COLLOQUIUM MATHEMATICUM

VOL. 73 1997 NO. 1

SYSTEMS OF FINITE RANK

BY

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Ergodic theory, and particularly its measure-theoretic part, was developed to model physical systems; but a growing part of it is devoted to the building of new "hand-made" systems, generally to provide a specific example, or, in the vast majority of cases, counter-example; the object of this survey (which, in a reduced form and in French, was originally a part of the author's habilitation thesis) is the study of a special class of these examples, rich enough to allow us to see general structures and common properties.

When Ornstein finally cracked the problem of isomorphism of Bernoulli shifts ([ORN1]), he also, independently but using related techniques, wrote a less noticed paper ([ORN2]), producing an example of a transformation with no square root; this paper had perhaps more consequences than the isomorphism theorem, as it gave a new life to a class of systems which had been defined by Chacon ([CHA1]) as "a class of geometric constructions", and was then formalised under the name "class one" before being eventually called "rank one" ([ORN-RUD-WEI]). Later it was also useful to speak of rank two, three . . . , and also to use some related notions like local or joining rank. Most of these systems admit a very simple constructive definition, and, all together, they form a very rich zoo of examples and counter-examples, to meet almost all needs of the ergodicians, while the class is small enough to allow some general results. This field is still very active nowadays, as the multiple bibliographic references less than five years old testify.

The aim of this survey is, first, to gather, in a very scattered existing literature, all the definitions related to these notions of rank, to describe the relations between them and their basic properties; this may sometimes look "technical", but the lack of such a dictionary was, in the author's humble opinion, deeply felt, and was the prime reason for writing this study. Then we aim to give at least a general, though necessarily not exhaustive, overview of the richness of the zoo; and finally, both to see how the definitions work and to emphasize that these classes are also "natural", to show how these notions apply to several classical systems. As even these modest aims tended

1991 *Mathematics Subject Classification*: 28D.

to produce a lengthy paper, we gave ourselves two severe restrictions: we do not give proofs, except in the not so rare cases when they do not appear anywhere in the literature and are much needed; and, though the notions we describe have been extended to wider fields, such as systems with an infinite measure, or non-measure-preserving systems, or actions of $\mathbb{R}$, $\mathbb{Z}^n$ and $\mathbb{R}^n$, we restrict ourselves to **finite measure-preserving actions of** $\mathbb{Z}$.

Before beginning the core of this work, we gather the fundamental notions which are needed in the sequel.

0.1. Measure-theoretic dynamical systems. Throughout this survey, we shall deal with **measure-theoretic dynamical systems** and they will be **finite measure-preserving**. So, unless otherwise stated, we consider systems of the form $(X, \mathcal{A}, T, \mu)$ where X is a Lebesgue space, $\mathcal{A}$ its Lebesgue σ-algebra, T an invertible mapping, and μ a probability measure such that $T\mu = \mu$. All sets, mappings, partitions ... considered are tacitly assumed to be **measurable**, and all relations, equalities, etc... involving only measurable (as opposed to continuous) quantities are tacitly assumed to hold only almost everywhere. In this category, we use the notion of **measure-theoretic isomorphism**:

Definition 1. Two systems $(X, \mathcal{A}, T, \mu)$ and $(X', \mathcal{A}', T', \mu')$ are **measure-theoretically isomorphic** if there exist $X_1 \subset X$, $X_1' \subset X'$, and a (bimeasurable) bijection ϕ from X_1 to X_1' such that $\mu(X_1) = \mu'(X_1') = 1$, $\phi\mu = \mu'$ and $T'\phi = \phi T$.

We shall use freely the fact that a system is defined up to measure-theoretic isomorphism.

In particular, it is possible to reduce measure-theoretic ergodic theory to the study of sequences over a finite alphabet; as soon as we have a partition P of X (all partitions will be supposed **finite** unless otherwise stated), we may look at the P-names; the P**-name** $PN(x)$ of a point x under the transformation T is the sequence such that $PN(x)_i = j$ whenever $T^i x \in P_j$; the P-n**-name** of a point x is the sequence $(PN(x)_i, 0 \leq i \leq n)$, the **positive** P**-name** is the sequence $(PN(x)_i, i \geq 0)$. And T may be seen as the **shift** on the P-names, $(Ty)_n = y_{n+1}, n \in \mathbb{Z}$. Krieger's theorem ([KRI1], [KRI2]) ensures that, provided the entropy is finite, which will always be the case here, there always exists a *generating partition* (see 0.2.3), and so every system may be represented as the shift on some set of symbolic sequences.

0.2. A small glossary of measure-preserving ergodic theory. All our systems will be interesting by their situation regarding the "classical" properties of measure-preserving dynamical systems; we feel our duty to give here the appropriate definitions, though we beg the reader to refer to these only in case of urgent need. In the main course of this study, new notions will also be introduced; they will be written in **bold** if defined explicitly, and in *italics* if the reader is referred to a later part of the text, or to existing literature (i.e. any book of ergodic theory, for example [COR-FOM-SIN], if no precise reference is given).

We recall that $\mathbb{N}$ is the set of nonnegative integers and $\mathbb{N}^\star$ is the set of positive integers.

0.2.1. *Distances, words.* Let us recall two notions of **distance** which we shall often use (and which are of course related through the association of P-names to points): the **distance between partitions**,

$$|P - P'| = \sum_i \mu(P_i \triangle P'_i),$$

and the **Hamming distance** between finite sequences,

$$\bar{d}((x_1, \dots, x_n), (y_1, \dots, y_n)) = \frac{1}{n}\#\{i \in \{1, \dots, n\} : x_i \neq y_i\}.$$

We also recall that a finite sequence $x_1 \dots x_n$, over a finite or infinite alphabet, is called a **word**, or a **block** of **length** n; the concatenation of two words is denoted multiplicatively. In the set $\{0,1\}^{\mathbb{Z}}$, we call the sets of the form $(x_{i_1} = a_1, \dots, x_{i_k} = a_k)$ **cylinders**.

0.2.2. *Ergodicity, mixing, rigidity.* A system is **ergodic** if, when a set A satisfies $\mu(A \triangle TA) = 0$, then $\mu(A) = 0$ or 1. This is equivalent to saying that the associated **spectral operator** on $\mathcal{L}^2(X)$, $U_T f = f \circ T$, has 1 as a simple eigenvalue. If U_T has no eigenvalue except 1, T is **weakly mixing**.

T is **strongly mixing** if for every set A and every set B, $\mu(A \cap T^n B)$ tends to $\mu(A)\mu(B)$ as $n \to \infty$. It is **mixing of order** p if $\mu(T^{k_1}A_1 \cap \dots \cap T^{k_p}A_p)$ tends to $\mu(A_1)\dots\mu(A_p)$ as $(k_1, k_2 - k_1, \dots, k_p - k_{p-1})$ tend to infinity independently (it is an old open question whether mixing of order 2 implies mixing of higher order).

On the opposite side, T is **rigid** if there exists a sequence k_n such that $\mu(T^{k_n} \triangle A) \to 0$ for every $A \subset X$.

0.2.3. *Partitions, entropy.* A partition P **refines** a partition Q if each atom of Q is a union of atoms of P; this is an order relation, and we denote by $P \vee Q$ the supremum of P and Q; P is called a **generating partition** if the partitions $\bigvee_{i=-n}^{n} T^i P$ increase to the whole σ-algebra $\mathcal{A}$.

The **entropy** of a partition $P = (P_1, \dots, P_k)$ is defined by

$$h(P) = -\sum_{i=1}^{k} \mu(P_i) \log \mu(P_i);$$

the **mean entropy** of a partition P is

$$h(P,T)=\lim_{n\to\infty}\frac{1}{n}h\Big(\bigvee_{i=0}^{n-1}T^iP\Big).$$

The (measure-theoretic) **entropy of the system** is the supremum of $h(P,T)$ over the set of all partitions, or else the entropy of any generating partition.

0.2.4. *Induction, Kakutani equivalence, Loose Bernoulli.* For a map T, a set A of positive measure and a point x in A, we define the **return time** $\tau_A(x)$ of x in A as the smallest positive integer n such that T^nx is in A again; this time is finite almost surely (by Poincaré's recurrence theorem) and we may define the **induced transformation** T_A of T on A by $T_A(x)=T^{\tau_A(x)}x$; it preserves the **induced measure** on A defined by $\mu_A(E)=\mu(E)/\mu(A)$ for $E\subset A$.

