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BOUNDEDNESS OF L^1 SPECTRAL MULTIPLIERS FOR AN EXPONENTIAL SOLVABLE LIE GROUP

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Introduction. Let M be a measure space and let L be a positive definite operator on $L^2(M)$. By the spectral theorem, for any bounded Borel measurable function $m : [0,\infty) \to \mathbb{C}$ the operator $m(L)f = \int_0^\infty m(\lambda)\,dE(\lambda)f$ is bounded on $L^2(M)$.

We are interested in sufficient conditions on m for $m(L)$ to be bounded on $L^p(M)$, $p \neq 2$.

A very general theorem of Stein [18] and its strengthening by Cowling [8] give such conditions under the assumption that L generates a semigroup of contractions on L^1. Then, indeed, $m(L)$ is bounded on $L^p(M)$, $1 < p < \infty$, if m is of the form $m(\lambda) = \lambda\int_0^\infty e^{-\lambda\xi}F(\xi)\,d\xi$ with $F \in L^\infty(\mathbb{R}^+)$. The assumption on L is satisfied by many examples, but the condition on m implies that m is holomorphic in the half-plane $|\arg(\lambda)| < \pi/2$.

In the case of a Riemannian manifold of exponential volume growth of Riemannian balls (or a noncompact semisimple Lie group) stronger positive results have been proved. However, it is usually assumed that m is holomorphic in a parabolic region around the real half-axis and in some cases it is known that this assumption is necessary [2], [3], [7], [19].

In contrast to the above, what follows from a 1960 result by Hörmander [13] is that if $M = \mathbb{R}^n$ and L is the (minus) Laplace operator on $\mathbb{R}^n$, then a bound on only a finite number of derivatives of m suffices for $m(L)$ to be bounded on $L^p(M)$, $1 < p < \infty$. This has been generalized to M being a group of polynomial growth [1], or a compact manifold [6], L being the laplacian. In these works polynomial growth was crucial in the proof and the growth rate determined the required regularity of m.

It turns out, however, that the relation of the volume growth of the balls to the regularity of m which guarantees boundedness of $m(L)$ on L^p is not straightforward. For some nilpotent Lie groups regularity of m is related to the topological dimension of the group rather than to its growth

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[11], [12], [17]. Also on groups NA in the Iwasawa decomposition of a semisimple Lie group which are of exponential growth for some specific elliptic L, estimates on only a finite number of derivatives of m imply boundedness of $m(L)$ on L^p, $p \neq 2$ ([9], [10]). As these groups and the distinguished laplacian on them are rather special, we would like to present another example, this time of a unimodular group of exponential growth and an elliptic laplacian L on it for which $m \in C_c^6(\mathbb{R}^+)$ implies that $m(L)$ is bounded on all L^p, $1 \le p < \infty$.

Our methods allow us to handle a little more general groups but, at the moment, seem to be too crude to solve the problem in general. In fact, Christ and Müller [5] give an example of a solvable Lie group on which m must be holomorphic. There is a large disparity between our methods and the ones of [5]. This is the reason why some arguments are presented in a more general form then actually needed for the proof here. Also in Lemmas (1.1) and (1.2) we give sharp estimates for the constants involved, as those might be of interest of their own.

Preliminaries. Let G be equal to $\mathbb{R}^3$, the multiplication being given by the formula

$$(t_1, x_1, y_1)(t_2, x_2, y_2) = (t_1 + t_2, e^{t_2} x_1 + x_2, e^{-t_2} y_1 + y_2).$$

We consider left invariant vector fields X, Y, T,

$$X = \partial_x, \quad Y = \partial_y, \quad T = \partial_t + x\partial_x - y\partial_y.$$

The corresponding right invariant vector fields $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{T}$ are

$$\widetilde{X} = e^t \partial_x, \quad \widetilde{Y} = e^{-t}\partial_y, \quad \widetilde{T} = \partial_t.$$

Let

$$L = X^2 + Y^2 + T^2, \qquad \widetilde{L} = \widetilde{X}^2 + \widetilde{Y}^2 + \widetilde{T}^2$$

and let p_t be the convolution kernel of $\exp(tL)$:

$$\exp(tL)f = f * p_t.$$

We need the formula

$$\exp(t\widetilde{L})f = p_t * f.$$

This formula is valid for all sublaplacians on Lie groups. For the proof one may consider the distribution K_L supported at identity such that $Lf = f * K_L$. Then $\widetilde{L}f = K_L * f$ and

$$\partial_t(f * p_t) = \partial_t \exp(tL)f = L\exp(tL)f = \exp(tL)Lf = f * K_L * p_t = f * (\widetilde{L}p_t).$$

