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WILD TILTED ALGEBRAS REVISITED

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In this paper wild tilted algebras are studied. Following [6] an algebra B is called *tilted* (*of type* A) if there exists a finite-dimensional hereditary algebra A over some field k and a tilting module T in the category A-mod of finite-dimensional left A-modules with $B = \mathrm{End}_A(T)$. The tilting module T has a structure as an (A, B)-bimodule and induces in B-mod a splitting torsion pair $(\mathcal{X}, \mathcal{Y})$, where the torsion-free class $\mathcal{Y}$ is the full subcategory of B-mod, defined by the objects M with $\mathrm{Tor}^1_B(T, M) = 0$, whereas the torsion class $\mathcal{X}$ is defined by the objects N with $T \otimes_B N = 0$.

A tilted algebra B of type A is only wild if A is wild hereditary. It was shown in [9] that the study of $\mathcal{Y}$ (respectively, $\mathcal{X}$) can be reduced to the case of tilting modules without nonzero direct summands in the preinjective component $\mathcal{I}(A)$ (respectively, preprojective component $\mathcal{P}(A)$). Only this case will be considered here, and it was shown in [9] that in this situation B is wild if and only if A is wild. In this paper the torsion-free class $\mathcal{Y}$ is studied, dual results hold for $\mathcal{X}$. For basic terminology and general results we refer to [6, 16]. The main result of this paper is:

THEOREM 1. *Let A be connected wild hereditary*, *T a tilting module in* A-mod *without indecomposable preinjective direct summand and* $B = \mathrm{End}_A(T)$. *If* $F = \mathrm{Hom}_A(T, -)$ *denotes the tilting functor and* $(\mathcal{Y}, \mathcal{X})$ *the torsion pair in* B-mod *induced by* T, *we have*:

1. *The Auslander–Reiten quiver* $\Gamma(B)$ *of* B *has exactly one preprojective component* $\mathcal{P}(B)$.
   (a) $C = B/\mathrm{ann}\mathcal{P}(B)$ *is connected wild concealed.*
   (b) *If* T_0 *is a preprojective direct summand of* T, *then* $F(T_0)$ *is preprojective in* B-mod.
2. *If* $X \in \mathcal{Y}$ *is indecomposable and not in the connecting component*, *then*:
   (a) $\tau_B^{-m} X$ *is in* C-mod *for* $m \gg 0$.

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(b) $\tau_B^{-m} X = \tau_C \tau_B^{-m-1} X$ *for* $m \gg 0$.
(c) *If* X *is not in* $\mathcal{P}(B)$, *then* $\tau_B^{-m} X$ *is a regular* C*-module for* $m \gg 0$.

3. *All regular* C*-modules are in* $\mathcal{Y}$. *If* X *is a regular* C*-module, then* $\tau_C^{-m} X = \tau_B \tau_C^{-m-1} X$ *for* $m \gg 0$.

The first part of the theorem was the main result in the paper [17] of Strauss. The remaining parts had been first shown in [11].

The original proofs are quite complicated. A unified, shorter and more conceptual proof will be given here. Many of the ideas for this proof can be found in [2, 11, 17].

Additionally it turned out that rather similar results hold for some classes of quasi-tilted algebras (see for example [13, 14], and [4] for the concept of quasi-tilted algebras).

It should also be mentioned that by parts 2 and 3 of the theorem there is a bijection between the set $\Omega(C)$ of regular components of the Auslander–Reiten quiver of C and the set $\Omega(\mathcal{Y})$ of those components of $\Gamma(B)$ which are completely contained in $\mathcal{Y}$ and are different from the preprojective component. In particular, no component in $\Omega(\mathcal{Y})$ has empty stable part. Hence by [9] there is a bijection between $\Omega(\mathcal{Y})$ and the set $\Omega(A)$ of regular components of the Auslander–Reiten quiver of A, too. For more details see [2, 9, 11].

In order to make the proof less technical, the theorem will be reformulated. The tilting module T defines in A-mod a torsion pair $(\mathcal{G}, \mathcal{F})$ where the torsion class $\mathcal{G}$ consists of the A-modules generated by the tilting module T. The torsion-free class $\mathcal{F}$ is defined by the modules Y with $\mathrm{Hom}(T, Y) = 0$. The torsion class $\mathcal{G}$ is equivalent to $\mathcal{Y}$ under the functor F. In $\mathcal{G}$ there exist relative Auslander–Reiten sequences; the relative Auslander–Reiten translation in $\mathcal{G}$ will be denoted by $\tau_{\mathcal{G}}$. If t is the torsion-radical associated with $\mathcal{G}$, then $\tau_{\mathcal{G}} = \mathrm{t}\tau_A$, and $\tau_{\mathcal{G}}$ is a full functor. Moreover, one has $F\tau_{\mathcal{G}} = \tau_B F$. The relative Auslander–Reiten quiver of $\mathcal{G}$ is denoted by $\Gamma(\mathcal{G})$ and its preprojective component or components by $\mathcal{P}_{\mathcal{G}}$. The image of $\mathcal{P}_{\mathcal{G}}$ under the tilting functor F is $\mathcal{P}(B)$.

If A is hereditary with n simple modules and U is a partial tilting module with m pairwise nonisomorphic indecomposable direct summands, we denote by $U^{\perp}$ the full subcategory of A-mod defined by the objects Y with $\mathrm{Hom}(U, Y) = 0$ and $\mathrm{Ext}(U, Y) = 0$. In this case $U^{\perp}$ is an exact abelian subcategory of A-mod which is closed under extensions. Moreover, $U^{\perp} \cong H$-mod, where H is a hereditary algebra with $n - m$ simple modules (see [3, 5, 18]). Hence the Auslander–Reiten translations in $U^{\perp}$, denoted by $\tau_{U^{\perp}}, \tau_{U^{\perp}}^{-}$ or τ_H, τ_H^{-}, are full functors in $U^{\perp}$.

In terms of the torsion class $\mathcal{G}$ in A-mod, Theorem 1 reads as follows.

THEOREM 2. *Let A be connected wild hereditary*, T *a tilting module in* A-mod *without indecomposable preinjective direct summands*, $\mathcal{G}$ *the class of* A-*modules generated by* T *and* $\Gamma(\mathcal{G})$ *its relative Auslander–Reiten quiver.*

1. *There exists exactly one preprojective component* $\mathcal{P}_{\mathcal{G}}$ *in* $\Gamma(\mathcal{G})$. *If* T_1 *is the direct sum of all indecomposable direct summands* X *of* T *contained in* $\mathcal{P}_{\mathcal{G}}$ *and* $T = T_1 \oplus T_2$ *then*:
   (a) $C = \mathrm{End}_A(T_1)$ *is connected wild concealed.*
   (b) T_2 *is regular in* A-mod.
   (c) T_1 *is a preprojective tilting module in* $T_2^{\perp}$.
2. *Denote by* $\widetilde{\mathcal{G}}$ *the torsion class* $\mathcal{G} \cap T_2^{\perp}$ *in* $T_2^{\perp}$. *If* $X \in \mathcal{G}$ *is indecomposable and not preinjective in* A-mod, *then*:
   (a) $\tau_{\mathcal{G}}^{-m} X$ *is in* $T_2^{\perp}$ *for* $m \gg 0$.
   (b) $\tau_{\mathcal{G}}^{-m} X = \tau_{\widetilde{\mathcal{G}}} \tau_{\mathcal{G}}^{-m-1} X$ *for* $m \gg 0$.
   (c) *If* X *is not in* $\mathcal{P}_{\mathcal{G}}$, *then* $\tau_{\mathcal{G}}^{-m} X$ *is a regular* $T_2^{\perp}$-*module for* $m \gg 0$.
3. *If* X *is regular in* $T_2^{\perp}$, *then* $\tau_{T_2^{\perp}}^{-m} X = \tau_{\mathcal{G}} \tau_{T_2^{\perp}}^{-m-1} X$ *for* $m \gg 0$.

