated R-submodule of eR, and eR/I is $\boldsymbol{\delta}_R$-torsion, then $\mathrm{Hom}_R(eR/I, R) = 0$. Since eR/I is finitely presented then $\mathrm{Hom}_R(eR/I, F) = 0$ for every flat right R-module F [26, 36.8]. This means that eR/I is $\boldsymbol{\tau}_R$-torsion.

Take an epimorphism

$$L' = \bigoplus_{\lambda \in \Phi} e_\lambda R \xrightarrow{\xi} I \longrightarrow 0$$

in $\mathrm{Mod}(R)$, where Φ is a finite set of (possibly repeated) indices of Λ. By composing ξ with the inclusion $I \subseteq eR$, we get a morphism $\zeta : L' \to eR$, which is easily seen to be an epimorphism in the category $\mathrm{Fl}(R)$. By the first part of the proof, L' and eR are finitely presented in the category $\mathrm{Fl}(R)$, and hence $K = \mathrm{Ker}\, \xi$ is finitely generated. Hence there is an exact sequence

$$0 \to K' \to L' \to eR \to eR/I \to 0$$

in the category $\mathrm{Mod}(R)$ and, by condition (iv), the module K' is flat. This entails that $K' = K$ is a finitely generated object in the category $\mathrm{Fl}(R)$. But then, there is another finitely generated projective module L'' with a morphism $\epsilon : L'' \to K$ which is an epimorphism in $\mathrm{Fl}(R)$.

Let $L := \mathrm{Im}\, \varepsilon$ (in $\mathrm{Mod}(R)$). Then L is finitely generated and, for any flat right R-module F, $\mathrm{Hom}_R(K/L, F) = 0$. In particular, $\mathrm{Hom}_R(K/L, E(R_R))$ $= 0$ and therefore the module K/L is $\boldsymbol{\delta}_R$-torsion. Hence we infer that K is $\boldsymbol{\delta}_R$-finitely generated. Since the sequence $0 \to K \to L' \to I \to 0$ is exact in $\mathrm{Mod}(R)$, we conclude that I is $\boldsymbol{\delta}_R$-finitely presented.

The proof of (v) is analogous to the proof of a corresponding property in [7, Theorem 3].

In order to prove the converse we assume that R satisfies the conditions (i)–(v). By (i), (ii) and Proposition 2.5, R is a right FTF ring, with the torsion radical $\boldsymbol{\tau}_R$. Since any injective flat module is clearly $\boldsymbol{\tau}_R$-closed then according to (v) each module $e_\lambda R$ is also $\boldsymbol{\tau}_R$-closed, because the localization functor preserves kernels. It follows that any finitely generated projective module is $\boldsymbol{\tau}_R$-closed.

We now prove that direct limits of $\boldsymbol{\tau}_R$-closed modules are again $\boldsymbol{\tau}_R$-closed; then, it will follow from [26, 36.5] that every flat right R-module is $\boldsymbol{\tau}_R$-closed.

We denote by $\mathrm{Mod}(R,\boldsymbol{\tau}_R)$ the quotient category of $\mathrm{Mod}(R)$ with respect to $\boldsymbol{\tau}_R$ (see [24]). We shall show that each $e_\lambda R$ is finitely presented in $\mathrm{Mod}(R,\boldsymbol{\tau}_R)$.

Note first that, since R is a right FTF ring, by hypothesis, an argument analogous to [9, Proposition 1.3.6] shows that for each $e_\lambda R$, the filter of submodules $L \subseteq e_\lambda R$ such that $e_\lambda R/L$ is $\boldsymbol{\tau}_R$-torsion, has a basis consisting of finitely generated submodules of $e_\lambda R$. As in [24, Proposition XIII.1.1] we show that each $e_\lambda R$ is a finitely generated object of the category $\mathrm{Mod}(R,\boldsymbol{\tau}_R)$.

For any $\lambda \in \Lambda$, fix an epimorphism $\xi : F \to e_\lambda R$ in the category $\mathrm{Mod}(R,\boldsymbol{\tau}_R)$, with F a finitely generated object, and set $K := \mathrm{Ker}\,\xi$. By applying the inclusion functor, we get an exact sequence

$$0 \to K \to F \to L \to 0$$

in $\mathrm{Mod}(R)$ with $L \subseteq e_\lambda R$. Note that F contains a finitely generated submodule F' such that F/F' is $\boldsymbol{\tau}_R$-torsion, and there exists an exact sequence

$$0 \to K' \to F' \to L' \to 0$$

where $K' \subseteq K$, $L' \subseteq L$ and K/K', L/L' are $\boldsymbol{\tau}_R$-torsion.

Consequently, without loss of generality, we may assume that F is finitely generated also as a right R-module. The condition that ξ is an epimorphism implies that $\mathrm{Hom}_R(e_\lambda R/L, F') = 0$ for any flat right R-module F', because all such modules F' are $\boldsymbol{\tau}_R$-torsionfree. In particular, $\mathrm{Hom}_R(e_\lambda R/L, E(R_R)) = 0$, by assumptions (i), (ii) and Proposition 2.5, and thus $e_\lambda R/L$ is $\boldsymbol{\delta}_R$-torsion. By our hypothesis (iii), the module L is $\boldsymbol{\delta}_R$-finitely presented. By using the same arguments as in [7, Theorem 3], we get another exact sequence

$$0 \to X \to Y \to L \to 0$$

with Y a finitely presented right R-module and X a $\boldsymbol{\delta}_R$-torsion module. Hence we deduce that X is also $\boldsymbol{\tau}_R$-torsion, so that L is $\boldsymbol{\tau}_R$-finitely presented. Then, K has to be $\boldsymbol{\tau}_R$-finitely generated and therefore it is a finitely generated object of the category $\mathrm{Mod}(R,\boldsymbol{\tau}_R)$. This means that $e_\lambda R$ is finitely presented.

Suppose now that $\{M_i\}$ is a direct system of $\boldsymbol{\tau}_R$-closed objects and let $M := \varinjlim M_i$. There is also a direct limit $M_{\boldsymbol{\tau}_R}$ of the system $\{M_i\}$ in the category $\mathrm{Mod}(R, \boldsymbol{\tau}_R)$. Since $e_\lambda R$ is finitely presented, we have

$$\varinjlim \mathrm{Hom}_{\mathrm{Mod}(R,\tau_R)}(e_\lambda R, M_i) \cong \mathrm{Hom}_{\mathrm{Mod}(R,\tau_R)}(e_\lambda R, M_{\boldsymbol{\tau}_R}) \cong M_{\boldsymbol{\tau}_R} e_\lambda.$$

But there exist analogous isomorphisms in $\mathrm{Mod}(R)$ with M and $M_{\boldsymbol{\tau}_R}$ interchanged. Hence we deduce that each canonical homomorphism $Me_\lambda \to M_{\boldsymbol{\tau}_R} e_\lambda$ is an isomorphism. It follows that

$$M = \bigoplus_{\lambda\in\Lambda} Me_\lambda \cong \bigoplus_{\lambda\in\Lambda} M_{\boldsymbol{\tau}_R} e_\lambda = M_{\boldsymbol{\tau}_R}.$$

This shows that direct limits of $\boldsymbol{\tau}_R$-closed objects are $\boldsymbol{\tau}_R$-closed and consequently every flat right R-module is $\boldsymbol{\tau}_R$-closed.

It remains to prove that all $\boldsymbol{\tau}_R$-closed modules are flat, to conclude that $\mathrm{Fl}(R)$ is equivalent to $\mathrm{Mod}(R, \boldsymbol{\tau}_R)$, and, hence, is a Grothendieck category. But this follows from the condition (iv) in a similar way to [7, Theorem 3]. ∎

Now we shall show that, in some sense, condition (v) of Theorem 2.7 is not essential.