So

$$\partial_t p_t = \widetilde{L}p_t \quad \text{and} \quad \partial_t(p_t * f) = (\partial_t p_t) * f = (\widetilde{L}p_t) * f = \widetilde{L}(p_t * f).$$

As this equation has a unique solution for f in the domain of $\widetilde{L}$, we have $p_t * f = \exp(t\widetilde{L})f$.

Schrödinger operators. As we shall see toward the end of the paper, our considerations here cannot be restricted to sublaplacians. We also have to consider Schrödinger operators, that is, operators of the form

$$H = \sum U_j^* U_j$$

where $U_j = X_j + iV_j$, X_j are vector fields and V_j are locally square integrable real functions. Our operators U_j are closed operators with a common core $C_c^\infty(G)$. Consequently, by the von Neumann theorem (cf. for example [20], Theorem 5.39) H is selfadjoint.

From now on we will assume that the X_j generate G. For the proof of the main theorem of the paper we assume even more. Then it is assumed that the X_j form a linear basis of the Lie algebra of G. Let $d(x)$ be the optimal control distance from $x \in G$ to e associated with the X_j's.

(1.1) Lemma. *For every f in $C_c^\infty(G)$ and $\gamma > 0$ we have*

$$\gamma |\Im(Hf, e^{2sd}f)| \le \Re(Hf, e^{2sd}f) + s^2(1+\gamma)^2 \|e^{sd}f\|^2.$$

Proof. We have

$$\begin{aligned}(U_j^* U_j f, e^{2sd}f) &= (U_j f, U_j(e^{2sd}f)) = (U_j f, e^{2sd}U_j f) + (U_j f, 2sX_j(d)e^{2sd}f)\\ &= \|e^{sd}U_j f\|^2 + (U_j f, 2sX_j(d)e^{2sd}f)\end{aligned}$$

and

$$\begin{aligned}|(U_j^* U_j f, e^{2sd}f) - \|e^{sd}U_j f\|^2| &= |(U_j f, 2sX_j(d)e^{2sd}f)|\\ &\le 2s\|e^{sd}U_j f\| \cdot \|X_j(d)e^{sd}f\|\\ &\le \alpha\|e^{sd}U_j f\|^2 + (s^2/\alpha)\|X_j(d)e^{sd}f\|^2.\end{aligned}$$

Next, by the inequality $\sum |X_j(d)(x)|^2 \le 1$,

$$\sum \|X_j(d)e^{sd}f\|^2 \le \|e^{sd}f\|^2$$

so

$$\Re(Hf, e^{2sd}f) + (s^2/\alpha)\|e^{sd}f\|^2 \ge (1-\alpha)\sum \|e^{sd}U_j f\|^2$$

and

$$|\Im(Hf, e^{2sd}f)| \le \alpha \sum \|e^{sd}U_j f\|^2 + (s^2/\alpha)\|e^{sd}f\|^2.$$

Putting $\alpha = (1+\gamma)^{-1}$ we have

$$\begin{aligned}\gamma|\Im(Hf, e^{2sd}f)| &\le \frac{\gamma}{1+\gamma}\sum \|e^{sd}U_j f\|^2 + \gamma(1+\gamma)s^2\|e^{sd}f\|^2\\ &= (1-\alpha)\sum \|e^{sd}U_j f\|^2 + \gamma(1+\gamma)s^2\|e^{sd}f\|^2\\ &\le \Re(Hf, e^{2sd}f) + ((1+\gamma)+\gamma(1+\gamma))s^2\|e^{sd}f\|^2,\end{aligned}$$

which is our conclusion.

For a real s let

$$(f,g)_s = \int_G f(x)\overline{g}(x)e^{2sd(x)}\,dx$$

be the inner product in $L^2(e^{2sd})$. We write $\|f\|_s$ for the corresponding norm.