Two systems (X,T) and (Y,S) are **Kakutani-equivalent** if there exist subsets A of X and B of Y, of positive measure, such that $T_A=S_B$ up to measure-theoretic isomorphism. Now, among the systems of entropy zero, we define the class of **Loosely Bernoulli**, or **LB** systems to be the smallest class which is closed under Kakutani equivalence and contains all the irrational rotations.

0.2.5. *Factors, centralizer, joinings, Veech simplicity.* A **factor** of a system $(X,\mathcal{A},T,\mu)$ is the same system restricted to a T-invariant sub-σ-algebra $\mathcal{B}$: we say that two points are equivalent if they are not separated by the sets of $\mathcal{B}$, and we let T act on the quotient set; if every equivalence class has p elements, we say that the factor has **fiber** p. A map is **prime** if it has no nontrivial factor.

The **centralizer** of T is the set of all measure-preserving maps S such that $TS=ST$ (if T is ergodic, we get the same set if we require only that S preserves the equivalence class of the measure and $ST=TS$); it is called **trivial** if it reduces to the powers T^n.

A **joining** of two systems (X,T,μ) and (Y,S,ν) is an ergodic measure on $X\times Y$, invariant under $T\times S$ and whose marginals are μ and ν. Two systems are **disjoint** (in the sense of Furstenberg) if their only joining is the product measure.

A system has **minimal self-joinings** if the only joinings of this system with itself are the product measure $\mu\times\mu$ and measures of the form $\varrho(A\times B)=\mu(A\cap T^nB)$. A system with minimal self-joinings is prime, and its centralizer reduces to its powers.

A system has **minimal self-joinings of order** p if for every p-uple $(k_1,\dots,k_p)$ of positive integers, the ergodic measures which are invariant for $T^{k_1}\times\ldots\times T^{k_p}$, and have μ as marginals, are all the products of diagonal measures, that is, measures defined by $\nu(A_1\times\ldots\times A_n)=\mu(T^{l_1}A_1\cap\ldots\cap T^{l_n}A_n)$.

A system is **simple** (in the sense of Veech) if its self-joinings are the product measure and measures of the type $\mu(A \cap S^{-1}B)$ for elements S of the centralizer; this is a weaker property than the minimal self-joinings, since it allows T to have factors, and a big centralizer, which is the case for example if T is rigid. **Simplicity of order** p is defined like minimal self-joinings of order p by replacing, in the definition of diagonal measures, $T^{l_1}, \ldots, T^{l_p}$ by arbitrary elements $S_1, \ldots, S_p$ of the centralizer.

0.2.6. *Horocycle flows.* The **horocycle flow** can be defined as the left multiplication by the matrix

$$h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

on some compact quotient group of $SL_2\mathbb{R}$, for example $SL_2\mathbb{R}/SL_2\mathbb{Z}$. These define of course many different horocycle flows, but they will have the same properties as far as we are concerned.

0.2.7. *Spectral properties.* The map T, or its associated spectral operator, has **discrete spectrum** if the space $\mathcal{L}^2(X)$ is generated by eigenvectors of U_T (see 0.2.2).

The **cyclic space** generated by an element f of $\mathcal{L}^2(X)$, denoted by $H(f)$, is the closed linear space generated by the $U_T^n f$, $n \in \mathbb{Z}$. The **spectral type** of f is the measure σ_f on the torus S^1 defined by $\widehat{\sigma}_f(n) = (U_T^n f, f)$. There is a standard decomposition of $\mathcal{L}^2(X)$ in a finite or infinite orthogonal sum of cyclic spaces $H(f_i)$, where $\sigma_{f_{i+1}}$ is absolutely continuous with respect to σ_{f_i}; the number of cyclic spaces and the equivalence classes of the spectral types σ_{f_i} are invariants of the spectral system $(\mathcal{L}^2(X), U_T)$. The **maximal spectral type** is the first measure σ_{f_1}; we say that T has **singular**, resp. **absolutely continuous spectrum** if the maximal spectral type is singular, resp. absolutely continuous with respect to the Lebesgue measure on the torus (if none of these cases occurs, we say T has **mixed spectrum**); if there are p cyclic spaces in the decomposition, $p = 1, 2, \ldots, \infty$, we say that T has **spectral multiplicity** equal to p; if $p = 1$ we say that T has **simple spectrum** (in $\mathcal{L}^2$).

The notions of multiplicity and simple spectrum may be extended to the spaces $\mathcal{L}^q(X)$ for every $q > 0$: we just define the cyclic spaces with the $\mathcal{L}^q$-topology, and require that p cyclic spaces generate $\mathcal{L}^q(X)$.

0.2.8. *Minimality and unique ergodicity.* A *topological system* (that is, a continuous map acting on a topological space) (X, T) is **minimal** if every orbit under T is dense in X. It is **uniquely ergodic** if there exists only one probability measure invariant under T. Unique ergodicity implies minimality if the invariant measure gives nonzero measure to every nonempty open set.

1. RANK ONE

1.1. The lecturer's nightmare: how to define a rank one system. Though it is easy to define any particular rank one system (see 1.4), several different definitions of what is a rank one system appear in the literature; they are (modulo one small snag, see 1.1.4 and 1.3.1) all equivalent, each of them may be useful in some context, and none of them is short and easy to explain. We try to give them all, and emphasize the links between them, as this is sadly missing in the existing literature.

1.1.1. *Nonconstructive geometric definition.* The class of systems now known as the rank one systems appeared gradually between 1970 and 1975; it evolved from the notion of *periodic approximation* used previously by the Russian school ([KAT-STE]) by a weakening of the requirements; after the first examples by Chacon ([CHA1]), Ornstein ([ORN2]) gave the first definition of the rank one systems as a coherent family of systems.

Definition 2 (NG). A system is of **rank one** if for any partition P of X and any positive ε, there exist a subset F of X, a positive integer h and a partition P' of X such that:

- the T^kF, $0 \le k \le h-1$, are disjoint,
- $|P - P'| < \varepsilon$,
- P' is refined by the partition formed by the sets F, TF, $\dots$, $T^{h-1}F$, and $X \setminus \bigcup_{i=0}^{h-1} T^iF$.

We then say that $(F, TF, \dots, T^{h-1}F)$ is a **Rokhlin tower** (or **stack**), of **height** h, approximating P; as we need to approximate arbitrarily sharp partitions in a nonatomic space, it follows from the definition that the towers must be arbitrarily high and fill a part of the space of measure arbitrarily close to one.

1.1.2. *Nonconstructive symbolic definition.* The translation of this definition in a symbolic setting is not immediate, it was done in ([deJ2]):

Definition 3 (NS). A system is of **rank one** if for every partition P of X, every positive ε, and every natural l, there exists a word B of length $l(B)$ greater than or equal to l such that, for all n large enough, on a subset of X of measure at least $1-\varepsilon$, the P-n-names of points are of the form $\delta_1 W_1 \dots \delta_p W_p \delta_{p+1}$ with $l(\delta_1) + \dots + l(\delta_p) < \varepsilon n$ and $\bar{d}(W_i, B) < \varepsilon$ for all i.

1.1.3. *Constructive geometric definition.* These nonconstructive definitions were of course satisfied by the first examples of Chacon ([CHA1]); conversely, it was shown in [BAX] or [FER0] that each rank one system can be explicitly built by a sequence of nested Rokhlin towers generating the whole σ-algebra; this may be written as:

Definition 4 (CG). A system is of **rank one** if there exist sequences of positive integers q_n, $n \in \mathbb{N}$, and $a_{n,i}$, $n \in \mathbb{N}$, $1 \le i \le q_n - 1$, such that, if h_n is defined by

$$h_0 = 1, \qquad h_{n+1} = q_n h_n + \sum_{j=1}^{q_n - 1} a_{n,i},$$

then

$$\sum_{n=0}^{\infty} \frac{h_{n+1} - q_n h_n}{h_{n+1}} < \infty, \tag{1.1}$$

and subsets of X, denoted by F_n, $n \in \mathbb{N}$, by $F_{n,i}$, $n \in \mathbb{N}$, $1 \le i \le q_n$, and by $C_{n,i,j}$, $n \in \mathbb{N}$, $1 \le i \le q_n - 1$, $1 \le j \le a_{n,i}$ (if $a_{n,i} = 0$, and if $i = q_n$, there are no $C_{n,i,j}$), such that for all n:

- $(F_{n,i}, 1 \le i \le q_n)$ is a partition of F_n,
- the $T^k F_n$, $1 \le k \le h_n - 1$, are disjoint,
- $T^{h_n} F_{n,i} = C_{n,i,1}$ if $a_{n,i} \ne 0$ and $i < q_n$,
- $T^{h_n} F_{n,i} = F_{n,i+1}$ if $a_{n,i} = 0$ and $i < q_n$,
- $TC_{n,i,j} = C_{n,i,j+1}$ if $j < a_{n,i}$,
- $TC_{n,i,a_{n,i}} = F_{n,i+1}$ if $i < q_n$,
- $F_{n+1} = F_{n,1}$,

and the partitions $\{F_n, TF_n, \dots, T^{h_n - 1} F_n, X \setminus \bigcup_{k=0}^{h_n - 1} T^k F_n\}$ are increasing to $\mathcal{A}$.