THEOREM 2.8. *Assume that R is a unitary ring with the following properties*:

(a) *R is a right FTF ring.*
(b) *R is right weakly $\boldsymbol{\delta}_R$-coherent.*
(c) *The weak global dimension* $\mathrm{w.gl.dim}\, R$ *of R is ≤ 2.*

If Q is the ring of quotients of R with respect to the hereditary torsion theory $\boldsymbol{\tau}_R$ (see [24]) then Q is a (unitary) right panoramic ring.

Proof. It is well-known (see e.g. [24, Chapter X.2]) that the quotient category $\mathrm{Mod}(R, \boldsymbol{\tau}_R)$ of $\mathrm{Mod}(R)$ with respect to $\boldsymbol{\tau}_R$ is also a quotient category of $\mathrm{Mod}(Q)$, namely the quotient category corresponding to a hereditary torsion theory, say $\boldsymbol{\tau}_R^*$ of $\mathrm{Mod}(Q)$. By [9, Teorema 2.3.4], Q is also a right FTF ring, with corresponding torsion theory $\boldsymbol{\tau}'_R$ such that a right Q-module X_Q is $\boldsymbol{\tau}'_R$-torsion (respectively, $\boldsymbol{\tau}'_R$-torsionfree) if and only if X_R is $\boldsymbol{\tau}_R$-torsion (resp. $\boldsymbol{\tau}_R$-torsionfree). By [24, Proposition X.2.2], for any right ideal I of Q, one has: Q/I is $\boldsymbol{\tau}_R^*$-torsion if and only if $R/R \cap I$ is $\boldsymbol{\tau}_R$-torsion, and this happens if and only if Q/I is $\boldsymbol{\tau}_R$-torsion in $\mathrm{Mod}(R)$. Now, our previous remark shows that $\boldsymbol{\tau}_R^* = \boldsymbol{\tau}'_R$.

We now claim that every object X_R of $\mathrm{Mod}(R, \boldsymbol{\tau}_R)$ is a flat right R-module. This is clear for X_R being injective, because $\boldsymbol{\tau}_R$-torsionfree objects embed in flat modules. Now, if X_R is not injective, then its injective hull $E(X)$ has to be flat and the factor module $E(X)/X$ is $\boldsymbol{\tau}_R$-torsionfree. If we apply the hypothesis that $\mathrm{w.gl.dim}\, R \le 2$, we see that X_R is also flat.

We show next that all flat right Q-modules belong to the category $\mathrm{Mod}(Q, \boldsymbol{\tau}_R^*)$. This is true for the module Q_Q, as Q is the ring of quotients of R, and hence, this is also true for every finitely generated projective right Q-module. Now, the assumption that R is right weakly $\boldsymbol{\delta}_R$-coherent implies, as in the proof of Theorem 2.7, that Q is a finitely presented object of the category $\mathrm{Mod}(Q, \boldsymbol{\tau}_R^*) = \mathrm{Mod}(R, \boldsymbol{\tau}_R)$, and that direct limits in $\mathrm{Mod}(Q)$ of $\boldsymbol{\tau}_R^*$-closed objects are still $\boldsymbol{\tau}_R^*$-closed. This implies that all flat right Q-modules are $\boldsymbol{\tau}_R^*$-closed, i.e., they belong to $\mathrm{Mod}(Q, \boldsymbol{\tau}_R^*)$.

Finally, let us prove that every $\boldsymbol{\tau}_R^*$-closed object is a flat right Q-module. We have already seen that, if X_Q is in $\mathrm{Mod}(Q, \boldsymbol{\tau}_R^*) = \mathrm{Mod}(R, \boldsymbol{\tau}_R)$, then X_R is flat. This means that there exists a direct system $\{P_i, f_{ji}\}_{i\in I}$ of finitely generated projective right R-modules such that $\varinjlim P_i \cong X_R$. For each $i \in I$, there exist a positive integer n_i and a module Q_i such that $P_i \oplus Q_i = R^{n_i}$. By applying the localization functor

$$\mathbf{a} : \mathrm{Mod}(R) \to \mathrm{Mod}(R, \boldsymbol{\tau}_R)$$

and keeping in mind that $\mathbf{a}(R^{n_i}) \cong Q^{n_i}$, as $\mathbf{a}$ commutes with finite direct sums, we see that $\mathbf{a}(P_i) = P_i'$ is a finitely generated projective right Q-module. Thus, there is a corresponding direct system $\{P_i', \mathbf{a}(f_{ji})\}$, both in the categories $\mathrm{Mod}(R)$ and $\mathrm{Mod}(Q)$.

Now, we may compute the direct limit of this system either in $\mathrm{Mod}(R)$ or in the quotient category $\mathrm{Mod}(R, \boldsymbol{\tau}_R)$. The fact that direct limits of $\boldsymbol{\tau}_R$-closed objects are $\boldsymbol{\tau}_R$-closed (that we have already seen, and depends on Q being finitely presented in the quotient category), implies that these two limits coincide. But the functor $\mathbf{a}$ is left adjoint and commutes with direct limits, so that $X \cong \mathbf{a}(\varinjlim P_i) \cong \varinjlim(\mathbf{a}(P_i))$ in $\mathrm{Mod}(R)$. Hence there is an isomorphism $X \cong \varinjlim P_i'$ in any of these two categories. But then, $X \cong \varinjlim P_i'$ also in $\mathrm{Mod}(Q)$, as one sees by applying the restriction of scalars functor to this last direct limit. And this shows finally that X_Q is isomorphic to a direct limit of finitely generated projective right Q-modules, from which it follows that X_Q is flat. This completes the proof. ■

We now conclude from the theorem above some important consequences (see [7, Corollary 5] for the unitary case).

COROLLARY 2.9. *A left and right locally $\boldsymbol{\delta}_R$-coherent ring $R = \bigoplus_{\lambda\in\Lambda} Re_\lambda = \bigoplus_{\lambda\in\Lambda} e_\lambda R$ is right panoramic if and only if* $\mathrm{w.gl.dim}\, R \le 2$ *and for each $\lambda \in \Lambda$ the flat-dominant dimension of $e_\lambda R$ is ≥ 2.*

Proof. The "only if" part follows from Theorem 2.7. In order to prove the converse implication we note first that by the assumption on the flat-dominant dimension of the $e_\lambda R$ the injective hull $E(e_\lambda R)$ is flat for all $\lambda \in \Lambda$. At the same time, by [12, Proposition 1.6] (translating their proof into the case of non-unitary rings) we may deduce that the left R-module

$E(Re_\lambda)$ is flat for every $\lambda \in \Lambda$. An argument similar to that in [11, Proposition 2.2] shows that R is a right FTF ring. Thus R is right panoramic by Theorem 2.7. ■

We can also use the preceding result along with some new facts on FTF rings to get a left-right symmetry of panoramic rings.

COROLLARY 2.10. *Every right panoramic and right locally $\boldsymbol{\delta}_R$-coherent ring $R = \bigoplus_{\lambda\in\Lambda} Re_\lambda = \bigoplus_{\lambda\in\Lambda} e_\lambda R$ is also left panoramic.*

Proof. Since R is right panoramic, it is left locally ($\boldsymbol{\delta}_R$-)coherent, by Theorem 2.7. By an obvious version of [10, Corollary 1.11] for non-unitary rings, R is also a left FTF ring and the injective hull $E(Re_\lambda)$ is a flat module for every $\lambda \in \Lambda$. Therefore R is left panoramic by Theorem 2.7. ■

Note that if the conditions in Corollary 2.10 do hold, then R has to be in fact a left and right coherent ring. We will see in the next section that if one does not assume this condition of being locally $\boldsymbol{\delta}_R$-coherent, then right panoramic rings need not be left panoramic.