If M is regular in $T_2^{\perp}$, then $M \in \widetilde{\mathcal{G}}$ with $\tau_{\widetilde{\mathcal{G}}} M = \tau_{T_2^{\perp}} M$ by 1(c). It should be mentioned that the theorem trivially holds if T is a preprojective tilting module, in particular, if A has only two simple modules. Therefore, we assume that T is not preprojective and A has $n > 2$ simple modules. The proof will be by induction on n.

1. The Strauss decomposition of T. We assume that T is a squarefree tilting module with n pairwise nonisomorphic indecomposable direct summands, none of them preinjective and not all of them preprojective in A-mod. By $\mathcal{P}_{\mathcal{G}}$ we denote the preprojective component or components of the relative Auslander–Reiten quiver $\Gamma(\mathcal{G})$. Then T has a decomposition, usually called the *Strauss decomposition*,

$$T = T_1 \oplus T_2$$

where T_1 is the sum of all indecomposable direct summands of T which are $\mathcal{G}$-preprojective, that is, which are in $\mathcal{P}_{\mathcal{G}}$. It has to be shown that $T_1 \neq 0$, that $\mathrm{End}_A(T_1)$ is a connected wild concealed algebra and that all A-preprojective direct summands of T are in T_1. The second summand T_2 has a decomposition $T_2 = P \oplus R$ where P is preprojective and R is regular in A-mod. It is easy to show

LEMMA 1.1. $T_1 \in T_2^{\perp}$ *and* $T_1 \oplus P \in R^{\perp}$.

In the sequel the summand R will be studied in detail.

LEMMA 1.2. $R \neq 0$.

P r o o f. The statement is obvious if $T_1 = 0$, since T has regular direct summands by assumption. Suppose $T_1 \neq 0$ but $R = 0$. Since $\mathrm{End}(T)$ is not concealed one has $P \neq 0$. The algebra $\mathrm{End}(T)$ is connected and $T_1 \in T_2^{\perp}$ by 1.1. Consequently, there exist indecomposable direct summands X of T_1 and Y of P with $\mathrm{Hom}(X, Y) \neq 0$. Since only X is in $\mathcal{P}_{\mathcal{G}}$, each nonzero homomorphism $f : X \to Y$ has an arbitrary long factorisation through $\mathcal{G}$-preprojectives, that is, there exist infinitely many indecomposable modules M with $\mathrm{Hom}(M, Y) \neq 0$, an absurdity.

An indecomposable regular A-module Y is uniquely determined by its quasi-length r and its quasi-socle X (respectively, quasi-top Z) (see [15]). We write $Y = X(r)$ (respectively, $Y = [r]Z$) in this case. If Y is quasi-simple we have $Y = Y(1) = [1]Y$ with this convention.

If $Y = X(r)$ is an indecomposable regular A-module of quasi-length r and with quasi-socle X, the *wing* $\mathcal{W}(Y)$ with top Y and length r is the mesh complete full subquiver of the regular component $\mathcal{C}$ containing Y, which consists of the vertices $\{\tau_A^{-i} X(j) \mid 1 \le j \le r,\ 1 \le i + j \le r\}$ (see [16]).

If $X = X(r)$ is a direct summand of R, the wing $\mathcal{W}(Y)$ contains exactly r indecomposable direct summands of T (see [16, 17]). Since these r summands are connected by $\mathcal{G}$-irreducible maps, all of them are direct summands of R. We therefore get a decomposition

$$R = \bigoplus_{i=1}^{l} W_i$$

where all r_i indecomposable direct summands of W_i are contained in the same wing $\mathcal{W}(S_i(r_i))$ with S_i quasi-simple and $\mathcal{W}(S_i(r_i)) \cap \mathcal{W}(S_j(r_j)) = \emptyset$ for $i \neq j$ (see for example [11]). The tops $S_i(r_i)$ of the wings $\mathcal{W}(S_i(r_i))$ are summands of R. The class $\mathcal{G}$ and the relative Auslander–Reiten quiver $\Gamma(\mathcal{G})$ remain unchanged outside the wings $\mathcal{W}(S_i(r_i))$ if we additionally assume that W_i is $\mathcal{W}(S_i(r_i))$-projective, that is, $W_i = \bigoplus_{j=1}^{r_i} S_i(j)$ (see [11], 2.5). In particular, $\mathcal{P}_{\mathcal{G}}$ remains unchanged.

We therefore assume $W_i = \bigoplus_{j=1}^{r_i} S_i(j)$ for the rest of the paper. In [11] this was called the *normalised form* of T.

We will frequently use

LEMMA 1.3. (a) *For* X, Y *regular in* A-mod *we have* $\mathrm{Hom}_A(X, \tau^{-m} Y) = 0$ *for* $m \gg 0$.

(b) $\mathrm{Hom}_A(S_i, \tau_A^{-m} S_i) = 0$ *for all* $m > 0$.

P r o o f. (a) was shown in [9] and (b) follows from [11], 1.2, since the S_i are quasi-simple bricks.

2. The wing quiver $\mathcal{Q}_{\mathcal{W}}(T)$. We call the decomposition

$$T = T_1 \oplus P \oplus \Big(\bigoplus_{i=1}^{l} W_i\Big)$$

with $W_i = \bigoplus_{j=1}^{r_i} S_i(j)$ and S_i quasi-simple regular the (*normalised*) *wing decomposition* of T. Moreover, we decompose $P = \bigoplus_{j=1}^{t} P_j$ with P_j indecomposable preprojective in A-mod. This decomposition will be used throughout the paper.

The *wing quiver* $\mathcal{Q}_{\mathcal{W}}(T)$ of T has $\{1, \ldots, l\}$ as set of vertices and no loops. For $1 \le i \ne j \le l$ there exists an arrow $i \to j$ exactly if we have $\mathrm{Hom}_A(S_i, \tau_A^{-m} S_j) \ne 0$ for some $m \ge 0$. Let $m(i,j) \ge 0$ be in this case the smallest natural number m with $\mathrm{Hom}_A(S_i, \tau_A^{-m} S_j) \ne 0$.