If the q_n and $a_{n,i}$ are known, it is easy to build sets $F_{n,i}$ and $C_{n,i,j}$ and a transformation T satisfying all the above rules: the sets are chosen to be intervals (open or closed; being in a measure-theoretic setting we are not concerned by the set of endpoints which is of measure zero), and we build a piecewise affine T by simply following the instructions; this defines T by stages: at stage n, T is defined everywhere except on the top of the last column (which is $T^{h_n - 1} F_{n,q_n}$); the last condition ensures that this eventually defines a measure-theoretic system, and, given the sequences q_n and $a_{n,i}$ the system defined is unique up to measure-theoretic isomorphism. Note that the condition (1.1) ensures that the total measure is finite, but that without it every system would be of rank one, as was pointed out to the author by Arnoux. Note also that for a given rank one mapping T we can find sequences q_n and $a_{n,i}$ but they are by no means unique.

1.1.4. *Constructive symbolic definition.* The constructive definition translates in the symbolic setting ([KAL]) but we have an equivalent definition only when the system is **totally ergodic**, that is, when every power T^n is ergodic; this is nowadays the most widely used definition of rank one (for what may happen with non-totally ergodic systems, see 1.3.1):

Definition 5 (CS). A totally ergodic system is of **rank one** if there exist sequences of positive integers q_n, $n \in \mathbb{N}$, and $a_{n,i}$, $n \in \mathbb{N}$, $1 \le i \le q_n - 1$, satisfying (1.1) such that, if we define the words B_n on the alphabet $\{0,1\}$ by

$$(1.2) \qquad B_0 = 0, \qquad B_{n+1} = B_n 1^{a_{n,1}} B_n 1^{a_{n,2}} B_n \ldots B_n 1^{a_{n,q_n-1}} B_n,$$

then T is the shift on the set X of sequences (x_n) of $\{0,1\}^{\mathbb{Z}}$ such that for every pair of integers $s < t$, $(x_s \ldots x_t)$ is a subsequence of B_n for some n.

This definition does not state the measure but, again because of (1.1), the topological system defined above admits only one nonatomic invariant probability measure, the measure μ which gives to the cylinder set $(x_1 = a_1, \ldots, x_p = a_p)$ a measure equal to the limit as n goes to infinity of the frequency of occurrences of the word $a_1 \ldots a_p$ in the bloc B_n. The definition given in [KAL] contains this expression, the simplified definition we give above appears in [deJ-KEA], [deJ-RUD] for particular cases. Note that there may be other invariant measures, the measure δ_1 which gives mass one to the sequence identically equal to 1, and all convex combinations of μ and δ_1. Similarly, the topological system defined above is not minimal whenever the numbers $a_{n,j}$ are not bounded, but the only nondense orbit is then reduced to the sequence identically equal to 1.

A rank one system is completely known if we know the **recursion formula** (1.2) giving B_{n+1}. The digit 1 in it is often replaced by the letter s, for **spacer**. The words B_n are called the n**-blocks**.

1.2. First properties and the reduced geometric definition. A rank one system is ergodic and of entropy zero ([CHA1] or [BAX]).

This allows us to give a useful new definition of rank one systems:

Definition 6 (RG). A system is of **rank one** if for any subset A of X and any positive ε, there exist a subset F of X, a positive integer h and a subset A' of X such that:

- the $T^k F, 0 \le k \le h-1$, are disjoint,
- $\mu(A \triangle A') < \varepsilon$,
- $\mu(\bigcup_{i=0}^{h-1} T^i F) > 1 - \varepsilon$,
- A' is measurable with respect to the partition formed by the sets $F, TF, \ldots, T^{h-1}F$, and $X \setminus \bigcup_{i=0}^{h-1} T^i F$.

This is simply the nonconstructive geometric definition, but restricted to two-set partitions; it looks weaker, but is in fact equivalent, as h has to be arbitrarily large; this allows us to approximate by towers also partitions of the form $\bigvee_{i=-n}^{n} T^i P$ for any partition P of X in two sets; this is enough to prove that the system has entropy zero, and so, by Krieger's theorem, has a generating partition with two elements, and thus every partition can be approximated by towers.

Every rank one system is Loosely Bernoulli ([ORN-RUD-WEI]). The LB property has also a "technical" definition which may be found in [FEL] and will not be given here; it uses some new distance between names. The LB class includes all rank one systems, and, as we shall see, most related systems, but also systems like the time-one map of the horocycle flow. Every rank one is Kakutani-equivalent to every other rank one, and induces every irrational rotation.

Every factor of a rank one system is of rank one ([deJ2] as an easy consequence of the nonconstructive symbolic definition).

1.3. First examples, and the last definition

1.3.1. *Von Neumann–Kakutani's transformation.* It is defined in [voN], and is also called **van der Corput's transformation**, or **dyadic odometer**. We can define it by

$$T\left(1-\frac{1}{2^n}+x\right)=\frac{1}{2^{n+1}}+x \quad \text{for } 0\le x<\frac{1}{2^{n+1}},\ n\in\mathbb{N},$$

or, isomorphically, as the translation $Tx = x+1$ on the group of dyadic integers, or as a vehicle odometer: we take $X=\{0,1\}^{\mathbb{N}}$ and, if $x=(x_0,\dots)$ with $x_i=1$ for all $i<k$ and $x_k=0$, then $Tx=(x'_0,\dots)$ with $x'_i=0$ for all $i<k$, $x'_k=1$, and $x'_l=x_l$ for all $l>k$. We put on X the product measure giving mass $1/2$ to 0 and 1 on each copy of $\{0,1\}$.

It is clear that T satisfies the constructive geometric definition with all the q_n equal to 2 and all the $a_{n,i}$ equal to 0. However, if T did satisfy the constructive symbolic definition with these same parameters, it would have the recursion formula $B_{n+1}=B_nB_n$, which gives a periodic sequence, while T is ergodic and hence aperiodic; in fact, it is not known whether T, which is not totally ergodic, satisfies the constructive symbolic definition with any family of parameters; which explains why it is not a satisfactory definition of rank one in this case.

T is in some sense the simplest rank one system; it has discrete spectrum, the eigenvalues being all the dyadic rationals.

1.3.2. *The general odometer.* It is defined for any sequence q_n of positive integers, by taking X to be the set $\prod_{n\in\mathbb{N}}\{0,\dots,q_n-1\}$; if $x=(x_0,\dots)$ with $x_i\ne q_i-1$ for all $i<k$ and $x_k=q_k-1$, then $Tx=(x'_0,\dots)$ with $x'_i=0$ for all $i<k$, $x'_k=x_k+1$, $x'_l=x_l$ for all $l>k$.

T preserves a finite measure, has a discrete spectrum, and satisfies the constructive geometric definition with all the $a_{n,i}$ equal to zero. Conversely, every rank one system may be written as a **Rokhlin–Kakutani tower**

over an odometer, and this is used in [HOS-MEL-PAR] as a definition of rank one systems:

1.3.3. *Definition by towers over odometers*

DEFINITION 7 (TO). A system (X, T, μ) is of **rank one** if there exists an odometer (Y, S, ν) and a map n from Y to $\mathbb{N}^\star$ such that:

- $n(y)$ depends only on $k(y) = \inf\{i : y_i \neq q_i - 1\}$,
- n is integrable,
- $X = ((y, z) \in Y \times \mathbb{Z} : z \leq n(y) - 1)$,
- μ is the measure defined by ν and the counting measure on $\mathbb{Z}$,
- $T(y, z) = (y, z + 1)$ if $z < n(y) - 1$,
- $T(y, n(y) - 1) = (Sy, 0)$.