We finish this section by the following result.

COROLLARY 2.11. *A ring $R = \bigoplus_{\lambda\in\Lambda} Re_\lambda = \bigoplus_{\lambda\in\Lambda} e_\lambda R$ is right and left panoramic if and only if the following three conditions are satisfied*:

(a) *R is left and right locally ($\boldsymbol{\delta}_R$-)coherent*,
(b) w.gl.dim $R \le 2$, *and*
(c) *the flat-dominant dimension of $e_\lambda R$ is ≥ 2 for every $\lambda \in \Lambda$.*

Proof. The sufficiency follows from Corollaries 2.10 and 2.11. The necessity is an immediate consequence of Theorem 2.7, because every right panoramic ring is left locally coherent. ■

3. Right panoramic rings and locally finitely presented Grothendieck categories. We start with the following definition.

DEFINITION 3.1. Let $\mathcal{C}$ be a locally finitely presented Grothendieck category. A family $\{G_\lambda\}_{\lambda\in\Lambda}$ of finitely presented objects of $\mathcal{C}$ is called a *family of finitely presented strong generators* in case every finitely presented object of $\mathcal{C}$ is isomorphic to a direct summand of a finite direct sum of objects G_λ.

If G is a finitely presented object of $\mathcal{C}$ such that $\{G\}$ is a family of finitely presented strong generators for $\mathcal{C}$, we shall say that G is a *strong finitely presented generator* of $\mathcal{C}$.

It is clear that every locally finitely presented Grothendieck category $\mathcal{C}$ has a family of finitely presented strong generators, namely the family of representatives of isomorphism classes of all finitely presented objects in $\mathcal{C}$ has this property (this is easily seen to be a set). However, even module categories over unitary rings could fail to have a strong finitely presented generator (as is the case with the category of abelian groups).

Recall also the following construction (see [5], [26, p. 483] or [6, p. 138]).

DEFINITION 3.2. Let $\mathcal{X} = \{X_\lambda\}_{\lambda\in\Lambda}$ be a family of objects of an abelian category $\mathcal{C}$. The (*Gabriel*) *functor ring* of the family $\mathcal{X}$ is the abelian group

$$R_{\mathcal{X}} = \bigoplus_{\lambda\in\Lambda}\bigoplus_{\mu\in\Lambda}\mathrm{Hom}_{\mathcal{C}}(X_\lambda, X_\mu)$$

equipped with an obvious addition and with the multiplication induced by the morphism composition in $\mathcal{C}$.

We are able to prove now our basic result of this section (compare with [19] and [20]).

THEOREM 3.3. *The map* $R \mapsto \mathrm{Fl}(R)$ *induces a one-to-one correspondence between*:

(a) *Morita equivalence classes of right panoramic rings* $R = \bigoplus_{\lambda\in\Lambda} Re_\lambda = \bigoplus_{\lambda\in\Lambda} e_\lambda R$; *and*

(b) *Equivalence classes of locally finitely presented Grothendieck categories.*

The inverse correspondence is given by attaching to any family $\mathcal{G} = \{G_\lambda\}_{\lambda\in\Lambda}$ *of finitely presented strong generators of the locally finitely presented Grothendieck category* $\mathcal{C}$ *the Gabriel functor ring* $R_{\mathcal{G}}$ *of the family* $\mathcal{G}$ (*see* 3.2). *The functor*

$$\mathbf{h}_\bullet : \mathcal{C} \longrightarrow \mathrm{Fl}(R_{\mathcal{G}}), \tag{3.4}$$

$C \mapsto \mathbf{h}_C = \bigoplus_{\lambda\in\Lambda}\mathrm{Hom}_{\mathcal{C}}(G_\lambda, C)$, *is an equivalence of categories.*

Proof. If R is a right panoramic ring then, according to Theorem 2.7, $\{e_\lambda R\}$ is a family of finitely presented generators for the category $\mathrm{Fl}(R)$. This shows that $\mathrm{Fl}(R)$ is a locally finitely presented Grothendieck category. Thus, the equivalence class of $\mathrm{Fl}(R)$ will be the image of the Morita equivalence class of R, because it is easy to see that for any pair R and S of Morita equivalent right panoramic rings the categories of flat modules $\mathrm{Fl}(R)$ and $\mathrm{Fl}(S)$ are equivalent.

All this shows that $R \mapsto \mathrm{Fl}(R)$ defines a map from the Morita equivalent classes of right panoramic rings $R = \bigoplus_{\lambda\in\Lambda} Re_\lambda = \bigoplus_{\lambda\in\Lambda} e_\lambda R$ to the equivalence classes of locally finitely presented Grothendieck categories $\mathcal{C}$.

It remains to show that this mapping is a bijection. In order to show that this map is injective we suppose that R and S are right panoramic rings and there exists an equivalence of categories

$$F : \mathrm{Fl}(R) \to \mathrm{Fl}(S).$$

What we need to do is to lift F to an equivalence of the categories $\mathrm{Mod}(R)$ and $\mathrm{Mod}(S)$. To this end, we are going to determine all finitely presented objects of the category $\mathrm{Fl}(R)$. We recall that there exists a localization functor

$$\mathbf{a} : \mathrm{Mod}(R) \to \mathrm{Fl}(R)$$

which is a left adjoint of the inclusion functor, so that $\mathbf{a}$ preserves direct limits.

Assume that $R = \bigoplus_{\lambda\in\Lambda} e_\lambda R = \bigoplus_{\lambda\in\Lambda} Re_\lambda$. We know that each $e_\lambda R$ is a finitely presented object of the category $\mathrm{Fl}(R)$. Now, let X be a finitely presented object in $\mathrm{Fl}(R)$. Since X is a flat right R-module, it is a direct limit

$$X \cong \varinjlim M_i$$

in the category $\mathrm{Mod}(R)$ of finitely presented projective modules and each of the modules M_i is isomorphic to a direct summand of a finite direct sum of objects of the form $e_\lambda R$. Since the functor $\mathbf{a}$ preserves direct sums, we deduce that

$$X \cong \varinjlim M_i$$

also in the category $\mathrm{Fl}(R)$, and that each M_i is isomorphic (also in $\mathrm{Fl}(R)$) to a direct summand of a finite direct sum of objects of the form $e_\lambda R$. Now, since X is finitely presented in $\mathrm{Fl}(R)$, there are isomorphisms

$$\mathrm{Hom}(X, X) \cong \mathrm{Hom}(X, \varinjlim M_i) \cong \varinjlim(\mathrm{Hom}(X, M_i)).$$

Therefore, the isomorphism $X \cong \varinjlim M_i$ has a factorization through some M_i. It follows that X is isomorphic to a direct summand of a finite direct sum of modules of the form $e_\lambda R$. This describes (up to isomorphism) the family of finitely presented objects of the category $\mathrm{Fl}(R)$. Indeed, this shows that $\mathbf{a}$ restricts to an equivalence

$$\mathrm{proj}(R) \to \mathrm{fp}(\mathrm{Fl}(R))$$

between the category $\mathrm{proj}(R)$ of all finitely generated projective objects of $\mathrm{Mod}(R)$ and the category $\mathrm{fp}(\mathrm{Fl}(R))$ of finitely presented objects of $\mathrm{Fl}(R)$.