Lemma 2.1. *$\mathcal{Q}_{\mathcal{W}}(T)$ has no oriented cycles. Therefore it has sinks.*

Proof. Suppose, first, $\mathcal{Q}_{\mathcal{W}}(T)$ has an oriented cycle $i \to j \to i$ of length 2. Since $\mathrm{Hom}(S_r, \tau_A S_t) = 0$ for all $1 \le r, t \le l$, all nonzero maps $f \in \mathrm{Hom}(S_i, \tau^{-m(i,j)} S_j)$ and $g \in \mathrm{Hom}(S_j, \tau^{-m(j,i)} S_i)$ are injective or surjective (see [6], 4.1). If f is surjective, then $f\tau^{-m(i,j)} g : S_i \to \tau^{-(m(i,j)+m(j,i))} S_i$ is nonzero. From 1.3(b), $m(i,j) + m(j,i) = 0$ follows and f therefore is a split mono, hence an isomorphism, a contradiction to $i \ne j$. A similar argument works for f injective.

Suppose next that $\mathcal{Q}_{\mathcal{W}}(T)$ has an oriented cycle, say

$$i_1 \to i_2 \to \ldots \to i_r \to i_1$$

of minimal length $r > 2$, therefore with $i_x \ne i_y$ for $1 \le x \ne y \le r$. Again we use [6], 4.1. If $0 \ne f \in \mathrm{Hom}(S_{i_1}, \tau^{-m(i_1,i_2)} S_{i_2})$ is surjective, we get $\mathrm{Hom}(S_{i_1}, \tau^{-(m(i_1,i_2)+m(i_2,i_3))} S_{i_3}) \ne 0$. Then $i_1 \to i_3 \to \ldots \to i_r \to i_1$ is a cycle of smaller length $r-1$, a contradiction. If f is injective, we construct a cycle $i_2 \to \ldots \to i_r \to i_2$ of length $r-1$.

For the rest of the paper we assume that l is a sink of $\mathcal{Q}_{\mathcal{W}}(T)$.

Lemma 2.2. *Let $X(r)$ be indecomposable regular of quasi-length $r \ge 1$.*

(a) *If Y is indecomposable and not in $\mathcal{W}(X(r))$, then* $\mathrm{Hom}_A(X(r), Y) = 0$ (*respectively*, $\mathrm{Hom}_A(Y, X(r)) = 0$) *if and only if* $\mathrm{Hom}_A(U, Y) = 0$ (*respectively*, $\mathrm{Hom}_A(Y, U) = 0$) *for all* $U \in \mathrm{add}\, \mathcal{W}(X(r))$.

(b) *The wing $\mathcal{W}(X(r))$ is a standard wing*, *that is*, $\mathrm{rad}^{\infty}(U, V) = 0$ *for all* $U, V \in \mathrm{add}\, \mathcal{W}(X(r))$, *if and only if $X(r)$ is a brick.*

Proof. See [11], 1.4 and 1.6.

It should be mentioned that it is 2.2(a) which allows us to consider only the normalised form $W_i = \bigoplus_{j=1}^{r_i} S_i(j)$ $(1 \le i \le l)$ of T.

Lemma 2.3. (a) $\mathrm{Hom}(S_l, W_i) = 0$ *for $i < l$.*

(b) $\mathrm{Hom}(W_l, W_i) = 0$ *for* $i < l$.

(c) $\mathrm{Hom}(W_l, \tau_A^{-j} W_i) = 0$ *for* $i < l$ *and* $j \geq 0$.

(d) $\mathrm{Hom}(W_l, \tau_A^{-j} W_l) = 0$ *for* $j \geq r_l$.

(e) *If* $T = W_l \oplus U$, *then* $U \in W_l^{\perp}$.

(f) *For* $X \in \mathrm{add}\, T$ *we have* $\mathrm{rad}^{\infty}(W_l, X) = 0$.

Proof. (a) $\mathrm{Hom}(S_l, W_i) \neq 0$ for some $i < l$ is equivalent to $\mathrm{Hom}(S_l, \tau_A^{-j} S_i) \neq 0$ for some j with $0 \leq j < r_i$ (see [11], 1.4). This cannot happen by definition of l.

(b) Consider for $1 < j \leq r_l$ the exact sequence $0 \to S_l \to S_l(j) \to \tau_A^{-} S_l(j-1) \to 0$. From $\mathrm{Hom}(\tau_A^{-} S_l(j-1), W_i) \cong \mathrm{Hom}(S_l(j-1), \tau_A W_i) = 0$ and $\mathrm{Hom}(S_l, W_i) = 0$ we get $\mathrm{Hom}(S_l(j), W_i) = 0$, hence $\mathrm{Hom}(W_l, W_i) = 0$.

(c) From $\mathrm{Hom}(S_l, \tau_A^{-j} S_i) = 0$ for all $j \geq 0$ and $i < l$ we get, again by [11], 1.4 or Lemma 2.2(a), $\mathrm{Hom}(S_l, \tau_A^{-j} W_i) = 0$ for all $j \geq 0$. Assume $\mathrm{Hom}(W_l, \tau_A^{-j} W_i) \neq 0$ for some j. Take j minimal with this property, hence $j > 1$ by (b). Let $m > 1$ be minimal with $\mathrm{Hom}(S_l(m), \tau_A^{-j} W_i) \neq 0$. As in (b) we get a contradiction if we apply $\mathrm{Hom}(-, \tau_A^{-j} W_i)$ to the short exact sequence $0 \to S_l \to S_l(m) \to \tau_A^{-} S_l(m-1) \to 0$.

(d) follows from (1.3) and [11], 1.4, whereas (e) follows from 1.1 and part (b) of the lemma.

(f) Let $X = X_1 \oplus X_2$ with $X_1 \in \mathrm{add}\, U$ and $X_2 \in \mathrm{add}\, W_l$. Since $\mathrm{Hom}(W_l, X_1) = 0$, we have $\mathrm{rad}^{\infty}(W_l, X) = \mathrm{rad}^{\infty}(W_l, X_2) = 0$ by 2.2.

3. Relative Auslander–Reiten translations. If $\mathcal{T}$ is a torsion class in Λ-mod, where Λ is some finite-dimensional algebra and X is indecomposable in $\mathcal{T}$, not Ext-projective, then the relative Auslander–Reiten translate $\tau_{\mathcal{T}} X$ of X in $\mathcal{T}$ is the $\mathcal{T}$-torsion submodule $\mathrm{t}\tau_{\Lambda} X$ of $\tau_{\Lambda} X$ (see [1, 7]). If A is hereditary and $\mathcal{T}$ a torsion-class, the cokernel of the embedding $\tau_{\mathcal{T}} X \to \tau_A X$ is Ext-injective in the corresponding torsion-free class $\mathcal{F}$, see [10, 11]. If $\mathcal{G}$ is a tilting torsion class induced by a tilting module this implies (see [11], 2.2):

LEMMA 3.1. *Let* A *be hereditary and* T *a tilting module without preinjective direct summand. If* X *is in* $\mathcal{G}$, *not* Ext*-projective, then there is a short exact sequence* $0 \to \tau_{\mathcal{G}} X \to \tau_A X \to F \to 0$ *with* $F \in \mathrm{add}\, \tau_A T$. *If* X *is not in* $\mathcal{P}_{\mathcal{G}}$, *then* F *is in* $\mathrm{add}\, \tau_A T_2$.

From 3.1 we deduce (see for example [12], 3.2):

LEMMA 3.2. *Let* $X \in \mathcal{G}$ *be indecomposable and* $r > 0$.