(1.2) Lemma. *Let H and d be as above. For s negative* (*resp. positive*) *the semigroup e^{-zH}, $\Re z > 0$, extends* (*resp. restricts*) *to a holomorphic semigroup on the weighted space $L^2(e^{2sd})$. Moreover*,

$$\|e^{-zH}\|_{L^2(e^{2sd}),L^2(e^{2sd})} \le \exp\left(\Re z\left(1+\left|\frac{\Im z}{\Re z}\right|\right)^2 s^2\right).$$

Proof. This is a consequence of (1.1). Indeed, we rewrite (1.1) as

$$\gamma|\Im(Hf,f)_s| \le \Re(Hf,f)_s + C\|f\|_s^2,$$

where $C = s^2(1+\gamma)^2$. In general f ranges over $C_c^\infty(G)$, but if $s \le 0$ we may extend the inequality above to the domain of H. To see this we write

$$\langle f,g\rangle_{C,s} = \tfrac{1}{2}(((H+CI)f,g)_s + (f,(H+CI)g)_s).$$

Then $\langle f,g\rangle_{C,s}$ is a new scalar product on a dense subspace of $L^2(e^{2sd})$ and our inequality extends to the domain of the corresponding quadratic form. From the proof of (1.1) we have

$$\langle f,f\rangle_{C,s} = \Re(Hf,f)_s + C\|f\|_s^2 \approx \sum \|e^{sd}U_j f\|^2 + C\|f\|_s^2$$

so $\langle f,f\rangle_{C,s}$ is finite for f in the domain of H. We then have

$$\Re(z(H+CI)f,f) \ge 0$$

for f in the domain of H and z in the sector $S_\gamma = \{z : |\Im z| \le \gamma\Re z\}$. This means that the operator $A = -z(H+CI)$ is dissipative in the $L^2(e^{2sd})$-norm. Moreover,

$$e^{tA} = \lim_{n\to\infty}\left(I - \frac{t}{n}A\right)^{-n}.$$

On the right hand side we have a sequence of contractions on $L^2(e^{2sd})$ which is convergent on functions from L^2, which is a dense subset of $L^2(e^{2sd})$, $s<0$. Therefore e^{tA} extends to contractions on $L^2(e^{sd})$. This completes the proof of the lemma for negative s. For $s>0$ we obtain the lemma by duality.

(1.3) LEMMA. *If $t > 0$ and $f \in L^2$ then*
$$|e^{-tH} f| \le \exp\Big(-t \sum X_j^* X_j\Big)|f|$$
almost everywhere.

Remark. The lemma is valid in general, but we will prove it only when $\{\exp(tX_j) : t \in \mathbb{R}\}$ is unbounded (for example, if G is exponential). Then for every locally integrable function V_j there exists a Borel function F_j such that $X_j F_j = V_j$.

Proof of Lemma (1.3). By the Trotter formula [15] it is enough to prove the lemma for a single $U = X + iV$. By assumption, there is an F such that $XF = V$. Then
$$\exp(-iF)X\exp(iF) = X + iV = U, \qquad \exp(-iF)X^*\exp(iF) = X^* - iV = U^*$$
and
$$\exp(-iF)X^*X\exp(iF) = U^*U$$
so
$$\exp(-iF)\exp(-tX^*X)\exp(iF) = \exp(-tU^*U).$$
Since $\exp(-iF)$ has absolute value 1 the assertion follows.

In the sequel we will denote the kernel of the semigroup by $e^{-zH}\delta_e$, the second argument being replaced by e.

(1.4) LEMMA. *There is a constant C independent of V_j such that for all real l and s,*
$$\|e^{-(1+il)H}\delta_e\|_{L^2(e^{sd})} \le C\exp(C(1+l^2)s^2).$$

Proof. Note that (1.3) implies $|e^{-(1/2)H}\delta_e| \le \exp(-(1/2)\sum X_j^* X_j) = p_{1/2}$. As for real z and p_z the estimate is known, the lemma follows from (1.2).