On the other hand, the equivalence $F : \mathrm{Fl}(R) \to \mathrm{Fl}(S)$ carries finitely presented objects of the category $\mathrm{Fl}(R)$ exactly onto the finitely presented objects of the category $\mathrm{Fl}(S)$ and therefore induces a new category equivalence

$$G : \mathrm{proj}(R) \to \mathrm{proj}(S).$$

For each $\lambda \in \Lambda$ we set $G(e_\lambda R) = N_\lambda$, and consider the projective right S-module

$$P_S := \bigoplus_{\lambda\in\Lambda} N_\lambda.$$

Since every object of proj(R) is, in this category, a summand of a finite direct sum of objects of the form $e_\lambda R$, we see that every object in proj(S) is also a summand of a finite direct sum of N_λ's. By applying this to S we deduce that P_S generates S_S, and, therefore, P_S is a projective generator of the category $\mathrm{Mod}(S)$.

We also remark that, since G preserves finite direct sums and so does the inclusion functor $\mathrm{proj}(R) \to \mathrm{Mod}(R)$, we have $G(R) = P_S$. Since G is an (additive) equivalence, there is a ring isomorphism $\mathrm{End}(P_S) \cong \mathrm{End}(R_R)$.

According to the notation and terminology of [8, p. 52] we denote by $f\,\mathrm{End}(P_S)$ the subring of $\mathrm{End}(P_S)$ defined by the formula

$$\alpha \in f\,\mathrm{End}(P_S) \Leftrightarrow \alpha = g \circ h,\ h : P_S \to S^n,\ g : S^n \to P_S,\ \text{and}$$
$$g(s_1, \ldots, s_n) = \sum_{i=1}^{n} x_i s_i \text{ for some fixed } x_i \in P_S.$$

It is easy to see that an S-endomorphism $\alpha : P_S \to P_S$ belongs to $f\,\mathrm{End}(P_S)$ if and only if $\mathrm{Im}\,\alpha$ is contained in a finitely generated submodule of P_S, that is,

$$\alpha \in f\,\mathrm{End}(P_S) \Leftrightarrow \mathrm{Im}\,\alpha \subseteq \bigoplus_{\lambda\in J} N_\lambda \quad \text{with } J \subseteq \Lambda,\ J \text{ finite.}$$

Let $E = \mathrm{End}(P_S)$ and let $\theta : E \to \mathrm{End}(R_R)$ be the isomorphism induced by the functor G defined above. Since this is the isomorphism induced by G and $G(e_\lambda R) = N_\lambda$, it is easy to see that the image $\theta(f\,\mathrm{End}(P_S)) = R'$ of θ is described by the formula

$$\beta \in R' \Leftrightarrow \mathrm{Im}\,\beta \subseteq \bigoplus_{\lambda\in J} e_\lambda R, \quad \text{with } J \subseteq \Lambda,\ J \text{ finite.}$$

On the other hand, we may view R as a left ideal of the ring $\mathrm{End}(R_R)$ in an obvious way. Under this identification, it is clear that for every $\beta \in R'$, there exists $t \in R$ such that t fixes (acting on the left) all elements of $\mathrm{Im}\,\beta$. Hence $\beta t = \beta$ and it follows that $R'R = R'$. Moreover, it is clear that if $t \in R$, then $t \in R'$, because $\mathrm{Im}\,t = tR$ is contained in the finite direct sum $\bigoplus_{\lambda\in J} e_\lambda R$ of modules $e_\lambda R$, if $t \in \bigoplus_{\lambda\in J} e_\lambda R$ and $J \subseteq \Lambda$ is finite. This shows that $RR' = R$. If we now apply [16, Proposition 1.3], we infer that the rings R and R' are Morita equivalent. On the other hand, in view of the isomorphism θ the ring R' is equivalent to $f\,\mathrm{End}(P_S)$.

Finally, the ring $f\,\mathrm{End}(P_S)$ is, in turn, equivalent to S, because the natural mappings

$$P \otimes_S \mathrm{Hom}(P, S) \to f\,\mathrm{End}(P_S), \qquad \mathrm{Hom}(P, S) \otimes_E P \to S$$

give a Morita context with surjective homomorphisms and [2, Theorem 2.2] applies. Consequently, the rings R and S are Morita equivalent, as we required.

It remains to prove that the mapping $R \to \mathrm{Fl}(R)$ is a surjection. For this purpose we shall describe the inverse map.

Let $\mathcal{C}$ be a locally finitely presented Grothendieck category and let $\mathcal{G} = \{G_\lambda\}_{\lambda\in\Lambda}$ be a set of finitely presented strong generators. We have to construct a ring R with enough idempotents in such a way that the category $\mathrm{Fl}(R)$ is equivalent to the given category $\mathcal{C}$. We take for R the functor ring $R_{\mathcal{G}}$ of the family $\mathcal{G}$. It follows from [19, Sec. 3] and [20, Corollary 2.9] that the functor $\mathbf{h}_\bullet : \mathcal{C} \longrightarrow \mathrm{Fl}(R_{\mathcal{G}})$, $C \mapsto \mathbf{h}_C = \bigoplus_{\lambda\in\Lambda} \mathrm{Hom}_{\mathcal{C}}(G_\lambda, C)$, is an equivalence of categories (see also [13] and [6, Proposition 1.3 and following comments]), because it is easy to check that the functor $\mathrm{Add}(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathcal{A}b) \longrightarrow \mathrm{Mod}(R_{\mathcal{G}})$, $T \mapsto \bigoplus_{\lambda\in\Lambda} T(G_\lambda)$, is an equivalence of categories, where $\mathrm{Add}(\mathrm{fp}(\mathcal{C})^{\mathrm{op}}, \mathcal{A}b)$ is the category of additive contravariant functors from $\mathrm{fp}(\mathcal{C})$ to the category of abelian groups. The equivalence carries flat functors to flat modules.

This shows that the mapping $R \mapsto \mathrm{Fl}(R)$ is a bijection and its inverse is given by $\mathcal{C} \mapsto R_{\mathcal{G}}$, where $R_{\mathcal{G}}$ is the functor ring of a family $\mathcal{G}$ of finitely presented strong generators of $\mathcal{C}$. This finishes the proof. ∎

We want to emphasize a couple of properties that can be derived from the proof of Theorem 3.3, and so we present them now as a corollary.

COROLLARY 3.5. (a) *In the one-to-one correspondence of Theorem* 3.3, *given a locally finitely presented Grothendieck category* $\mathcal{C}$ *with a family* $\mathcal{G} = \{G_\lambda\}_{\lambda\in\Lambda}$ *of finitely presented strong generators*, *the* (*Morita equivalence class of the*) *corresponding right panoramic ring is the functor ring* $R_{\mathcal{G}}$ *of the family* $\mathcal{G}$.

(b) *The category equivalence* $\mathbf{h}_\bullet : \mathcal{C} \to \mathrm{Fl}(R)$ (see (3.4)) *induces an equivalence*

$$\mathbf{h}'_\bullet : \mathrm{fp}(\mathcal{C}) \to \mathrm{proj}(R)$$

from the category of finitely presented objects of $\mathcal{C}$ *to the category of finitely generated projective objects of* $\mathrm{Mod}(R)$. ∎

COROLLARY 3.6. *There exists a right panoramic ring which is not left panoramic.*

Proof. Let $\mathcal{C}$ be a locally finitely presented Grothendieck category which is not locally coherent. Let $\mathcal{G} = \{G_\lambda\}_{\lambda\in\Lambda}$ be a family of finitely presented strong generators of $\mathcal{C}$. Let $R_{\mathcal{G}} = \bigoplus_{\lambda\in\Lambda} e_\lambda R = \bigoplus_{\lambda\in\Lambda} Re_\lambda$ be the functor ring of the family $\mathcal{G}$.

By Corollary 2.11, the category $\mathcal{C}$ is equivalent to the category $\mathrm{Fl}(R_{\mathcal{G}})$, and therefore the ring $R_{\mathcal{G}}$ is right panoramic.