(a) *If* $\tau_{\mathcal{G}}^r X \neq 0$ *there is a short exact sequence*

$$0 \to \tau_{\mathcal{G}}^r X \to \tau_A^r X \xrightarrow{\pi} S \to 0$$

where S has a filtration $S = S_r \supset S_{r-1} \supset \ldots \supset S_1 \supset S_0 = 0$ with $S_i/S_{i-1} \in \operatorname{add} \tau_A^i T$, or even $S_i/S_{i-1} \in \operatorname{add} \tau_A^i T_2$ for all i if $X \notin \mathcal{P}_{\mathcal{G}}$.

(b) *If $\tau_A^{-r} X \neq 0$ there is a short exact sequence*

$$0 \to \tau_A^{-r} X \to \tau_{\mathcal{G}}^{-r} X \xrightarrow{\pi} Q \to 0$$

where Q has a filtration $Q = Q_0 \supset Q_1 \supset \ldots \supset Q_{r-1} \supset Q_r = 0$ with $Q_i/Q_{i+1} \in \operatorname{add} \tau_A^{-i} T$, or even $Q_i/Q_{i+1} \in \operatorname{add} \tau_A^{-i} T_2$ for all i if $X \notin \mathcal{P}_{\mathcal{G}}$.

Note that 3.2(a) implies that for an indecomposable module $X \in \mathcal{G}$ and $r \gg 0$ either $\tau_{\mathcal{G}}^r X = 0$ or $\tau_{\mathcal{G}}^{r+1} X = \tau_A \tau_{\mathcal{G}}^r X$. Indeed, if $\tau_{\mathcal{G}}^r X$ is nonzero for all $r > 0$, consider the short exact sequences $0 \to \tau_{\mathcal{G}}^r X \to \tau_A^r X \to S \to 0$ and $0 \to \tau_{\mathcal{G}}^{r+1} X \to \tau_A \tau_{\mathcal{G}}^r X \to \tau \widetilde{T} \to 0$. They induce an infinite chain

$$X \supset \tau_A^{-1} \tau_{\mathcal{G}} X \supset \tau_A^{-2} \tau_{\mathcal{G}}^2 X \supset \ldots \supset \tau_A^{-r} \tau_{\mathcal{G}}^r X \supset \ldots$$

hence this chain becomes stationary [9, 2]. In particular, there are no regular tubes in $\Gamma(\mathcal{G})$.

Lemma 3.2 has the following application.

LEMMA 3.3. *Let $X \in \mathcal{G}$ be indecomposable not in $\mathcal{P}_{\mathcal{G}}$, and s an integer with $\tau_{\mathcal{G}}^s X \neq 0$. Then $\operatorname{Hom}_A(S_l, \tau_A^s X) = 0$ implies $\operatorname{Hom}_A(S_l, \tau_{\mathcal{G}}^s X) = 0$.*

Proof. For $s > 0$ the claim follows from 3.2(a), nothing is to show for $s = 0$.

Let $s = -r < 0$. Assume $\operatorname{Hom}_A(S_l, \tau_A^{-r} X) = 0$ but $\operatorname{Hom}_A(S_l, \tau_{\mathcal{G}}^{-r} X) \neq 0$. Take $0 \neq f \in \operatorname{Hom}_A(S_l, \tau_{\mathcal{G}}^{-r} X)$. From $\operatorname{Hom}_A(S_l, \tau_A^{-r} X) = 0$ we see by 3.2(b) that $f\pi : S_l \to Q$ is nonzero. Since $Q_i/Q_{i+1} \in \operatorname{add} \tau_A^{-i} T_2$ we deduce from the definition of l that $\operatorname{Hom}_A(S_l, Q_i/Q_{i+1}) = 0$ for $i > 0$ and therefore $\operatorname{Hom}_A(S_l, Q_1) = 0$. If $\pi_1 : Q \to Q/Q_1$ denotes the canonical surjection, we therefore have $0 \neq f\pi\pi_1 : S_l \to Q/Q_1$. But $\operatorname{rad}^\infty(S_l, Q/Q_1) = 0$ by 2.3(f), hence Q/Q_1 has a direct summand $Z \in \operatorname{add} W_l$ and the image of $f\pi\pi_1$ is contained in Z. Thus there exists a nonzero composition of maps $S_l \to \tau_{\mathcal{G}}^{-r} X \to S_l(i)$ for some $1 \leq i \leq r_l$. But $\operatorname{Hom}_A(S_l, S_l(i))$ is one-dimensional as $\operatorname{End}_A(S_l)$-module or $\operatorname{End}_A(S_l(i))$-module, by 2.2(b) and $\tau_{\mathcal{G}}^{-r} X$ is indecomposable. Therefore $\tau_{\mathcal{G}}^{-r} X \cong S_l(j)$ for some $1 \leq j \leq r_l$, which is impossible, since $r \geq 1$.

LEMMA 3.4. *For X indecomposable in $\mathcal{G}$ we have $\operatorname{Hom}_A(W_l, \tau_{\mathcal{G}}^{-r} X) = 0$ for $r \gg 0$.*

Proof. Since $\operatorname{Hom}_A(W_l, \mathcal{P}_{\mathcal{G}}) = 0$, the statement trivially holds for $X \in \mathcal{P}_{\mathcal{G}}$. If X is preinjective in A-mod we have $\tau_{\mathcal{G}}^{-r} X = \tau_A^{-r} X = 0$ for $r \gg 0$.

Suppose that $X \notin \mathcal{P}_{\mathcal{G}} \cup \mathcal{I}(A)$. If X is preprojective in A-mod we have $\operatorname{Hom}_A(S_l, \tau_A^{-r} X) = 0$ for all integers r. If X is regular, there exists r' with $\operatorname{Hom}_A(S_l, \tau_A^{-j} X) = 0$ for all $j \geq r'$ (see 1.3(a)). Hence there exists in both cases an integer r such that $\operatorname{Hom}_A(S_l, \tau_A^{-j} X) = 0$ for all

$j \geq r - r_l$. By 3.3 this implies $\mathrm{Hom}_A(S_l, \tau_{\mathcal{G}}^{-j} X) = 0$ for all $j \geq r - r_l$. We show by induction on $m \leq r_l$ that $\mathrm{Hom}_A(S_l(m), \tau_{\mathcal{G}}^{-j} X) = 0$ for all $j \geq r - r_l + m - 1$. Assume the statement holds for all $1 \leq m < r_l$. Consider the short exact sequence $0 \to S_l \to S_l(m+1) \to \tau_A^- S_l(m) \to 0$ and take $j \geq r - r_l + m$. We get $\mathrm{Hom}_A(S_l(m+1), \tau_{\mathcal{G}}^{-j} X) \cong \mathrm{Hom}_A(\tau_A^- S_l(m), \tau_{\mathcal{G}}^{-j} X)$. Take $f \in \mathrm{Hom}_A(\tau_A^- S_l(m), \tau_{\mathcal{G}}^{-j} X)$. Then $\tau_A f \in \mathrm{Hom}_A(S_l(m), \tau_A \tau_{\mathcal{G}}^{-j} X)$ has image in the torsion submodule $\tau_{\mathcal{G}}^{-j+1} X$ of $\tau_A \tau_{\mathcal{G}}^{-j} X$. Therefore $\tau_A f = 0$, by induction. Hence f is zero and the claim follows.