1.3.4. *The irrational rotations.* Like the odometers, they are translations of compact groups preserving the Haar measure; they will be studied at length in 3.1, but it is already useful to know that they are of rank one.

1.4. The famous rank one systems: a guided tour of the zoo

1.4.1. *Chacon's map*

$$B_{n+1} = B_n B_n 1 B_n$$

This is, of course, the constructive symbolic definition, given by the recursion formula; note that this definition is self-similar, the recursion formula is the same for all n; in fact, this map falls into the great family of *substitutions*, which deserve a whole section (3.3) of this study.

Chacon's map, with its very short and easy definition, hides a wealth of properties which make it a central point in ergodic theory: it is the first known example of a weakly mixing map which is not strongly mixing ([CHA2]); it was shown later to be prime and to have trivial centralizer ([deJ3]); in fact, it is the simplest known transformation to have minimal self-joinings of all orders ([deJ-RAH-SWA]), and so is a good (cheap) fuel for the *counter-example machine*, see 1.5.3. It is worth mentioning that the proof of minimal self-joinings for Chacon's map uses the presence of isolated spacers between blocks B_n; a similar property, called the *R-property*, was used in [RAT2] to compute the joinings of the horocycle flows, and was the basis of the famous papers of Ratner which culminate in the proof of Ragunathan's conjecture ([RAT3]).

Chacon's map also has a property of partial rigidity: for every set A, there exists a sequence r_n such that $\liminf \mu(T^{r_n} A \cap A) \geq (2/3)\mu(A)$; this implies that T has singular spectrum (Friedman, unpublished). Let us also mention that its **Cartesian square** has been the object of still unfinished studies; it is not known whether it is LB or not, and [deJ-KEA] contains a deep result about the explicit determination of its generic points.

1.4.2. *Generalized Chacon's maps*

$$B_{n+1} = B_n^{r_n} 1 B_n^{s_n}$$

with r_n and s_n sequences of positive integers and $(r_n, s_n) \neq (1,1)$ for infinitely many values of n. They share some of the properties of Chacon's map, particularly the weak mixing and the absence of strong mixing; if we let the sequences r_n and s_n vary suitably, we get a continuum of rank one systems which are nonisomorphic, and even disjoint in the sense of Furstenberg (Fieldsteel, unpublished, quoted in [deJ-RAH-SWA]).

One particular case, $r_n = s_n = 2^n$, was investigated by del Junco and Rudolph ([deJ-RUD]); their map is shown to be rigid, prime and simple.

1.4.3. *Katok's map*

$$B_{n+1} = B_n^{p_n}(B_n 1)^{p_n}$$

for a sequence p_n growing fast enough (the only published reference is [GER]).

Though it is almost immediate that a rank one is LB, Katok's map is the only known zero-entropy map whose Cartesian square is still LB. This square has other interesting properties (see 2.2 and 2.4.1). Katok's map is also weakly mixing and rigid.

1.4.4. *Ornstein's mixing rank one*

$$B_{n+1} = B_n 1^{a_{n,1}} B_n 1^{a_{n,2}} B_n \dots B_n 1^{a_{n,q_n-1}} B_n$$

It is proved in [ORN2] that if we fix a sequence q_n growing fast enough and if we draw the $a_{n,i}$ at random in an equidistributed way between two reasonable (taking (1.1) into account) integers K_n and L_n, then with arbitrarily high probability the rank one map T thus defined will be strongly mixing. The proof was simplified in [POL] and [RUD]. So, in a way, strongly mixing maps are generic in the class of rank one maps, though there is no explicit example of Ornstein's rank one maps.

"The" Ornstein map has been shown to have a trivial centralizer ([ORN2]) and to be prime ([POL]). One version of it, which uses an a priori smaller set of possible parameters, is mixing of all orders and has minimal self-joinings of all orders ([RUD]). However, all this was proved later to be true for all mixing rank one systems (see 1.5.2). One version of Ornstein's map has a non-LB Cartesian product ([ORN-RUD-WEI]); one version has singular spectrum (Thouvenot, unpublished); see also 1.6.2 for the general problem of the singularity of the spectrum.

1.4.5. *Smorodinsky–Adams' maps*

$$B_{n+1} = B_n B_n 1 B_n 11 B_n 111 B_n \dots 1^{n-2} B_n$$

This is the simplest explicit example of mixing rank one map ([ADA] proving a conjecture of Smorodinsky); others can be found more generally by taking (1.2) with $a_{n,i} = i-1$, $1 \le i \le q_n-1$ (the so-called **staircase construction**) for many (explicit) sequences q_n: in [ADA-FRI] the construction is done for sequences q_n having any prescribed polynomial growth, and in [ADA] for any q_n such that (1.1) is satisfied and

$$q_n/(\log h_n)^{1-a} \to 0$$

as $n \to \infty$, for some $a > 0$. They have singular spectrum ([KLE]).

1.4.6. *King's teratology.* In [KIN1], two examples are built, which have surprising properties for nonrigid rank one systems (though they are trivially satisfied by irrational rotations); one is a weakly mixing rank one T such that T^k is still of rank one for any $k \neq 0$; the other is a weakly mixing rank one system which is a denumerable cartesian product of (necessarily weakly mixing rank one) systems T_i, $i \in \mathbb{N}$. Both T and the T_i are built with recursion formulas using long concatenations of blocks B_n with few isolated spacers.

In [FRI-KIN] an example is built (derived from Chacon's map) of a rank one system which is **lightly mixing**, that is, $\liminf_{n\to\infty} \mu(T^n A \cap B) > 0$ whenever $\mu(A)\mu(B) > 0$, but not **partially mixing**, which would be $\liminf_{n\to\infty} \mu(T^n A \cap B) \ge \alpha\mu(A)\mu(B)$ for some $\alpha > 0$.

1.4.7. *Smooth models.* Though it is not our purpose to enter here the vast problem of smooth representations of abstract systems, it is worth mentioning that, among the examples in [ANO-KAT] of C^∞ transformations which may be realized on any two-dimensional compact manifold, there are rigid weakly mixing rank one systems.

Other smooth examples may be found among **Anzai skew products**, that is, extensions of irrational rotations by $\mathbb{R}$ or $\mathbb{S}^1$ with the Lebesgue measure ([ANZ]); after [IWA-SER] proved that many of these systems have rank one, [KWI-LEM-RUD] built an example of an Anzai skew product which is of rank one, *real analytic* and of nondiscrete spectral type; these properties are in fact generic among Anzai skew products ([IWA2]).

1.5. Measure-theoretic properties of rank one systems

1.5.1. *The weak closure theorem*

THEOREM 1 (WCT). *If T is of rank one, then every element of its centralizer is a weak limit of powers of T* ([KIN1]).

This creates a dichotomy among rank one systems; if T is not rigid, the centralizer is trivial, while it is uncountable when T is rigid ([KIN1]). This justifies the interest of rigid rank one systems, with their richer structure; they have been classified in [FRI-GAB-KIN].

A related problem is to find all measure-preserving maps such that $ST = T^{-1}S$, assuming of course that T is isomorphic to T^{-1}; it was proved in [GOO-deJ-LEM-RUD] that this relation implies $S^2 = I$ whenever T is of rank one, or has the weaker property (see 1.6.1) of simple spectrum. For a system satisfying the conclusion of the WCT, this relation implies $S^4 = I$.

1.5.2. *Mixing rank one systems.* It was shown successively that all mixing rank one systems have a trivial centralizer ([AKC-CHA-SCH]), are prime ([FER0]), and, finally, that they have minimal self-joinings of all orders ([KIN3]), which includes the previous properties. They are also mixing of order 3 ([KAL]), and, by the same proof, mixing of all orders; it is worth noticing that they form one of the very few classes of systems for which an answer to the old question about mixing of all orders is known.