We claim that $R_{\mathcal{G}}$ is not left panoramic. Assume to the contrary that $R_{\mathcal{G}}$ is also left panoramic. Then, by Corollary 3.5, $R_{\mathcal{G}}$ is locally right coherent and therefore $e_\lambda R_{\mathcal{G}}$ is a coherent object of $\mathrm{Mod}(R_{\mathcal{G}})$ for every $\lambda \in \Lambda$. Let L be a finitely generated subobject of $e_\lambda R_{\mathcal{G}}$ in the category $\mathrm{Fl}(R_{\mathcal{G}})$. There is an epimorphism $\varepsilon : M \to L$ in the category $\mathrm{Fl}(R_{\mathcal{G}})$, where M is a direct sum of summands of modules of the form $e_\mu R_{\mathcal{G}}$. By applying the inclusion functor, we derive an epimorphism $M \to L'$, where $L' \subseteq L$. If

$$\mathbf{a} : \mathrm{Mod}(R_{\mathcal{G}}) \to \mathrm{Fl}(R_{\mathcal{G}})$$

is the localization functor that corresponds to the torsion theory $\boldsymbol{\tau}_{R_{\mathcal{G}}}$ of $\mathrm{Mod}(R_{\mathcal{G}})$ then $\mathbf{a}(L') = L$. Since L' is finitely generated we derive an exact sequence

$$M' \to M \to L' \to 0$$

in the category $\mathrm{Mod}(R_{\mathcal{G}})$, where M' is a finitely generated projective module. By applying the exact functor $\mathbf{a}$ we get an exact sequence

$$M' \to M \to \mathbf{a}(L') = L \to 0$$

in the category $\mathrm{Fl}(R_{\mathcal{G}})$. This shows that L is a finitely presented object in $\mathrm{Fl}(R_{\mathcal{G}})$, and therefore $e_\lambda R_{\mathcal{G}}$ is a coherent object of the category $\mathrm{Fl}(R_{\mathcal{G}})$. Consequently, $\mathrm{Fl}(R_{\mathcal{G}})$ is a locally coherent category and we get a contradiction. It follows that the ring $R_{\mathcal{G}}$ is not left panoramic and the corollary follows. ■

We want now to restrict the bijection of Theorem 3.3 to particular subclasses of right panoramic rings and of locally finitely presented Grothendieck categories.

COROLLARY 3.7. *The map* $R \mapsto \mathrm{Fl}(R)$ *defines a one-to-one correspondence between*:

(a) *Morita equivalence classes of right panoramic unitary rings* R; *and*

(b) *Equivalence classes of Grothendieck categories* $\mathcal{C}$ *with a finitely presented strong generator* G.

The inverse map is defined by the formula $\mathcal{C} \mapsto \mathrm{End}_{\mathcal{C}}(G)$.

P r o o f. Assume that R is a right panoramic and unitary ring. By Corollary 3.5, the finitely presented objects of $\mathrm{Fl}(R)$ are exactly the finitely generated projective right R-modules. Then the module R_R is a finitely presented strong generator of the category $\mathrm{Fl}(R)$ and therefore the corollary is a consequence of Theorem 3.3. ■

R e m a r k 3.8. It follows from Theorem 3.3 and Corollary 3.7 that there exist right panoramic rings which are not unitary, because there exist locally finitely presented Grothendieck categories without a finitely presented strong generator.

4. The problem (JS) and panoramic rings. We now discuss the problem (JS) stated in [15] and presented in the introduction in connection with our results on panoramic rings given in the previous sections. While the following result cannot settle the problem (JS), it could help in finding an answer.

THEOREM 4.1. *The map* $R \mapsto \mathrm{Fl}(R)$ *defines a one-to-one correspondence between*:

(a) *Morita equivalence classes of unitary right panoramic rings* R *such that* R_R *is a direct sum of indecomposable right ideals*; *and*

(b) *Equivalence classes of locally finitely presented Grothendieck categories* $\mathcal{C}$ *with a finite family of indecomposable finitely presented strong generators.*

Proof. We shall apply the correspondence given in Theorem 3.3. For this purpose we assume that R is a unitary and right panoramic ring, and that $R_R = \bigoplus_{i=1}^{n} I_i$ is a direct sum of indecomposable right ideals $I_1, \ldots, I_n$. The obvious equivalence $F : \mathrm{proj}(R) \to \mathrm{fp}(\mathrm{Fl}(R))$ shows that every finitely presented object of the Grothendieck category $\mathcal{C} = \mathrm{Fl}(R)$ is isomorphic to a summand of a direct sum of finitely many objects of the form $F(I_1), \ldots, F(I_n)$. It follows that the modules $F(I_1), \ldots, F(I_n)$ form a family of finitely presented strong generators of the category $\mathrm{Fl}(R)$. By our assumption $F(I_1), \ldots, F(I_n)$ are indecomposable objects of $\mathrm{Fl}(R)$ and therefore the category $\mathcal{C} = \mathrm{Fl}(R)$ has the properties required in (b).

In order to finish the proof we take any locally finitely presented Grothendieck category $\mathcal{C}$ as in (b), and assume that $\mathcal{U} = \{U_1, \ldots, U_n\}$ is a set of indecomposable finitely presented strong generators of $\mathcal{C}$. If $U = U_1 \oplus \ldots \oplus U_n$ and $R = \mathrm{End}_{\mathcal{C}}(U)$, then according to [19], [20] (see also [13]) the functor $\mathcal{C} \to \mathrm{Fl}(R)$, $C \mapsto \mathrm{Hom}_{\mathcal{C}}(U, C)$, is an equivalence of categories, and therefore the ring R corresponds to $\mathcal{C}$ in the bijective correspondence of Theorem 3.3. It follows that $I_i := \mathrm{Hom}_{\mathcal{C}}(U, U_i)$ is an indecomposable right ideal of R and $R_R = I_1 \oplus \ldots \oplus I_n$. This finishes the proof. ∎

Comparing Theorem 4.1 with the problem (JS) leads to the following interesting question:

Is every category $\mathcal{C}$ *with the properties stated in* (b) *of Theorem* 4.1 *necessarily equivalent to a module category*?

In connection with this question we could try to determine the panoramic rings corresponding to module categories by the correspondence of Theorem 3.3. The following observation will be useful.

LEMMA 4.2. *Assume that* $R = \bigoplus_{\lambda \in \Lambda} Re_\lambda = \bigoplus_{\lambda \in \Lambda} e_\lambda R$ *is a right panoramic ring.*

(a) *Every projective object of the Grothendieck category* $\mathrm{Fl}(R)$ *is projective as a right R-module.*

(b) *A projective module* P_R *is a projective object of* $\mathrm{Fl}(R)$ *if and only if* P_R *has the following property*:

(P1) *Every proper submodule of* P_R *is contained in a proper flat submodule of* P_R.

(c) *A projective object* P_R *of* $\mathrm{Fl}(R)$ *is a projective generator of* $\mathrm{Fl}(R)$ *if and only if the following condition holds*:

(P2) *Every nonzero right R-module X such that* $\mathrm{Hom}_R(P, X) = 0$ *is of flat dimension* 2.

Proof. (a) Assume that P is a projective object of $\mathrm{Fl}(R)$. As $\{e_\lambda R\}_{\lambda\in\Lambda}$ is a family of generators of $\mathrm{Fl}(R)$, there is an epimorphism $\pi : \bigoplus_{\omega\in\Omega}(e_\omega R)^{t_\omega} \to P$ which splits, because P is projective in the category $\mathrm{Fl}(R)$. It follows that π is also a splitting epimorphism in $\mathrm{Mod}(R)$ and hence P is a direct summand of the module $\bigoplus_{\omega\in\Omega}(e_\omega R)^{t_\omega}$. Then (a) follows.