Recall that $P = \bigoplus_{j=1}^t P_j$ with P_j indecomposable preprojective.

COROLLARY 3.5. (a) $\mathrm{Hom}_A(W_l, \tau_{\mathcal{G}}^{-r} S_i) = 0$ *for all* $i < l$ *and all* $r \geq 0$.
(b) $\mathrm{Hom}_A(W_l, \tau_{\mathcal{G}}^{-r} P_j) = 0$ *for all* $1 \leq j \leq t$ *and all* $r \geq 0$.
(c) $\mathrm{Hom}_A(W_l, \tau_{\mathcal{G}}^{-r} S_l) = 0$ *for all* $r \geq r_l$.

Proof. Since $\mathrm{Hom}(W_l, \tau_A^{-j} W_i) = 0$ for all $i < l$ and all $j \geq -1$ by 2.3, we get $\mathrm{Hom}(S_l, \tau_A^{-j} S_i) = 0$ for all $j \geq -r_l$ by 2.2(a), and (a) follows from 3.4. (b) immediately follows from 3.4 and for (c) we use $\mathrm{Hom}_A(S_l, \tau_A^{-j} S_l) = 0$ for all $j > 0$ (see 1.3).

4. Comparison of relative Auslander–Reiten translations. The tilting module T has a decomposition $T = W_l \oplus U$ with $U \in W_l^{\perp}$ (see 2.3). If $W_l^{\perp} \cong A'$-mod, then A' is a wild connected hereditary algebra by [17] and we identify $W_l^{\perp}$ with A'-mod. In particular, we write $\tau_{A'}$ for the Auslander–Reiten translation in $W_l^{\perp}$. Moreover, we have $\tau_{\mathcal{G}}^{-r} X \in W_l^{\perp}$ for $X \in \mathcal{G}$ and $r \gg 0$ by 3.4. Notice that $\mathcal{P}_{\mathcal{G}}$ is in $W_l^{\perp}$, too.

The module U is a tilting module in A'-mod, so it defines a torsion pair $(\overline{\mathcal{G}}, \overline{\mathcal{F}})$ in A'-mod by $\overline{\mathcal{G}} = \{Y \in W_l^{\perp} \mid \mathrm{Ext}_{A'}(U, Y) = 0\}$ and $\overline{\mathcal{F}} = \{Y \in W_l^{\perp} \mid \mathrm{Hom}_{A'}(U, Y) = 0\}$. The Auslander–Reiten translation $\tau_{A'}$ in A'-mod induces a relative Auslander–Reiten translation $\tau_{\overline{\mathcal{G}}}$ in $\overline{\mathcal{G}}$.

The torsion class $\overline{\mathcal{G}}$ in A'-mod is a full, exact and extension-closed subcategory of A-mod, but it is not closed under factors in A-mod, hence it is not a torsion class in A-mod. The following can be shown easily.

LEMMA 4.1. (a) $\overline{\mathcal{G}} \subset \mathcal{G}$.
(b) $\overline{\mathcal{G}} = \{Y \in \mathcal{G} \mid \mathrm{Hom}_A(W_l, Y) = 0\}$.

The aim of this part is to describe for $X \in \overline{\mathcal{G}}$ the relation between $\tau_{\mathcal{G}} X$ and $\tau_{\overline{\mathcal{G}}} X$. For this Lemma 2 of [2] is used.

Let G be the minimal projective generator in $W_l^{\perp}$. Then $T' = W_l \oplus G$ is a tilting module. If $\mathcal{G}'$ denotes the torsion class of A-modules generated by T', as in [2] one has $\mathcal{G}' = \{Y \mid \mathrm{Ext}_A(W_l, Y) = 0\}$ thus $\mathcal{G} \subset \mathcal{G}'$ and A'-mod $= W_l^{\perp} \subset \mathcal{G}'$.

It is easy to check that $G \oplus (\bigoplus_{i=1}^{r_l-1} S_l(i))$ is the minimal projective generator in $S_l(r_l)^{\perp}$ and $S_l(r_l)^{\perp} = W_l^{\perp} \times \operatorname{add} \mathcal{W}(S_l(r_l - 1))$ (see for example [17], 4.5).

Lemma 4.2. *If M is an indecomposable A'-module, not projective, then $\tau_{A'} M$ is the middle term of the universal sequence*

$$0 \to \tau_A S_l(r_l) \otimes_{\operatorname{End}(\tau_A S_l(r_l))} D\operatorname{Ext}(\tau_{\mathcal{G}'} M, \tau_A S_l(r_l)) \to \tau_{A'} M \to \tau_{\mathcal{G}'} M \to 0.$$

Proof. It follows from $S_l(r_l)^{\perp} = W_l^{\perp} \times \operatorname{add} \mathcal{W}(S_l(r_l - 1))$, for $M \in A'$-mod that $\tau_{A'} M = \tau_{S_l(r_l)^{\perp}} M$. Since $G \oplus (\bigoplus_{i=1}^{r_l-1} S_l(i))$ is the minimal projective generator in $S_l(r_l)^{\perp}$, the claim follows from [2], Lemma 2.

Lemma 4.3. *Let M be indecomposable in $\overline{\mathcal{G}} \subset A'$-mod, not Ext-projective. Then $\tau_{\overline{\mathcal{G}}} M$ is the middle term V of the universal sequence*

$$0 \to \tau_A S_l(r_l) \otimes_{\operatorname{End}(\tau_A S_l(r_l))} D\operatorname{Ext}(\tau_{\mathcal{G}} M, \tau_A S_l(r_l)) \to V \to \tau_{\mathcal{G}} M \to 0.$$

Proof. Consider the universal sequence

$$0 \to \tau_A S_l(r_l)^t \to \tau_{A'} M \to \tau_{\mathcal{G}'} M \to 0$$

with $t = \dim_{\operatorname{End}(\tau_A S_l(r_l))} \operatorname{Ext}(\tau_{\mathcal{G}'} M, \tau_A S_l(r_l))$, given in 4.2.

Since $\tau_{\mathcal{G}} M$ and $\tau_{\mathcal{G}'} M$ are the torsion submodules of $\tau_A M$ with respect to the torsion classes $\mathcal{G}$ and, respectively, $\mathcal{G}'$, we get from $\mathcal{G} \subset \mathcal{G}'$ a short exact sequence

$$0 \to \tau_{\mathcal{G}} M \xrightarrow{\varepsilon} \tau_{\mathcal{G}'} M \to F \to 0$$

with $F \in \mathcal{F} = \mathcal{F}(T)$. But F is a factor module of $\tau_{\mathcal{G}'} M$, hence in $\mathcal{G}'$. Therefore $F \in W_l^{\perp}$, that is, $F \in \overline{\mathcal{F}}$.