1.5.3. *The counter-example machine.* Let T be a map with minimal self-joinings of all orders, which is true of Chacon's map or of every mixing rank one system (these and some horocycle flows are the only known examples). Let K be a finite or countable set, X_K and T_K the cartesian products of K copies of X and T respectively, π a compact permutation of K (that is, a permutation in which every cycle has finite length), l a map from K to $\mathbb{Z}$; then we define the map $U(\pi, l)$ on X_K by

$$U(\pi, l)(x_1, \ldots, x_n, \ldots) = (T^{l(\pi 1)}x_{\pi 1}, \ldots, T^{l(\pi n)}x_{\pi n}, \ldots).$$

Then [RUD] describes completely the centralizer and factors of the $U(\pi, l)$, and all possible isomorphisms between them, or elements of their centralizer (roughly, they are not trivial, but no more than we expect them to be: elements of the centralizer are some $U(\alpha, m)$ satisfying some cocycle relations, factors are given by subgroups of the centralizer or subsets of the coordinates, etc ...). This allows one, by choosing suitably π and l, to build a virtually unlimited number of counter-examples, among which:

- a map with a cubic root but no square root,
- two nonisomorphic maps T and S such that S^n is isomorphic to T^n for every n except -1 and $+1$,
- a map with a continuum of nonisomorphic square roots,
- a map T which has no roots while T^2 has roots of any order,
- two maps which are **weakly isomorphic** (each one is a factor of the other) but not isomorphic,
- a map with a continuum of nonisomorphic factors with *fiber two*,

• two maps which are **weakly disjoint** (they have no common factor) but not disjoint.

1.6. Spectral properties of rank one systems

1.6.1. *Simple spectrum.* We give here a proof of the following theorem due to Thouvenot; it is stated with another proof in [CHA3] but only for $p = 1$.

THEOREM 2. *A rank one system has simple spectrum in every* $\mathcal{L}^p$.

Proof. We want to show that $\mathcal{L}^p(X) = H(g)$ for some function g; or else, that

$$\bigcap_{n,p}\{g : d(f_n, H(g)) < 1/p\} \neq \emptyset$$

for a dense sequence of functions f_n; by Baire's theorem, it is enough to show that for given f and ε, the set $\{g : d(f, H(g)) < \varepsilon\}$ is dense; so we take f and a function k, which we may both suppose to have norm one; the definitions (NG) and (CG) imply easily the existence of a function ϕ such that $d(f, H(\phi)) < \varepsilon$ and $d(k, H(\phi)) < \varepsilon$.

Hence $\|k - P(U_T)\phi\|_p < \varepsilon$ for some polynomial P; hence $\|k - Q(T)\phi\|_p < 2\varepsilon$ for some polynomial Q which has no zero on the unit circle, which implies that $Q(T)$ is invertible and its inverse can be approximated by polynomials in T. So, if we take $g = Q(T)\phi$, we have $\|k - g\|_p < 2\varepsilon$ and $d(f, H(g)) < 2\varepsilon$. ∎

The converse of this theorem is false as we shall see in 2.4.2; however, it is true that discrete spectrum (which implies simple spectrum) plus ergodicity imply rank one ([deJ1]).

1.6.2. *Spectral type.* It is generally a difficult question to compute the maximal spectral type of a given system; for rank one systems, there is an algorithm, given in [HOS-MEL-PAR] in a more general setting (see also [CHO-NAD1]) to compute the spectral types of some particular functions, namely the indicator functions of the bases of the towers; as they are dense in $\mathcal{L}^2(X)$, these give access to the maximal spectral type ([CHA3]).

So let T be a rank one system defined by (1.2); let B_n be the word $b_1 \ldots b_{h_n}$; (1.2) ensures that B_{n+1} begins by B_n. Let A be the set defined in (CS) by $(x_0 = 0)$ (which means in (CG) the basis of the first tower; the computations are the same for the nth tower); let τ be its spectral type. It is easy to show, simply by identifying Fourier coefficients, that τ is a (vague) limit of the measures $\frac{1}{N}|A_N|^2 d\lambda$, where λ is the Lebesgue measure on the torus and $A_N(x) = \sum_{1 \le k \le N, b_k = 0} e^{ikx}$. This, together with (1.2), gives τ as a generalized Riesz product:

$$d\tau = \lim_{N\to\infty} \frac{1}{h_N} \prod_{n\le N} \Big| \sum_{j=0}^{q_n-1} e^{i(jh_n+c_{n,j})} \Big|^2 d\lambda,$$

where $c_{n,0} = 0$ and $c_{n,j} = a_{n,1} + \ldots + a_{n,j}$.

This expression was used in [CHO-NAD2] to compute the eigenvalues of rank one systems, and, spectacularly, in [BOU] to show that all known versions of Ornstein's map have singular spectrum, though this is not yet known to be true for every mixing rank one system; it was also used in [KLE-REI] to prove that rank one systems with a bounded sequence q_n have singular spectrum.

2. GENERALIZATIONS OF RANK ONE AND RELATED NOTIONS

2.1. Finite rank. The class of rank one systems, as we saw, does contain some natural systems, namely the two families of rotations: the odometers and the irrational rotations. But it appeared quickly ([ORN-RUD-WEI]) that, if we accept systems that can be approximated by two, three or r Rokhlin towers instead of one, we keep most properties but we include in our field two fundamental families of systems, the substitutions and the interval exchanges; hence the interest of defining a **finite rank**:

Definition 8 (NG). A system is of **rank at most** r if for every partition P of X and every positive ε, there exist r subsets F_i of X, r positive integers h_i and a partition P' of X such that

- $(T^j F_i, 1 \le i \le r, 0 \le j \le h_i - 1)$ are disjoint,
- $|P' - P| < \varepsilon$,
- P' is refined by the partition formed by the sets $(T^j F_i)$, $1 \le i \le r$, $0 \le j \le h_i - 1$, and $X \setminus \bigcup_{1\le i\le r, 0\le j\le h_i-1} T^j F_i$.

A system is of **rank** r if it is of rank at most r and not of rank at most $r-1$. A system is of **infinite rank** if no such finite r exists.

This definition admits a symbolic translation, which is a straightforward generalization of the (NS) definition of rank one; there are also constructive definitions, but they are somewhat tedious and are never stated except on examples; for once, we shall not make any exception to this rule.

Except for the aforementioned natural systems, we know one finite rank system which has been built completely "by hand"; it can be found in ([GOO-deJ-LEM-RUD]); it satisfies the weak closure theorem and is conjugate to its inverse by an isomorphism S such that S^4 is the identity but S^2 is not (see 1.5.1).

2.2. Local rank and covering number. It was remarked, as far back as [KAT-SAT], that most of the structure of finite rank systems does not lie in the fact that r Rokhlin towers approximate everything, but in the fact that one Rokhlin tower approximates everything on $1/r$ of the space. To formalize this weaker definition led to some new examples of the constructive sort, which are rich enough to justify the inclusion here of this apparently purely technical notion, first formalized in [FER1]:

DEFINITION 9 (NG). A system is of **local rank one** if there exists a positive number a such that for every partition P of X and every positive ε, there exist two subsets F and A of X, a positive integer h and a partition P' of A such that

- $(T^jF, 0 \leq j \leq h-1)$ are disjoint,
- $A = \bigcup_{j=0}^{h-1} T^jF$,
- $\mu(A) > a$,
- $|P' - P_{\setminus A}| < \varepsilon$,
- P' is refined by the partition $\{F, \ldots, T^{h-1}F\}$.

The supremum of all possible a's in this definition is called the **covering number** of the system ([KIN1]) and is written $F^\star(T)$, or simply $F^\star$.

Note that T is of local rank one iff $F^\star(T) > 0$, and of rank one iff $F^\star(T) = 1$; it is also easy to show that if T is of rank r, then $F^\star(T) \geq 1/r$; local finite rank would be the same as local rank one (in Ryzhikov's papers, $F^\star$ is called simply the *local rank*, and hence what we call local rank one becomes positive local rank).

The local rank one also has a symbolic translation, but no constructive definition.

The first local rank one systems which are not known to be of finite rank are the Cartesian square of Katok's map (see 1.4.3) with covering number at least $1/4$ ([GER]), and the author's own example (see 2.4.2 and [FER1]). The examples built in [FIL-KWI] show that the covering number behaves independently of the rank, given that $F^\star \geq 1/r$.

2.3. Metric properties

2.3.1. *General properties.* A system of rank r has at most r ergodic components ([KIN3]).