Main Theorems. Now we come back to our group G as defined in the preliminaries. Let d be a (left) invariant Riemannian metric on G. There is a constant C such that
$$\begin{aligned} B_r &= \{(t,x,y) : d((t,x,y),0) \le r\} \\ &\subset \{(t,x,y) : |t| < r,\ |x| < C(e^r+1),\ |y| < C(e^r+1)\}. \end{aligned}$$
Let a weight function w be defined as $w(t,x,y) = |xy|$. A straightforward calculation shows that for some C,
$$\int\limits_{d(g,0)<r} (1+w(g))^{-1}\,dg \le Cr^3.$$

(1.5) THEOREM. *There exists C such that for every $s \in \mathbb{R}$ we have*

$$\|p_{1+is}\|_{L^1(G)} \le C(1+|s|^5).$$

(1.6) THEOREM. *For every compactly supported $F \in C^6$ (or F in the Sobolev space $H^{(11/2+\varepsilon)}$) the operator $F(-L)$ is bounded on $L^1(G)$.*

Theorem (1.6) is a consequence of (1.5). Indeed, using the spectral theorem and the inversion formula for the Fourier transform we have

$$F(-L) = \frac{1}{2\pi}\int \widehat{f}(s)p_{1-is}\,ds,$$

where $f(x) = F(x)e^x$. Now $F \in H^{(11/2+\varepsilon)}$ implies that $\int|\widehat{f}(s)|(1+|s|)^5\,ds$ is convergent.

We now prove (1.5). From (1.4) (putting $V_j = 0$) we know that

$$\begin{aligned}\int|p_{1+is}(g)|e^{d(g,0)}\,dg& \\ &\le \Big(\int|p_{1+is}(g)|^2e^{(C+1)d(g,0)}\,dg\Big)^{1/2}\Big(\int e^{-2Cd(x,0)}\,dg\Big)^{1/2} \le C\exp(Cs^2).\end{aligned}$$

Consequently, if $r = Cs^2$ then

$$\begin{aligned}\int|p_{1+is}(g)|\,dg &\le \int\limits_{d(g,0)<r}|p_{1+is}(g)|\,dg + \int\limits_{d(g,0)\ge r}|p_{1+is}(g)|\,dg\\
&\le \int\limits_{d(g,0)<r}|p_{1+is}(g)(1+w(g))^{1/2}|(1+w(g))^{-1/2}\,dg\\
&\quad + e^{-r}\int\limits_{d(g,0)\ge r}|p_{1+is}(g)|e^{d(g,0)}\,dg\\
&\le \|p_{1+is}(1+w)^{1/2}\|_{L^2}\|(1+w)^{-1/2}\|_{L^2(B_r)}\\
&\quad + e^{-r}\int|p_{1+is}(g)|e^{d(g,0)}\,dg\\
&\le C(1+|s|^2)^{3/2}\|p_{1+is}(1+w)^{1/2}\|_{L^2} + e^{-Cs^2}Ce^{Cs^2}.\end{aligned}$$

We are going to show that

$$\|p_{1+is}(1+w)^{1/2}\|_{L^2}^2 \le C(1+|s|)^4 \tag{1.7}$$

and

$$|\partial_s\|p_{1+is}w^{1/2}\|_{L^2}^2| \le C(1+|s|)^3. \tag{1.8}$$

We note that $w \le Ce^{cd}$ for some C and c because $w \le (1+|x|)(1+|y|)$ and $(x,y,u) \to (1+|x|)(1+|y|)$ is submultiplicative, by [14]. Consequently, the left hand side of both (1.7) and (1.8) is finite, and (1.8) implies (1.7). To prove (1.8) we write

$$
\begin{aligned}
|\partial_s \|p_{1+is} w^{1/2}\|_{L^2}^2| &= 2|\Re\langle iLp_{1+is}, wp_{1+is}\rangle| \\
&= 2|\Im(\langle Xp_{1+is}, X(wp_{1+is})\rangle + \langle Yp_{1+is}, Y(wp_{1+is})\rangle \\
&\quad + \langle Tp_{1+is}, T(wp_{1+is})\rangle)|.
\end{aligned}
$$

Since $Tw = 0$, we have

$$
\begin{aligned}
2|\Im(\langle Xp_{1+is}, X(wp_{1+is})\rangle + \langle Yp_{1+is}, Y(wp_{1+is})\rangle + \langle Tp_{1+is}, T(wp_{1+is})\rangle)| \\
= 2|\Im(\langle Xp_{1+is}, (Xw)p_{1+is}\rangle + \langle Yp_{1+is}, (Yw)p_{1+is}\rangle)| \\
\le 2(|\langle Xp_{1+is}, \operatorname{sgn}(x)|y|p_{1+is}\rangle| + |\langle Yp_{1+is}, \operatorname{sgn}(y)|x|p_{1+is}\rangle|).
\end{aligned}
$$