Consider the following pullback along ε:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \tau_A S_l(r_l)^t & \to & V & \to & \tau_{\mathcal{G}} M & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \tau_A S_l(r_l)^t & \to & \tau_{A'} M & \to & \tau_{\mathcal{G}'} M & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F & = & F & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

Since $\tau_{A'} M$ and F are in $W_l^{\perp}$, also $V \in W_l^{\perp}$. Applying $\operatorname{Hom}(U, -)$ to the first row of the diagram, we get $0 = \operatorname{Ext}_A(U, V) = \operatorname{Ext}_{A'}(U, V)$, hence

$V \in \bar{\mathcal{G}}$. Applying $\mathrm{Hom}_A(-, \tau_A S_l(r_l))$ to the same exact sequence, we get

$$0 \to \mathrm{Hom}_A(\tau_A S_l(r_l)^t, \tau_A S_l(r_l)) \overset{\cong}{\to} \mathrm{Ext}_A(\tau_{\mathcal{G}} M, \tau_A S_l(r_l)) \to 0.$$

Hence

$$0 \to \tau_A S_l(r_l)^t \to V \to \tau_{\mathcal{G}} M \to 0$$

is a universal short exact sequence.

Since $V \in \bar{\mathcal{G}}$ and $F \in \bar{\mathcal{F}}$, the module V is the $\bar{\mathcal{G}}$-torsion submodule of $\tau_{A'} M$, that is, $V = \tau_{\bar{\mathcal{G}}} M$.

LEMMA 4.4. *For $X \in \mathcal{G}$ one has $\tau_{\mathcal{G}}^{-m} X = \tau_{\bar{\mathcal{G}}} \tau_{\mathcal{G}}^{-m-1} X$ for $m \gg 0$.*

Proof. By 3.4 there exists m_0 with $\mathrm{Hom}_A(W_l, \tau_{\mathcal{G}}^{-r} X) = 0$ for all $r \geq m_0$, that is, $\tau_{\mathcal{G}}^{-r} X \in W_l^{\perp}$ for all $r \geq m_0$.

Therefore, $D\mathrm{Ext}_A(\tau_{\mathcal{G}}^{-m} X, \tau_A S_l(r_l)) \cong \mathrm{Hom}_A(S_l(r_l), \tau_{\mathcal{G}}^{-m} X) = 0$ for all $m \geq m_0$, and the claim follows from 4.3.

5. The inductive setting

LEMMA 5.1. *The tilting module U in A'-*mod *has no nonzero A'-preinjective direct summands.*

Proof. We have $U = T_1 \oplus (\bigoplus_{j=1}^{t} P_j) \oplus (\bigoplus_{j<l} W_j)$. For an indecomposable module $X \in \mathcal{G}$ one has $\tau_{\mathcal{G}}^{-r} X = 0$ for some $r \geq 0$ if and only if X is A-preinjective. Therefore for each indecomposable direct summand X of U one has $\tau_{\mathcal{G}}^{-r} X \neq 0$ for all $r \geq 0$.

If X is a summand of T_1, one has $\tau_{\mathcal{G}}^{-r} X \in W_l^{\perp}$ for all r, since $\mathcal{P}_{\mathcal{G}} \in W_l^{\perp}$. For $X \in \{S_i, P_j \mid i < l, j \leq t\}$ one gets $\tau_{\mathcal{G}}^{-r} X \in W_l^{\perp}$ for all $r \geq 0$ by 3.5. If $0 \to \tau_{\mathcal{G}}^{-r} X \to E \to \tau_{\mathcal{G}}^{-r-1} X \to 0$ for $r \geq 0$ is the relative Auslander–Reiten sequence in $\mathcal{G}$, then also $E \in W_l^{\perp}$, since $W_l^{\perp}$ is closed under extensions.

Hence each indecomposable direct summand of T_1 and each of the modules P_j with $1 \leq j \leq t$ and S_i with $i < l$ has infinitely many successors in A'-mod. Consequently, it is not A'-preinjective.

The irreducible maps $S_i(j) \to S_i(j+1)$ for $1 \leq j < r_i$ and $i < l$ remain irreducible in A'-mod. Therefore the claim follows.

In the notation of [17] this means that W_l is a *special summand* of T.

Let $Z \to S_l(r_l)$ be the irreducible epimorphism in A-mod. If Y is the quasi-top of $S_l(r_l)$ we have $Z = [r_l + 1]Y$. Let m_l be such that $[m_l]Y$ is a brick with self-extensions (see [8, 11]).

LEMMA 5.2. (a) *$Z = \tau_{\bar{\mathcal{G}}} \tau_{\mathcal{G}}^{-r_l} S_l$.*
(b) *$\tau_{\mathcal{G}}^{i} Z = \tau_A^{i+1} S_l$ for $i > 0$.*
(c) *$[i]Y \in \mathcal{G}$ for all $i \geq 1$.*
(d) *$[j]Y \in \bar{\mathcal{G}}$ for $r_l + 1 \leq j \leq m_l$.*

Proof. (a) We have $\tau_{\mathcal{G}}^{-r_l+1}S_l = \tau_A^{-r_l+1}S_l = Y$ and $\tau_{\mathcal{G}}^{-r_l}S_l \in \overline{\mathcal{G}}$ by 3.5. By 4.3 there is a universal exact sequence

$$0 \to \tau_A S_l(r_l) \otimes D\mathrm{Ext}(Y, \tau_A S_l(r_l)) \to \tau_{\overline{\mathcal{G}}}\tau_{\mathcal{G}}^{-r_l}S_l \to Y \to 0.$$

By the Auslander–Reiten formula it follows from 2.2 that $\mathrm{Ext}_A(Y, \tau_A S_l(r_l))$ is one-dimensional as $\mathrm{End}_A(S_l(r_l))$-module with basis $0 \to \tau_A S_l(r_l) \to Z \to Y \to 0$.

(b) We first consider $i = 1$. We get $D\mathrm{Ext}_A(T, \tau_A^2 S_l) \cong \mathrm{Hom}_A(S_l, \tau_A^- T_1)$ from the Auslander–Reiten formula, since $\mathrm{Hom}_A(S_l, \tau_A^- T_2) = 0$ by definition of l (see 2.3). If $\mathrm{Hom}_A(S_l, \tau_A^- T_1) \neq 0$, then $\mathrm{Hom}_A(S_l, \tau_{\mathcal{G}}^- T_1) \neq 0$ by 3.2, which is impossible, since $\tau_{\mathcal{G}}^- T_1 \in \mathcal{P}_{\mathcal{G}}$. Therefore $\tau_A^2 S_l \in \mathcal{G}$. The relative Auslander–Reiten sequence ending in Z is $0 \to \mathrm{t}\tau_A Z \to \mathrm{t}[r_l+2]Y \to Z \to 0$. The first term $\tau_A^2 S_l$ of the short exact sequence $0 \to \tau_A^2 S_l \to \tau_A Z \to \tau_A S_l(r_l) \to 0$ is torsion and the last term is torsion free. Therefore $\tau_A^2 S_l = \mathrm{t}\tau_A Z$, which also implies $[r_l+2]Y \in \mathcal{G}$.

By induction on $i \geq 2$ one shows $\tau_A^i S_l \in \mathcal{G}$. If $\tau_A^i S_l$ is in $\mathcal{G}$, consider the universal sequence $0 \to \tau_{\mathcal{G}}\tau_A^i S_l \to \tau_A^{i+1} S_l \to \tau\widetilde{T} \to 0$ with $\widetilde{T} \in \mathrm{add}\, T_2$. The definition of l and 1.3 imply $\widetilde{T} = 0$, that is, $\tau_A^{i+1} S_l \in \mathcal{G}$.