(b) Suppose that P is a projective object in $\mathrm{Fl}(R)$ and $0 \to L \to P \to N \to 0$ is an exact sequence in $\mathrm{Mod}(R)$. Suppose also that L is not contained in any proper flat submodule of P. Let $\alpha : N \to F$ be a homomorphism, with F a flat right R-module, and set $C := \mathrm{Coker}\,\alpha$. Consider the exact sequence

$$0 \to L' \to P \to F \to C \to 0$$

in $\mathrm{Mod}(R)$. It follows from Theorem 2.7 that $\mathrm{w.gl.dim}R \le 2$. Hence we deduce that the module L' is flat. Since $L \subseteq L' \subseteq P$, our assumption yields $L' = P$ and $\alpha = 0$. This shows that $\mathrm{Hom}_R(N, F) = 0$ for any flat module F.

Let $M \to L$ be an epimorphism in $\mathrm{Mod}(R)$, where M is flat. Since $\mathrm{Hom}_R(N, F) = 0$ for all flat modules F, the composed morphism $\beta : M \to P$ is an epimorphism in the category $\mathrm{Fl}(R)$. Since P is projective in the category $\mathrm{Fl}(R)$, β is splittable and, therefore β is an epimorphism of right R-modules. Hence $\mathrm{Im}\,\beta = L = P$ and therefore L is not a proper submodule of P, a contradiction. This shows that P has the property (P1).

To prove the converse in (b), assume that P_R is a projective module in $\mathrm{Mod}(R)$ and (P1) holds. Let $\beta : M \to P$ be an epimorphism in $\mathrm{Fl}(R)$. By applying the inclusion functor $\mathrm{Fl}(R) \to \mathrm{Mod}(R)$, we get a homomorphism β, with $\mathrm{Im}\,\beta = L \subseteq P$. If the inclusion $L \subseteq P$ is proper, we deduce from (P1) that there exists a proper flat submodule $L' \subset P$ such that $L \subseteq L'$. But then, the inclusion morphism $L' \to P$ would be a factor of the epimorphism β in the category $\mathrm{Fl}(R)$, and therefore this inclusion would be an isomorphism. This implies that it would be also an isomorphism in $\mathrm{Mod}(R)$, and hence $L' = P$, which is a contradiction. It follows that the inclusion $L \subseteq P$ is not proper and β is an R-module epimorphism. Since P is projective in $\mathrm{Mod}(R)$ the morphism β splits in both categories. This finishes the proof of (b).

(c) Assume that P_R is a projective generator in $\mathrm{Fl}(R)$, and let $X_R \neq 0$ be any right R-module of flat dimension ≤ 1. Take a flat presentation

$$0 \longrightarrow F_1 \xrightarrow{\alpha} F_0 \longrightarrow X \longrightarrow 0$$

of X_R. Since $X \neq 0$, α is not an epimorphism in $\mathrm{Mod}(R)$. It follows that α is not an epimorphism in $\mathrm{Fl}(R)$ (if it were, it would be an isomorphism in both categories). Let $\beta : F_0 \to F'$ be the cokernel of α in the category $\mathrm{Fl}(R)$. Since P is a projective generator, there exists a non-zero morphism $h : P \to F'$ which can be lifted to $g : P \to F_0$. But β factors through $\pi : F_0 \to X$ and therefore there exists a non-zero homomorphism $P \to X$. This shows that $\mathrm{Hom}_R(P, X) \neq 0$ and (P2) holds.

To end the proof, we have to show that, if P_R is a projective object of $\mathrm{Fl}(R)$ such that (P2) holds, then P generates the category $\mathrm{Fl}(R)$. Let $\beta : F_1 \to F_0$ be a non-zero homomorphism of flat right R-modules. If $X := \mathrm{Im}\,\beta$, then obviously the flat dimension of X_R is ≤ 1. By (P2), there exists a non-zero morphism $h : P \to X$. Since P_R is projective in $\mathrm{Mod}(R)$, h can be lifted to a morphism $g : P \to F_1$. It follows that $\beta \circ g \neq 0$. This shows that P_R is a generator of the category $\mathrm{Fl}(R)$. ∎

We may now address the question of which right panoramic rings arise from module categories. For this purpose we recall that the *filter* of a torsion theory γ of a category $\mathrm{Mod}(R)$ is the set of right ideals J of R such that R/J is γ-torsion. A torsion theory γ is said to be *generated* by the right ideal I in case the filter of γ consists precisely of all right ideals which contain I. In such situation, a right R-module X is a γ-torsion module if and only if $XI = 0$.

PROPOSITION 4.3. *There is a one-to-one correspondence between*:

(a) *Morita equivalence classes of right panoramic rings* $R = \bigoplus_{\lambda\in\Lambda} e_\lambda R = \bigoplus_{\lambda\in\Lambda} Re_\lambda$ *for which the torsion theory* $\boldsymbol{\tau}_R$ *is generated by a right ideal* I, *that is*, *the trace on* R_R *of a finitely generated projective right* R*-module* P_R; *and*

(b) *Morita equivalence classes of unitary rings* A.

The correspondence is obtained by assigning $R \mapsto A = \mathrm{End}(P_R)$ *and*, *its inverse is defined by applying the correspondence of Theorem* 3.3 *to the category* $\mathcal{C} = \mathrm{Mod}(A)$.

Proof. Suppose first that R, I and P_R are as in statement (a). We will show that P_R is a projective generator of $\mathrm{Fl}(R)$. Since it is a finitely generated right R-module, it will also be finitely generated in $\mathrm{Fl}(R)$, because $\{e_\lambda R\}_{\lambda\in\Lambda}$ is a family of finitely presented generators of $\mathrm{Fl}(R)$ by Theorem 2.7, and P_R is finitely generated by this family both in $\mathrm{Mod}(R)$ and in $\mathrm{Fl}(R)$. This will prove that $\mathrm{Fl}(R)$ has a finitely generated projective generator and hence $\mathrm{Fl}(R)$ is a module category equivalent to $\mathrm{Mod}(\mathrm{End}(P_R))$, by [24, Example 2, p. 233]. This will give a half of the proof.

To see that P_R is a projective generator of $\mathrm{Fl}(R)$, we show that conditions (P1) and (P2) of Lemma 4.2 hold.

In order to show (P1) we assume to the contrary that $L \subseteq P$ is such that $L \neq P$ and L is not contained in a proper flat submodule of P_R, so that P/L is a $\boldsymbol{\tau}_R$-torsion module by [24, Proposition IX.4.2]. Hence $(P/L) \cdot I = (PI + L)/L = P/L = 0$. But this is a contradiction which proves (P1) for P_R.

Next, we show that also condition (P2) is fulfilled. Assume $\mathrm{Hom}_R(P, X) = 0$ for $X \neq 0$, and that there is a short exact sequence $0 \to P_1 \to P_0 \to X \to 0$, with P_0 a projective module and P_1 a flat module. By assumption, every homomorphism from P to P_0 lifts to P_1. But, since I is the trace of P, it is easy to see that the image of any homomorphism $P \to P_0$ is contained in $P_0 I$. This shows that $P_0 I \subseteq P_1 \subseteq P_0$, so that P_0/P_1 is $\boldsymbol{\tau}_R$-torsion. But, as P_0 and P_1 are both flat modules, they are $\boldsymbol{\tau}_R$-closed, from which it follows that $P_0 = P_1$ by [24, Proposition IX.4.2] again, and thus $X = 0$, a contradiction.