An ergodic system of finite rank or local rank one has zero entropy.

A factor of a system of local rank one has local rank one; a factor of a system of rank r has rank at most r.

Finite rank ([ORN-RUD-WEI]) or local rank one ([FER1]) imply the Loosely Bernoulli property.

The Weak Closure Theorem fails even for rank two systems (see the *Morse substitution* in 3.3.3 for a counter-example). In fact, for any integers $r \geq 2$ and $m \geq 1$, it is possible to find a transformation T of rank r such that m is the cardinality of the quotient group of the centralizer of T by the weak closure of the powers of T ([BU-KWI-SIE], [KWI-LAC]).

There exists one measure-theoretic property ensuring a system is bigger than a local rank one system: it is a strengthened form of mixing called the *Vershik property* (see [ROT] for a detailed study). It is proved in [FER1] that the Vershik property excludes local rank one, and hence that the class of Loosely Bernoulli systems, which has a nonempty intersection with the Vershik class, contains strictly the class of local rank one systems.

2.3.2. *Mixing, and joining rank.* For mixing transformations, the rank of T^k is k times the rank of T for any $k > 0$ ([KIN2]).

The structure of mixing systems of finite rank or local rank one has been studied in depth in [KIN3]. In this paper, the notion of **joining rank** is defined: a system has joining rank at most r if every measure on $X \times \ldots \times X$ (r copies), invariant under $T \times \ldots \times T$, has at least one trivial two-dimensional marginal, that is, a marginal equal to the product measure or carried by some graph of $I \times T^n$. For a mixing system, the joining rank is at most $1/F^\star$, which implies a generalization of the results in 1.5.2 (the number of factors is bounded, the centralizer is finite modulo the powers of T), and implies that mixing systems of local rank one are factors of finite extensions of systems with minimal self-joinings; furthermore, this structure is a canonical one ([KIN-THO]). This result extends also to some partially mixing (see 1.4.6 for a definition) systems.

The difficult questions of mixing of higher order are not completely solved, and are linked to the more general problems of self-joinings of higher orders; it is shown in [RYZ1] that mixing finite rank systems are mixing of all orders; this result is still true for mixing systems with $F^\star > 1/2$, or for local rank one systems satisfying already some weakened form of mixing of order 3 called $1+\varepsilon$*-mixing*; this is done by showing these systems have some form of simplicity of order 3 ([RYZ2]).

2.4. Spectral properties

2.4.1. *Rank versus spectral multiplicity.* The spectral multiplicity ($\mathcal{L}^2$) is not greater than the rank ([CHA3]).

This result generalizes in two different directions:

- the $\mathcal{L}^2$-spectral multiplicity is not greater than $1/F^\star$ (this result, attributed to the Russian school, was first published in [KIN3]).
- the $\mathcal{L}^p$-spectral multiplicity is not greater than the rank, for any $p > 0$ (Thouvenot, proof as in 1.6.1).

This implies for example that the Cartesian square of Katok's map (1.4.3) has spectral multiplicity at most 4; this is the only known Cartesian square with finite multiplicity.

The pair (multiplicity, rank) takes values in the upper half ($r \geq m$) of $\mathbb{N}^2$. Mentzen ([MEN1]) conjectured that for each possible pair, there exists a dynamical system realizing it; for the moment, the conjecture is proved for $(1,1)$ ([CHA1]), $(1,2)$ ([deJ2]), $(1,n)$ ([MEN1]), $(2,n)$ ([GOO-LEM]), (n,n) ([ROB1], [ROB2]), $(n,2n)$ ([MEN2]) and $(p-1,p)$ for prime p ([FER-KWI]); recently it was shown ([FER-KWI-MAU]) that it is true for all the pairs (d,n), for any $n \geq 2$ and any divisor d of $\psi(n)$, where, if $n = 2^{\alpha_0} p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ is the decomposition of n into prime factors ($\alpha_0 \geq 0, r \geq 0$ and $\alpha_i \geq 1$ for $1 \leq i \leq r$), then

$$\psi(n) = \mathrm{LCM}(2, 2^{\alpha_0 - 2}, p_i^{\alpha_i - 1}(p_i - 1); 1 \leq i \leq r)$$

if $\alpha_0 \geq 2$ and

$$\psi(n) = \mathrm{LCM}(1, p_i^{\alpha_i - 1}(p_i - 1); 1 \leq i \leq r)$$

if $\alpha_0 = 0, 1$.

As a corollary of the last result, for fixed d, the set of possible n in the pairs (d,n) is of density one.

2.4.2. *Simple spectrum versus rank.* The question whether simple spectrum is equivalent to rank one has been answered by the following chain of counter-examples:

- rank one systems have simple spectrum,
- the classical Morse sequence (see 3.3.3) has simple spectrum, is not of rank one, is of finite rank ([deJ2], a different proof comes from the computation of the covering number in [FER3]),
- the example in [FER1] has simple spectrum, is not of finite rank, is of local rank one,
- in [LEM-SIK], there is a system of simple spectrum, not of local rank one, but Loosely Bernoulli,
- the final (to date) example of the chain, with simple spectrum but not Loosely Bernoulli, will be defined in 2.6 ([FER2]).

2.4.3. *Spectral type.* We recall that an old problem attributed to Banach is to find a system with simple spectrum together with Lebesgue spectral type; the results in 1.6.2 leave little hope to achieve that with rank one systems, but some weaker results have been obtained by allowing bigger multiplicities. The computation in 1.6.2 can be generalized, by using *matrix Riesz products*; this was used in [QUE] to provide, with the *Rudin–Shapiro substitution* (see 3.3.3), the second known example of a system with finite spectral multiplicity (two, in this case) and a Lebesgue component in the spectrum.

This example, and also the first one due to [MAT-NAD], are of rank four, and not weakly mixing; a continuum of nonisomorphic weakly mixing systems with spectral multiplicity two, a Lebesgue component in the spectrum, and rank four, was built in [AGE]; another continuum of systems with these spectral properties was then built in [GOO], this time with the additional property that each one is isomorphic to its inverse; such constructions were generalized in [LEM3] in the framework of *Toeplitz extensions*: there exist finite rank systems with a Lebesgue component of multiplicity n in their spectrum for any even n (this result appears also in [AGE]).

2.5. Dictionary of other related notions

2.5.1. *Exact, uniform rank.* If, in the definition of finite rank, we add the requirement that all the h_i should be equal, we get **uniform finite rank**. If a system of finite rank admits one explicit construction where there are no spacers, we speak of finite rank **without spacers**; if also, in this construction, the measure of every tower is always bigger than some $a > 0$, then the system is of **exact finite rank**. These notions may be considered as rather poor: the exact finite rank excludes mixing ([ROS]), and every system of uniform and exact finite rank is a finite extension of an odometer ([MEN3]).

2.5.2. *Rank one with flat stacks.* It is defined by adding to the definition (NG) the condition that

$$\mu(F \triangle T^h F) < \varepsilon.$$

This is equivalent to one of the notions of periodic approximation in [KAT-STE]. Odometers, but also del Junco–Rudolph's map (see 1.4.2), and irrational rotations (see 3.1.1) are of rank one with flat stacks.

2.5.3. *Compact rank.* The closure of the class of local rank one systems for the inverse limit topology on transformations defines the **compact rank**. This notion, due to Thouvenot, has the following geometric definition:

DEFINITION 10. A system is of **compact rank** if for any partition P of X and any positive ε, there exist two subsets A and F of X, a positive integer h and a partition P' of A such that:

- the $T^k F$, $0 \leq k \leq h-1$, are disjoint,
- $A = \bigcup_{i=0}^{h-1} T^i F$,
- $\mu(A) \geq \varepsilon$,
- $|P' - P_{\setminus A}| < \varepsilon$,
- P' is refined by the partition formed by the sets F, TF, $\ldots$, $T^{h-1}F$, and $X \setminus \bigcup_{i=0}^{h-1} T^i F$.

This notion, though clearly weaker than local rank one, still implies the LB property: a proof of this may be found in [RAT1] though compact rank does not appear explicitly in the paper; to prove that the horocycle flow is LB, Ratner shows that one of its natural cross-sections is of compact rank (but of course, the final result implies that the horocycle flow has any LB system as a cross-section).

2.5.4. *Joining rank.* See 2.3.2.