(c) From $\tau_A^{1+i} S_l \in \mathcal{G}$ for $i > 0$ and $Z \in \mathcal{G}$ it follows by induction that the middle term $[r_l+1+i]Y$ of the short exact sequence $0 \to \tau_A^{1+i} S_l \to [r_l+1+i]Y \to [r_l+i]Y \to 0$ is in $\mathcal{G}$. Clearly $[j]Y \in \mathcal{G}$ for $j \leq r_l$, which proves (c).

(d) By [17] the modules $Z = [r_l+1]Y, \ldots, [m_l]Y$ are in $W_l^{\perp}$.

Lemma 5.2 also implies that the stable part of the relative component in $\Gamma(\mathcal{G})$ containing W_l is of type $\mathbb{Z}A_\infty$. A picture of this component is given in [11], Fig. 1.

6. The inductive step. The tilting module U in A'-mod has no A'-preinjective direct summand by 5.1. By induction on the number of nonisomorphic indecomposable direct summands of the tilting module, we get for the torsion class $\overline{\mathcal{G}}$ in A'-mod defined by U,

(ind1) *There exists exactly one preprojective component $\mathcal{P}_{\overline{\mathcal{G}}}$ in $\Gamma(\overline{\mathcal{G}})$. If U_1 is the direct sum of all indecomposable direct summands X of U contained in $\mathcal{P}_{\overline{\mathcal{G}}}$ and $U = U_1 \oplus U_2$ then:*
- (a) $C = \mathrm{End}(U_1)$ *is connected wild concealed.*
- (b) U_2 *is regular in* A'-mod.
- (c) U_1 *is a preprojective tilting module in* $U_2^{\perp} \subset A'$-mod.

(ind2) *Denote by $\widehat{\mathcal{G}}$ the torsion class of U_1 in $U_2^{\perp}$. If $X \in \overline{\mathcal{G}}$ is indecomposable and not preinjective in* A'-mod, *then*:
- (a) $\tau_{\overline{\mathcal{G}}}^{-m} X$ *is in* $U_2^{\perp}$ *for* $m \gg 0$.

(b) $\tau_{\overline{\mathcal{G}}}^{-m} X = \tau_{\widehat{\mathcal{G}}} \tau_{\overline{\mathcal{G}}}^{-m-1} X$ *for* $m \gg 0$.

(c) *If* X *is not in* $\mathcal{P}_{\overline{\mathcal{G}}}$, *then* $\tau_{\overline{\mathcal{G}}}^{-m} X$ *is a regular* $U_2^{\perp}$*-module for* $m \gg 0$.

(ind3) *If* X *is regular in* $U_2^{\perp}$, *then* $\tau_{U_2^{\perp}}^{-m} X = \tau_{\overline{\mathcal{G}}} \tau_{U_2^{\perp}}^{-m-1} X$ *for* $m \gg 0$.

LEMMA 6.1. *If* $\mathcal{P}$ *is a preprojective component in* $\Gamma(\mathcal{G})$, *then it is a preprojective component in* $\Gamma(\overline{\mathcal{G}})$.

Proof. If X is in $\mathcal{P}$, then it is in $W_l^{\perp}$, hence in $\overline{\mathcal{G}}$.

First we consider the module $Z = [r_l + 1]Y \in \overline{\mathcal{G}}$, where Y is the quasi-top of $S_l(r_l)$. It was shown already in [17] that Z is quasi-simple regular in A'-mod. We keep the notation of 5.2.

LEMMA 6.2. *The module* Z *is neither in* $\mathcal{P}_{\overline{\mathcal{G}}}$ *nor preinjective in* A'-mod.

Proof. The modules $[r_l + 1]Y, \ldots, [m_l]Y$, where $[m_l]Y$ is a brick with self-extensions, are in $\overline{\mathcal{G}}$ by 5.2. Therefore the chain of irreducible epimorphisms in A-mod

$$[m_l]Y \to [m_l - 1]Y \to \ldots \to Z$$

is also a chain of irreducible epimorphisms in $\mathcal{G}$ and $\overline{\mathcal{G}}$. Since $[m_l]Y$ has self-extensions, Z is neither in $\mathcal{P}_{\overline{\mathcal{G}}}$ nor in $\mathcal{I}(A')$.

LEMMA 6.3. $\mathcal{P}_{\overline{\mathcal{G}}}$ *is a full component in the relative Auslander–Reiten quiver* $\Gamma(\mathcal{G})$. *It is the unique preprojective component in* $\Gamma(\mathcal{G})$.

Proof. We show that $\tau_{\mathcal{G}}$ and $\tau_{\overline{\mathcal{G}}}$ coincide on $\mathcal{P}_{\overline{\mathcal{G}}}$. Let M be in $\mathcal{P}_{\overline{\mathcal{G}}}$, not Ext-projective. By 4.3 it has to be shown that $0 = D\mathrm{Ext}_A(\tau_{\mathcal{G}} M, \tau S_l(r_l)) \cong \mathrm{Hom}_A(S_l(r_l), \tau_{\mathcal{G}} M) = \mathrm{Hom}_A(S_l(r_l), \tau_A M)$.

From $M \in W_l^{\perp}$ we deduce $\mathrm{Hom}_A(\tau_A W_l, \tau_A M) = 0$. Considering the Auslander–Reiten sequences

$$0 \to \tau_A S_l \to \tau_A S_l(2) \to S_l \to 0$$

and

$$0 \to \tau_A S_l(i) \to \tau_A S_l(i+1) \oplus S_l(i-1) \to S_l(i) \to 0$$

for $1 < i < r_l$ we get by induction $\mathrm{Hom}_A(S_l(i), \tau_A M) = 0$ for $1 \leq i < r_l$. Since $Z \notin \mathcal{P}_{\overline{\mathcal{G}}}$ we get $0 = \mathrm{Ext}_{A'}(M, Z) = \mathrm{Ext}_A(M, Z)$. Using, finally, the Auslander–Reiten sequence $0 \to \tau_A S_l(r_l) \to Z \oplus S_l(r_l - 1) \to S_l(r_l) \to 0$ we get $0 = \mathrm{Ext}_A(\tau_{\mathcal{G}} M, \tau S_l(r_l))$, hence $\tau_{\mathcal{G}} M = \tau_{\overline{\mathcal{G}}} M$ for all $M \in \mathcal{P}_{\overline{\mathcal{G}}}$ and the claim follows.

The second statement follows from 6.1.

LEMMA 6.4. $T_1 = U_1$ *and* $\widetilde{\mathcal{G}} = \widehat{\mathcal{G}}$.