We are now set for the converse part of the proof. Start with a unitary ring A and take a family $\mathcal{U} = \{U_\lambda\}_{\lambda \in \Lambda}$ of representatives of the isomorphism classes of finitely presented right A-modules. We shall show that if R is the functor ring $R_{\mathcal{U}}$ of the family $\mathcal{U}$ as in 3.2 then R has the properties required in (a). We may assume that $A = U_\lambda$ for some $\lambda \in \Lambda$, and, under the equivalence $\mathbf{h}_\bullet : \mathrm{Mod}(A) \to \mathrm{Fl}(R)$ (see (3.4)), A corresponds to a finitely generated projective generator P_R of $\mathrm{Fl}(R)$. It follows from Lemma 4.2 that P_R satisfies (P1). Since P_R is a finitely generated object of $\mathrm{Fl}(R)$, there is an epimorphism $\bigoplus_{\lambda \in F} e_\lambda R \to P$ in the category $\mathrm{Fl}(R)$, with F being a finite set (with possibly repeated λ's), and the image L of this homomorphism in $\mathrm{Mod}(R)$ has to be such that P/L is $\boldsymbol{\tau}_R$-torsion. It follows from (P1) that $L = P$. Hence P is a finitely generated right R-module. If I is the trace of P_R on R_R, then obviously I is in the filter for $\boldsymbol{\tau}_R$, because P_R is a generator of the category $\mathrm{Fl}(R)$. Moreover, suppose that I' is in this filter, so that R/I' is a $\boldsymbol{\tau}_R$-torsion module. If there is a homomorphism $h : P \to R/I'$, then if $L = \mathrm{Ker}\, h$, we would see that P/L is also $\boldsymbol{\tau}_R$-torsion. But, by (P1) this implies that $L = P$ and therefore $\mathrm{Hom}_R(P, R/I') = 0$. It follows that $I \subseteq I'$, as required. ■

We may now characterize the right panoramic unitary rings that arise from module categories with the conditions stipulated in [15]. To this end, we introduce the following concept (compare with [27]).

Definition 4.4. Let R be a unitary ring. A right ideal I of R is said to be *left faithful* if there is no non-zero $r \in R$ such that $rI = 0$. Further, a finitely generated right ideal I will be said to be a *minimal finitely generated left faithful right ideal* if I is left faithful, and I is contained in every left faithful finitely generated right ideal I'.

Now we are able to prove the following result.

PROPOSITION 4.5. *There exists a one-to-one correspondence between*:

(a) *Morita equivalence classes of right panoramic unitary rings R such that*:
  - (a1) *There is an idempotent $e \in R$ such that ReR is a minimal finitely generated left faithful right ideal.*
  - (a2) *There exist primitive idempotents $e_1, \ldots, e_n$ such that every finitely generated projective right module is isomorphic to a direct sum of the modules $e_1R, \ldots, e_nR$; and*

(b) *Morita equivalence classes of unitary rings A such that*:
  - (b1) *There are finitely many indecomposable finitely presented right A-modules.*
  - (b2) *Every finitely presented right A-module is a direct sum of indecomposable modules.*

The correspondence is given by $R \mapsto A = eRe$.

Proof. In view of Theorem 3.3 the proposition will follow from the following two statements:

(1) If R is a right panoramic ring of the class described in (a) above, then $\mathrm{Fl}(R)$ is equivalent to a module category $\mathrm{Mod}(A)$ and A is a ring as described in (b).

(2) Conversely, if A is a ring as in (b) then there exists a finitely presented strong generator U in $\mathrm{Mod}(A)$ such that $\mathrm{End}_A(U)$ is a right panoramic ring satisfying the conditions (a1) and (a2) in (a).

In order to prove the statement (1) we first show that $\mathrm{Fl}(R)$ is a module category. To this end, it is sufficient to show that the torsion theory $\boldsymbol{\tau}_R$ of $\mathrm{Mod}(R)$ is generated by the ideal ReR, which is the trace on R of the finitely generated projective module eR. Then, by Proposition 4.3, we will infer that $\mathrm{Fl}(R) \approx \mathrm{Mod}(A)$ for $A = \mathrm{End}(eR) \cong eRe$.

Let I' be a right ideal of R_R that belongs to the filter of $\boldsymbol{\tau}_R$. Let us show that $ReR \subseteq I'$. We may assume that I' is finitely generated, in view of [24, Proposition XIII.1.1], because it follows from Theorem 2.7 that R is a finitely presented object of the quotient category $\mathrm{Mod}(R, \boldsymbol{\tau}_R) \approx \mathrm{Fl}(R)$. Since $\mathrm{Hom}(R/I', R) = 0$, the right ideal I' is left faithful and hence $ReR \subseteq I'$ by our hypothesis. By the same reason ReR belongs to the filter of $\boldsymbol{\tau}_R$ and it generates the torsion theory $\boldsymbol{\tau}_R$.

It follows that $\mathrm{Fl}(R) \approx \mathrm{Mod}(eRe)$ and according to Corollary 3.5, the equivalence induces the equivalence $\mathrm{proj}(R) \approx \mathrm{fp}(\mathrm{Mod}(eRe))$. Since by our hypothesis, all objects of $\mathrm{proj}(R)$ are direct sums of a finite number of indecomposable objects, it follows that there are only finitely many indecomposable finitely presented right eRe-modules, and every finitely presented right eRe-module is a direct sum of indecomposables. This shows that the ring $A = eRe$ has the properties required in (1).

In order to prove the statement (2) we assume that A is a unitary ring as stated, and we denote by U_A the direct sum of representatives of finitely many isomorphism classes of indecomposable finitely presented right A-modules. By the hypotheses, U_A is a finitely presented strong generator for $\mathrm{Mod}(A)$ and hence the correspondence of Theorem 3.3 shows that $R = \mathrm{End}(U_A)$ is the associated right panoramic ring. Again, the conditions assumed on the finitely presented right A-modules imply that there exist finitely many indecomposable objects $e_1R, \ldots, e_nR$ of $\mathrm{proj}(R)$ such that each finitely generated projective right R-module is a direct sum of modules of the form $e_1R, \ldots, e_nR$, that is, it is isomorphic to a module $\bigoplus_{i=1}^{n} (e_iR)^{m_i}$, $m_i \ge 0$.

On the other hand, by Proposition 4.3, the torsion theory $\boldsymbol{\tau}_R$ is generated by a right ideal I which is the trace of a finitely generated projective module P_R. But we know that $P_R \cong \bigoplus_{i\in F} (e_iR)^{m_i}$, where $m_i > 0$ and F is a subset of $\{1, \ldots, n\}$. It follows that the trace of P_R on R is just ReR, where $e = \sum_{i\in F} e_i$. It remains to show that ReR is a minimal finitely generated left faithful right ideal of R.

Let I' be a finitely generated right ideal of R which is left faithful. Then $\mathrm{Hom}(R/I', R) = 0$, and hence, since R/I' is finitely presented, $\mathrm{Hom}(R/I', F) = 0$ for any flat right R-module F. This means that R/I' is $\boldsymbol{\tau}_R$-torsion and therefore I' is in the filter of $\boldsymbol{\tau}_R$. It follows from Proposition 4.3 that $ReR \subseteq I'$. This proves (2) and finishes the proof of the proposition. ∎

We finish this section by considering right panoramic rings which are either semiperfect or right perfect. The second case is included in Tachikawa's situation (see [25] and Corollary 4.7 below).

THEOREM 4.6. *The map* $R \mapsto \mathrm{Fl}(R)$ *defines a one-to-one correspondence between*:

(a) *Morita equivalence classes of right panoramic unitary semiperfect rings* R; *and*

(b) *Equivalence classes of locally finitely presented Grothendieck categories* $\mathcal{C}$ *such that the number of isomorphism classes of indecomposable finitely presented objects of* $\mathcal{C}$ *is finite*, *and every finitely presented object of* $\mathcal{C}$ *has a direct sum decomposition that complements direct summands.*

Proof. We shall prove the theorem by applying Theorem 3.3.