2.5.5. *Approximate transitivity.* This notion, also called **propriété d'approximation convexe** in some French texts, is defined in [CON-WOO] in a more general context, and may be seen as an $\mathcal{L}^1$ version of rank one:

Definition 11. (X,T) is **approximately transitive** if for any finite set E of functions in $\mathcal{L}^{1+}(X)$ of norm one, for every $\varepsilon > 0$, there exists an element ϕ in $\mathcal{L}^{1+}(X)$ such that every element of E is at a distance not greater than ε from the closed convex hull of $(\phi \circ T^n, n \in \mathbb{Z})$, in the $\mathcal{L}^1$ topology.

This property is weaker than rank one, and even strictly weaker as it is implied by funny rank one (see 2.6). It implies simple spectrum in $\mathcal{L}^1$.

2.5.6. *Rank by intervals.* Let (X,T) be a topological system, on the interval $[0,1]$ for example; if, for some Borelian measure μ, the system (X,T,μ) is of rank one, we can approximate any partition by a Rokhlin tower; however, its basis, and hence all it levels, are only measurable sets, and may be topologically very unpleasant (with empty interiors, for example). If we can approximate any partition, in the sense of the (NG) definition, by a tower whose basis F is an interval, we say that T is of **rank one by intervals**; we may notice that this is a mixed notion, involving both the measure-theoretic and topological structures. Similarly, we may define local, finite, funny rank by intervals.

2.6. Funny rank. The notion of funny rank one was introduced by Thouvenot, and is discussed extensively in [FER2].

Definition 12. A system is of **funny rank one** if for every partition P of X and every positive ε, there exist a subset F of X, positive integers h and $k_1, \ldots, k_h$ and a partition P' of X such that

- the $T^{k_i}F, 1 \le i \le h$, are disjoint,
- $|P' - P| < \varepsilon$,
- P' is refined by the partition formed by the sets $T^{k_1}F, \ldots, T^{k_h}F$, and $X \setminus \bigcup_{i=1}^{h} T^{k_i}F$.

There is no equivalent symbolic definition, nor any constructive definition. By replacing the sequence $(0, 1, \ldots, h-1)$ by $(k_1, \ldots, k_h)$ in 2.1 and 2.2, we can also define systems of **funny finite rank** and **funny local rank**, along with a **funny covering number**, denoted by $T^\star$; but these notions have been used as yet mainly in conjectures.

Funny rank one implies ergodicity, zero entropy, simple spectrum $\mathcal{L}^p$ for every $p > 0$ (same proof as in 1.6.1). It also implies approximate transitivity ([CON-WOO]). Also, the spectral multiplicity ($\mathcal{L}^2$) is not greater than $1/T^\star$.

Funny rank one is mainly used for its spectral properties; in [FER2] there is an example of a non-LB funny rank one system.

Conversely, it is probable that simple spectrum $\mathcal{L}^2$ is not equivalent to funny rank one, as there should exist systems with $1/2 < T^\star < 1$; however, to get a precise estimate for $T^\star$, or to show directly a given system is not of funny rank one, is a difficult problem; in [FER3], it is proved that for the classical Morse sequence, $T^\star \geq 5/6$, while $F^\star = 2/3$. The question, due to Thouvenot, whether funny rank one is equivalent to simple spectrum $\mathcal{L}^p$ for every $p > 0$, is still completely open.

2.7. Other constructions by cutting and stacking. The geometric method of construction described in (CG) is called a contruction by **cutting and stacking**. It can be generalized, not only to a finite number of towers (finite rank), but also to an unbounded number of towers: a complete symbolic definition of these may be found in [ROT], and a geometric one in [FER2]; a very similar kind of construction, though worded differently, is what Vershik calls an *adic system* ([VER-LIV]). In fact, every system may be built in this way, and most examples in ergodic theory were explicitly built by cutting and stacking; among those, we may mention the famous Ornstein example of a non-Bernoulli K-automorphism ([ORN3]).

3. RANK PROPERTIES OF CLASSICAL SYSTEMS

3.1. Irrational rotations

$$Tx = x + \alpha \bmod 1$$

for an irrational number α, on the set $X = \mathbb{R}/\mathbb{Z}$. We shall use the well-known fact that irrational rotations are minimal. It is also well known, since [deJ1] or before, that, as measure-theoretic systems, rotations are of rank one, and this is of little use for their study. To see if they satisfy some of the finer notions, however, gives some insight into the arithmetic properties of the number α.

3.1.1. *Rank properties*

THEOREM 3. *Every irrational rotation is of rank one with flat stacks.*

Proof (Del Junco, unpublished; a similar proof using a different vocabulary appears in [IWA1]). Let P be a generating partition for the system, which we can choose to be a partition of the circle in two intervals; ε and s are given, and we choose K such that

$$\forall x,\ \exists 0 \le l < K : \quad |T^l x - 1| < \varepsilon/s, \tag{3.1}$$

which is an easy consequence of the minimality of T and the compactness of X. We also choose an $m \gg K$ such that

$$|T^m 1 - 1| < \varepsilon/s.$$

Let B be the P-m-name of the point 1.

We now look at the P-N-name of the point 1 for N large enough: the first m symbols form the word B; then $|T^{m+i}1 - T^i 1| = |T^m 1 - 1| < \varepsilon/s$. Hence, if we have chosen m large enough, the ergodic theorem ensures that the next m symbols, from $m+1$ to $2m$, will form a word B' such that $\bar{d}(B, B') < \varepsilon/s$, the following m symbols will form a word B" such that $\bar{d}(B, B") < 2\varepsilon/s$, and, up to the index sm, the P-name of 1 is formed by words B_i such that $\bar{d}(B, B_i) < \varepsilon$. When we reach the index sm, we use (3.1) to find $j < K$ such that $|T^{sm+j}1 - 1| < \varepsilon/s$, and we continue as after the mth symbol. Thus we can write the P-N-name of 1, but also of any point in view of (3.1), as a sequence of cycles of at least $s-1$ blocks ε-close to B, separated by strings of at most K symbols, hence T satisfies the (NS) definition, for the chosen generating partition, and so for any partition; the additional condition of flat stacks is ensured by the fact that s is any prescribed integer, independently of ε. ■

3.1.2. *Rank properties by intervals*

THEOREM 4. *Every irrational rotation is of funny rank one by intervals.*

Proof. Let $I = [a, b[$ be an interval of small length l. For given ε, we choose $n_1 = 0$, n_2 an integer such that $0 < T^{n_2}a - b < \varepsilon/l$, $\ldots$, n_{i+1} an integer such that $T^{n_{i+1}}I$ is an interval, disjoint from $\bigcup_{j=0}^{i-1} T^j I$ and situated at most ε/l to the right of this union. We get, in no more than $1/l$ steps, a Rokhlin "funny" tower $I, T^{n_1}I, \ldots, T^{n_p}I$ filling more than $1-\varepsilon$ of the space, and approximating any interval; hence a generating partition formed by two intervals can be approximated in this way, and so any partition can be approximated. ■

THEOREM 5. *Every irrational rotation is of rank at most two by intervals.*

To prove this theorem, we look at the rotation T on the fundamental domain $X' = [0,1[$; it has the following form:

$$Tx = \begin{cases} x+\alpha & \text{if } x \in [0, 1-\alpha[, \\ x+\alpha-1 & \text{if } x \in [1-\alpha, 1[. \end{cases}$$

We shall see in the next section that, in this form, T is an exchange of two intervals, and so this theorem is a particular case of our Theorem 7 (see 3.2), to the proof of which we refer the reader.

THEOREM 6. *If α has unbounded partial quotients, the rotation of argument α is of rank one by intervals, with flat stacks.*

Proof. In this case, if p_n/q_n are the reduced continued fractions of α, we have, after restricting n to a subsequence,

$$q_n^2 \left| \frac{p_n}{q_n} - \alpha \right| \to 0$$

and we can take the Rokhlin towers of basis

$$F_n = \left[q_n \left| \frac{p_n}{q_n} - \alpha \right|, \frac{1}{q_n} - q_n \left| \frac{p_n}{q_n} - \alpha \right| \right[,$$

which approximate a generating partition formed by two intervals, and so any partition. ∎

The converse of this theorem is true, and straightforward, thanks to the hypothesis of flat stacks; it is not known, however, whether the rank one by intervals of the rotation is equivalent to the unbounded partial quotients of α.