Proof. The first claim follows from 6.3. Since $T_2 = U_2 \oplus W_l$, we get

$$\begin{aligned} T_2^{\perp} &= \{X \in A\text{-mod} \mid \mathrm{Hom}(T_2, X) = 0 = \mathrm{Ext}(T_2, X)\} \\ &= \{X \in A'\text{-mod} = W_l^{\perp} \mid \mathrm{Hom}(U_2, X) = 0 = \mathrm{Ext}(U_2, X)\}. \end{aligned}$$

This gives

$$\begin{aligned}\widetilde{\mathcal{G}} &= \{X \in \mathcal{G} \mid \mathrm{Hom}(T_2, X) = 0 = \mathrm{Ext}(T_2, X)\}\\ &= \{X \in \bar{\mathcal{G}} \mid \mathrm{Hom}(U_2, X) = 0 = \mathrm{Ext}(U_2, X)\} = \widehat{\mathcal{G}}.\end{aligned}$$

LEMMA 6.5. *T_2 is regular in A-mod.*

Proof. By (ind1) the module U_2 is regular in A'-mod, and consequently it is regular in A-mod. Since $T_2 = U_2 \oplus W_l$, by 6.4, it is regular in A-mod. In particular, $P = 0$ and $T_2 = \oplus_{i=1}^{l} W_i$.

LEMMA 6.6. *If $X \in \mathcal{G}$ is indecomposable and not preinjective in A-mod, then*:

(a) *$\tau_{\mathcal{G}}^{-m} X$ is in $\widetilde{\mathcal{G}}$ for $m \gg 0$.*
(b) *$\tau_{\mathcal{G}}^{-m} X = \tau_{\widetilde{\mathcal{G}}} \tau_{\mathcal{G}}^{-m-1} X$ for $m \gg 0$.*
(c) *If X is not in $\mathcal{P}_{\mathcal{G}}$, then $\tau_{\mathcal{G}}^{-m} X$ is a regular $T_2^{\perp}$-module for $m \gg 0$.*

Proof. Take $X \in \mathcal{G}$ indecomposable and not preinjective in A-mod. Then $\tau_{\mathcal{G}}^{-m} X \neq 0$ for all $m \geq 0$ and by 3.4 there is an m_0 with $\tau_{\mathcal{G}}^{-m} X \in W_l^{\perp}$ hence in $\bar{\mathcal{G}}$ for all $m \geq m_0$. Let $Y = \tau_{\mathcal{G}}^{-m_0} X$. By 4.3, we have $\tau_{\mathcal{G}}^{-t} Y = \tau_{\bar{\mathcal{G}}}^{-t} Y$ for all $t \geq 0$ and Y is not preinjective in A'-mod since $\tau_{\bar{\mathcal{G}}}^{-t} Y \neq 0$ for all $t \geq 0$. The claim now follows from 6.4 and (ind2).

LEMMA 6.7. *For $X_1, X_2 \in \mathcal{G}$, not in $\mathcal{P}_{\mathcal{G}}$, we have $\mathrm{Hom}_A(X_1, \tau_{\mathcal{G}}^{-m} X_2) = 0$ for $m \gg 0$.*

Proof. It is enough to consider X_1, X_2 not preinjective in A-mod. By 6.6(b,c) there is an integer $s > 0$ with $\tau_{\mathcal{G}}^{-r} X_i$ a regular $T_2^{\perp}$-module for $i = 1, 2$ and all $r \geq s$ such that $\tau_{\mathcal{G}}^{-r} X_i = \tau_{\widetilde{\mathcal{G}}}^{s-r} \tau_{\mathcal{G}}^{-s} X_i = \tau_{T_2^{\perp}}^{s-r} \tau_{\mathcal{G}}^{-s} X_i$. By 1.3(a) we therefore get $\mathrm{Hom}(\tau_{\mathcal{G}}^{-s} X_1, \tau_{\mathcal{G}}^{-s-m} X_2) = 0$ for $m \gg 0$. Since $\tau_{\mathcal{G}}$ is a full functor, the claim follows.

The third statement of Theorem 2 is shown by induction on l. We start with the case $l = 1$.

LEMMA 6.8. *Let $T = T_1 \oplus W_l$. If X is a regular module in $W_l^{\perp}$, then $\tau_{W_l^{\perp}}^{-m} X = \tau_{\mathcal{G}} \tau_{W_l^{\perp}}^{-m-1} X$ for $m \gg 0$.*

Proof. Since T_1 is a preprojective tilting module in $W_l^{\perp}$, all regular $W_l^{\perp}$-modules are in $\widetilde{\mathcal{G}} = \bar{\mathcal{G}}$ and $\tau_{W_l^{\perp}} X = \tau_{\bar{\mathcal{G}}} X$, for all X regular in $W_l^{\perp}$.

Choose m_0 with $\mathrm{Hom}(Z, \tau_{W_l^{\perp}}^{-m} X) = 0$ for all $m \geq m_0$ (see 1.3). By 4.3, we have a universal sequence

$$0 \to \tau_A S_l(r_l) \otimes D\mathrm{Ext}(\tau_{\mathcal{G}} \tau_{W_l^{\perp}}^{-m-1} X, \tau_A S_l(r_l)) \xrightarrow{f} \tau_{W_l^{\perp}}^{-m} X \xrightarrow{g} \tau_{\mathcal{G}} \tau_{W_l^{\perp}}^{-m-1} X \to 0.$$

We show $\mathrm{Hom}(\tau_A S_l(r_l), \tau_{W_l^\perp}^{-m} X) = 0$, for $m \geq m_0$, which implies $f = 0$. Therefore g is an isomorphism.

Consider the Auslander–Reiten sequence

$$0 \to \tau_A S_l(r_l) \to Z \oplus S_l(r_l - 1) \to S_l(r_l) \to 0.$$

Applying $\mathrm{Hom}(-, \tau_{W_l^\perp}^{-m} X)$ to this sequence, we get $\mathrm{Hom}(\tau_A S_l(r_l), \tau_{W_l^\perp}^{-m} X) \cong \mathrm{Hom}(Z, \tau_{W_l^\perp}^{-m} X) = 0$ for $m \geq m_0$.

The proof of the inductive step is quite similar. Let X be regular in $T_2^\perp$. By (ind3) and 6.4 we get $\tau_{\overline{\mathcal{G}}}^r \tau_{T_2^\perp}^{-m-r} X = \tau_{T_2^\perp}^{-m} X$ for $m \geq m_0$ and $r \geq 0$. As in the proof of 6.8 we get $\mathrm{Hom}(\tau_A S_l(r_l), \tau_{T_2^\perp}^{-m} X) \cong \mathrm{Hom}(Z, \tau_{T_2^\perp}^{-m} X)$. Since Z is in $\overline{\mathcal{G}}$ by 5.2, it follows that $0 = \mathrm{Hom}(Z, \tau_{T_2^\perp}^{-m-r} X) = \mathrm{Hom}(Z, \tau_{\overline{\mathcal{G}}}^{-r} \tau_{T_2^\perp}^{-m} X)$ for $r \gg 0$, by 6.7. In particular, $\mathrm{Hom}(\tau_A S_l(r_l), \tau_{\overline{\mathcal{G}}} \tau_{T_2^\perp}^{-m-r-1} X) = 0$. Considering the universal exact sequence

$$0 \to \tau_A S_l(r_l)^t \to \tau_{\overline{\mathcal{G}}} \tau_{T_2^\perp}^{-m-r-1} X \to \tau_{\mathcal{G}} \tau_{T_2^\perp}^{-m-r-1} X \to 0$$

the claim follows.

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