Assume first that R is right panoramic, unitary and semiperfect. Then $\mathcal{C} = \mathrm{Fl}(R)$ is a Grothendieck category such that $\mathrm{fp}(\mathcal{C}) = \mathrm{proj}(R)$. Then the required condition on $\mathcal{C}$ is easily obtained from the fact that R is semiperfect, by [1, Theorem 27.12].

Conversely, suppose that $\mathcal{C}$ is as in (b). Then the direct sum U of the representatives of indecomposable finitely presented objects of $\mathcal{C}$ is clearly a strong finitely presented generator for $\mathcal{C}$. It follows from Corollary 3.6 that $\mathrm{End}(U) = R$ is a right panoramic unitary ring, which is semiperfect by [1, Theorem 27.12]. Hence the theorem follows from Theorem 3.3. ∎

Finally, we consider the case in which the ring R is right perfect.

COROLLARY 4.7 (Tachikawa [25]). *There is a one-to-one correspondence between*:

(a) *Morita equivalence classes of right panoramic unitary right perfect rings* R;

(b) *Morita equivalence classes of unitary rings* A *of finite representation type.*

Proof. We shall define the required correspondence by applying Proposition 4.3.

First, we assume that R is a right panoramic and right perfect unitary ring. By Theorem 2.7 and Proposition 2.5, R is a right FTF ring. Since every flat right R-module is projective the torsion theory $\boldsymbol{\tau}_R$ coincides with the Lambek torsion theory $\boldsymbol{\delta}_R$. From the fact that R is a right FTF ring and from [9, Proposition 2.4.1] it follows that R is a semiprimary ring. In particular, R is left perfect, and by [24, Example 2, p. 192] the torsion theory $\boldsymbol{\tau}_R = \boldsymbol{\delta}_R$ is generated by an ideal of the form ReR. By Proposition 4.3, $\mathrm{Fl}(R) = \mathrm{Mod}(A)$ for a unitary ring A, and according to Theorem 3.3 the ring R is Morita equivalent to the functor ring of the finitely presented right A-modules. By [4], [19], [20], [13] the ring A is right pure semisimple, because R is right perfect. Since $\mathrm{fp}(\mathrm{Mod}(A)) \approx \mathrm{proj}(R)$, the number of the isomorphism classes of indecomposable finitely presented right A-modules is finite and therefore A is of finite representation type (see [19], [20]).

Conversely, assume that A is of finite representation type. Then A satisfies the hypothesis in (b) of Proposition 4.5, and therefore the corresponding right panoramic ring R is unitary. Again by [20, Theorem 6.3], the right pure semisimplicity of A implies that R is right perfect. This finishes the proof. ∎

5. Concluding remarks and open problems. We start with an example analogous to that of [7, Example].

EXAMPLE 5.1. Let k be a field and let A be a k-algebra that is not right coherent (for instance, one could take $A = B[t]$, the polynomial ring with one indeterminate over the ring $B = \prod_{\mathbb{N}} \mathbb{Q}[X,Y]$, considered in [23]). It is shown there that A is a non-coherent commutative ring, which is also an algebra over the field $k = \mathbb{Q}$ of rational numbers.

Let T be the functor ring $R_{\mathcal{G}}$ of the set $\mathcal{G}$ of representatives of isomorphism classes of finitely presented right A-modules (see 3.2). Then, T is clearly a right panoramic ring, but, as we have seen in the proof of Corollary 3.6, T is not left panoramic and T is not right locally coherent, because A is not right coherent. As in the Example of [7, p. 89], we may construct the unitary ring

$$R = T \times k$$

with multiplication given by the formula $(t_1, a_1) \cdot (t_2, a_2) = (t_1t_2 + t_1a_2 + t_2a_1,\ a_1a_2)$, where $t_1, t_2 \in T$ and $a_1, a_2 \in k$. Note that T is a two-sided ideal of R satisfying the conditions of [24, Proposition XI.3.13] and $R/T \cong k$. Arguing like in [7, p. 90], we infer that R is a right panoramic ring with the identity element $(0,1)$. On the other hand, since T is not right locally coherent, we may easily deduce that R is not right coherent, either. Then, by Corollary 2.11, R is not left panoramic.

Summarizing the above we get the following.

COROLLARY 5.2. *There exists a unitary ring which is right panoramic, but not left panoramic and not right coherent.* ∎

As a consequence of Corollary 5.2 we get an analogous result on FTF rings.

COROLLARY 5.3. *There exists a unitary right FTF ring which is not a left FTF ring.*

Proof. Let R be a unitary right panoramic ring which is not left panoramic (Corollary 5.2). By Corollary 2.11, R is neither right $\boldsymbol{\delta}_R$-coherent, nor right $\boldsymbol{\tau}_R$-coherent (notice that $\boldsymbol{\tau}_R$-coherence always implies $\boldsymbol{\delta}_R$-coherence, because every $\boldsymbol{\tau}_R$-torsion module is $\boldsymbol{\delta}_R$-torsion). By [9, Proposition 2.2.9], R is not a left FTF ring. ∎

In a connection with the results above the following problem arises.

PROBLEM 5.4. *Find a right panoramic and semiperfect unitary ring R such that* $\mathrm{Fl}(R)$ *is not a module category.*

Note that every such a ring R is not right perfect, by a result of Tachikawa [25]. Moreover, in this case the category $\mathrm{Fl}(R)$ has no projective generators, because if it did, the torsion theory $\boldsymbol{\tau}_R$ of $\mathrm{Mod}(R)$ would be generated by a finitely generated right ideal which is the trace of a projective module; but then, it would also be the trace of a finitely generated projective module and $\mathrm{Fl}(R)$ would be a module category.

By restricting the one-to-one correspondence of Theorem 4.6 we get the following result.

PROPOSITION 5.5. *There is a one-to-one correspondence between*:

(a) *Morita equivalence classes of right panoramic unitary semiperfect rings R with a minimal finitely generated left faithful right ideal ReR*; *and*

(b) *Morita equivalence classes of unitary semiperfect rings A such that there are only finitely many isomorphism classes of indecomposable finitely presented right A-modules and the endomorphism ring of every indecomposable finitely presented right A-module is a local ring.*

Proof. Let R be as stated in (a). It follows from Proposition 4.5 and Theorem 4.6 that the category $\mathcal{C} = \mathrm{Fl}(R)$ which corresponds to R by the correspondence in Theorem 4.6 is a module category over a unitary ring A, there is only a finite number of indecomposable finitely presented right A-modules and every finitely presented right A-module has a decomposition that complements direct summands.

By [1, Proposition 12.10], the endomorphism ring of every indecomposable finitely presented right A-module is a local ring. It follows that A is a semiperfect ring, and consequently A is of the type given in (b).

Conversely, assume that A is as in (b). It follows from Proposition 4.5 and Theorem 4.6 that the corresponding right panoramic ring R is unitary semiperfect and has a minimal finitely generated left faithful right ideal. This finishes the proof. ■

Proposition 5.5 suggests the following problem.

PROBLEM 5.6. *Find a semiperfect right panoramic unitary ring R with a minimal finitely generated left faithful right ideal*, *such that R is not right perfect. Equivalently*, *find a unitary semiperfect ring A such that*:

(a) *the number of finitely presented indecomposable right A-modules up to isomorphism is finite*,

(b) *the endomorphism ring of any finitely presented indecomposable right A-module is local*,

(c) *the ring A is not of finite representation type.*

Problem 5.6 is closely related with the central problem in [15], and is also related to the following question that is dealt with in [18]:

PROBLEM 5.7. *Describe all unitary semiperfect rings R for which every indecomposable finitely presented R-module has a local endomorphism ring.*

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