3.2. Interval exchanges. An **exchange of s intervals** is defined in the following way: given s real numbers $l_i > 0$ with $\sum_{i=1}^{s} l_i = 1$, and a permutation π on s letters, let X be the interval $[0,1[$, partitioned into s semi-open intervals I_i, of lengths $l_1, \ldots l_s$ (in that order), and also into s semi-open intervals J_i of lengths $l_{\pi^{-1}1}, \ldots l_{\pi^{-1}s}$ (in that order); T is the piecewise affine map sending each I_i onto $J_{\pi i}$. We take as measure μ any one of the (possibly several) T-invariant nonatomic measures.

As explained in 3.1.2, an irrational rotation is an exchange of two intervals.

Interval exchange maps have been extensively studied, since [KEA2]; for example, they provide the first example of nonuniquely ergodic minimal map ([KEY-NEW]); [deJ4] builds an exchange of four intervals which is simple; though this does not enter the framework of this study, let us mention that any transformation can be realized as an exchange of an infinite number of intervals ([ARN-ORN-WEI]).

THEOREM 7. *An ergodic exchange of s intervals is of rank at most s by intervals, without spacers.*

Proof. It is proved in [RAU2] that there exists a sequence of nested intervals D_n, of length tending to zero, on which the induced map of T is still an exchange of s intervals (these are obtained by the so-called *Rauzy induction*, which generalizes the continued fraction approximation for irrational rotations [RAU1]). If D is one of these intervals, it is then partitioned in s intervals F_k on which the return time to D is a constant h_k; hence the $T^i F_k$, $1 \leq k \leq s$, $0 \leq i \leq h_k - 1$, form s Rokhlin towers filling all the space; this, applied to each D_n, gives a sequence of s Rokhlin towers, made of intervals, filling all the space X, such that each level at stage $n+1$ is a union of levels of towers at stage n, and generating the whole σ-algebra; hence the result is proved by a (CG) argument. ∎

A deep result of [VEE] uses the Rauzy induction, together with *Teichmüller spaces*, to show, among other things, that generically (for any fixed primitive permutation π, and for the Lebesgue measure on the simplex of possible vectors of lengths) an ergodic interval exchange map is of rank one by intervals.

3.3. Substitutions

3.3.1. *Symbolic systems, languages, complexity.* Many systems are produced by putting a measure on some **(topological) symbolic system**, where X is a closed subset of $\{0,1\}^{\mathbb{N}}$, with the product topology, and T is the (one-sided) **shift**, $T(u_0, u_1, \ldots, u_n, \ldots) = (u_1, u_2, \ldots, u_n, \ldots)$. Among these symbolic systems, the simplest ones are the **systems defined by a sequence**: if $u = (u_n, n \in \mathbb{N})$, we take for T the shift and for X the closure of the orbit of u under T.

A word $w_1 \ldots w_k$ is said to **occur** at place i in the sequence u if $u_i = w_1, \ldots, u_{i+k-1} = w_k$; the language of length n, $L_n(u)$, is the set of all words of length n occurring in u; the **complexity** of u is the function $p_u(n) = \# L_n(u)$, $n \in \mathbb{N}$.

The minimality of the topological system associated with a sequence can be read easily from the sequence: the system associated with the sequence u is minimal if and only if every word occurring in u occurs infinitely often, with bounded gaps between occurrences. The sequence u is then said to be **minimal**.

3.3.2. *Definition and generalities.* A vast literature has been, and will be still, devoted to substitutions, whose bible is [QUE]. They appear as symbolic systems defined on a finite alphabet $A = \{0, 1, \ldots, k-1\}$; a **substitution** ζ is a map from A to the set $A^\star$ of all finite words of A. It extends naturally into a morphism of $A^\star$ for the concatenation. We restrict ourselves to the case when $\zeta 0$ begins by 0 and the length of $\zeta^n 0$ tends to infinity with n. The infinite sequence u beginning with $\zeta^n 0$ for all $n \in \mathbb{N}$ is then called the **fixed point** of u beginning with 0, and the symbolic system associated with u is called the **dynamical system associated with** ζ.

When ζ is **primitive**, that is, there exists m such that a appears in $\zeta^m b$ for any $a \in A$, $b \in A$, then the system is uniquely ergodic, and we can consider the measure-preserving system associated with ζ, to which we refer for short, by abuse of notation, as "the substitution ζ".

Every primitive substitution on k letters is of rank at most k without spacers ([QUE]). Moreover, an explicit set of recursion formulas for a (CS) definition is given by

$$B^i_{n+1} = B_n^{(\zeta i)_1} \cdots B_n^{(\zeta i)_{q(i)}}$$

if ζi is the word $(\zeta i)_1 \dots (\zeta i)_{q(i)}$, $1 \leq i \leq k$. This precise structure was used in [FER-MAU-NOG] to compute the eigenvalues of the dynamical system.

The general structure of the substitution dynamical systems is not yet fully known; they can, in all known cases, be described as finite extensions of odometers or irrational rotations, but it is generally not clear whether the substitution is or not isomorphic to the underlying translation.

More precise rank estimates may be given for famous substitutions.

3.3.3. *Examples.* The **Morse substitution**, $\zeta 0 = 01$, $\zeta 1 = 10$, whose fxed point is the famous *Prouhet–Thue–Morse sequence*, first defined in [PRO], has rank two ([deJ2]), and its covering number is 2/3 ([FER3]); it is not rigid, and does not satisfy the Weak Closure Theorem ([LEM1]). It has simple, nondiscrete, singular spectrum ([deJ2]).

The **Rudin–Shapiro substitution**, $\zeta 0 = 01$, $\zeta 1 = 02$, $\zeta 2 = 31$, $\zeta 3 = 32$, has had its spectral properties mentioned in 2.4.3; its rank is four, and its covering number is 1/4.

The **Fibonacci substitution**, $\zeta 0 = 01$, $\zeta 1 = 0$, gives a system which is isomorphic to an irrational rotation ([QUE]), and is hence of rank one.

Also Chacon's map of 1.4.1 is associated with the nonprimitive substitution $\zeta 0 = 0010$, $\zeta 1 = 1$, or, after a topological (hence stronger than measure-theoretic) isomorphism, to the primitive substitution $\zeta 0 = 0012$, $\zeta 1 = 12$, $\zeta 2 = 012$ ([FER4]).

In [LEM-MEN], it is proved that, when ζa has the same length for every $a \in A$, then the system is of rank one if and only if it has discrete spectrum. Other computations of rank may be found in [MEN2], [GOO-LEM].

3.3.4. *Generalizations of substitutions.* Among systems generalizing the substitutions are the *generalized Morse sequences*, introduced by [KEA1], whose rank is computed in [LEM2]. The *generalized Rudin–Shapiro se-*

quences, and more generally the *Toeplitz extensions of translations of compact groups*, or *Toeplitz systems* ([LEM3]) are also of finite rank. These categories of systems provide most of the examples in 2.4.1 and 2.4.3. Some Toeplitz systems, built in [IWA-LAC], which are minimal and uniquely ergodic, have rank one with a continuous part in their spectrum.

A rather new interesting category of systems consists of the **S-adic systems**; they are the systems associated with sequences u of the form $u = \lim_{n\to\infty} \zeta_{i_1}\zeta_{i_2}\ldots\zeta_{i_n}0$ (the limit being in the sense that u begins with $\zeta_{i_1}\zeta_{i_2}\ldots\zeta_{i_n}0$ for all n) for a sequence (i_n) and a finite set of substitutions $\zeta_1, \ldots, \zeta_m$ on k letters, with the condition that the length of $\zeta_{i_1}\zeta_{i_2}\ldots\zeta_{i_n}0$ tends to infinity, ensuring this definition makes sense.

The rank of an S-adic system on k letters is still bounded by k. It is proved in [FER5] that if u is a minimal sequence of complexity $p(n)$ smaller than $an+b$ for all n, then the system associated with u is S-adic and, equipped with any nonatomic invariant probability, has rank at most $2[a]$; there is no bound on the number of letters on which the substitutions live, except if $p(n+1)-p(n) \le 2$ for all n, in which case this bound is equal to three, while if $p(n) = n+1$ for all $n \in \mathbb{N}$ the system is a rotation ([HED-MOR]) and hence of rank one.

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Received 11 July 1995;
revised 13 April 1996