We estimate the second term, the argument for the first being similar. We write

$$
|\langle Yp_{1+is}, \operatorname{sgn}(y)|x|p_{1+is}\rangle| \le \||x|^{1/2}p_{1+is}\|_{L^2}\||x|^{1/2}Yp_{1+is}\|_{L^2}.
$$

Now we note that

$$
\||x|^{1/2}Yp_{1+is}\|_{L^2} \le C(\sup_{|z-1|<1/2} \||x|^{1/2}p_{z+is}\|_{L^2} + 1).
$$

Indeed,

$$
\begin{aligned}
-\langle Lp_{1+is}, |x|p_{1+is}\rangle \\
&= \||x|^{1/2}Tp_{1+is}\|_{L^2}^2 + \||x|^{1/2}Xp_{1+is}\|_{L^2}^2 + \||x|^{1/2}Yp_{1+is}\|_{L^2}^2 \\
&\quad + \langle Tp_{1+is}, |x|p_{1+is}\rangle + \langle Xp_{1+is}, \operatorname{sgn}(x)p_{1+is}\rangle \\
&\ge \||x|^{1/2}Yp_{1+is}\|_{L^2}^2 + \||x|^{1/2}Tp_{1+is}\|_{L^2}^2 \\
&\quad - \||x|^{1/2}Tp_{1+is}\|_{L^2}\||x|^{1/2}p_{1+is}\|_{L^2} - \|Xp_{1+is}\|_{L^2}\|p_{1+is}\|_{L^2} \\
&\ge \||x|^{1/2}Yp_{1+is}\|_{L^2}^2 - \tfrac{1}{2}\||x|^{1/2}p_{1+is}\|_{L^2}^2 - C
\end{aligned}
$$

but since p_z depends analytically on z we have

$$
\||x|^{1/2}Lp_{1+is}\|_{L^2} = \||x|^{1/2}\partial_s p_{1+is}\|_{L^2} \le 2 \sup_{|z-1|<1/2} \||x|^{1/2}p_{z+is}\|_{L^2}.
$$

As for the z's with $|z-1| < 1/2$ the estimate trivially reduces to the case $z = 1$, we consider only this case.

Thus we end up with the task of proving

$$
\||x|^{1/2}p_{1+is}\|_{L^2}^2 \le C(1 + |s|^3).
$$

We have

$$
\begin{aligned}
|\partial_s \||x|^{1/2}p_{1+is}\|_{L^2}^2| &= |2\Re\langle i\widetilde{L}p_{1+is}, |x|p_{1+is}\rangle| \\
&= |2\Im\langle \widetilde{X}^2 p_{1+is}, |x|p_{1+is}\rangle| \\
&= |2\Im\langle e^{2t}\partial_x p_{1+is}, \operatorname{sgn}(x)p_{1+is}\rangle| \\
&= |2\langle e^{t3/2}\partial_x p_{1+is}, \operatorname{sgn}(x)e^{t/2}p_{1+is}\rangle| \\
&\le 2\|e^{t/2}\widetilde{X}p_{1+is}\|_{L^2}\|e^{t/2}p_{1+is}\|_{L^2}.
\end{aligned}
$$

We estimate the second factor; the estimate for the first is similar.

We decompose the left regular representation of G using the Fourier transform in x coordinate. Put

$$H_x = -\partial_t^2 - e^{-2t}\partial_y^2 + x^2 e^{2t}.$$

Then H_x is a Schrödinger operator on a two-dimensional solvable group G_0. We have

$$\begin{aligned}\|e^{t/2}p_{1+is}\|_{L^2}^2 &= \int_{-\infty}^{\infty} \|e^{t/2}\exp(-(1+is)H_x)\delta_0\|_{L^2(G_0)}^2\,dx \\ &= \int_{|x|<e^{-c|s|^2}} + \int_{e^{-c|s|^2}\le|x|<1} + \int_{|x|\ge 1}.\end{aligned}$$

By (1.4),

$$\|e^{t/2}\exp(-(1+is)H_x)\delta_0\|_{L^2}^2 \le Ce^{Cs^2}$$

independently of $x\in\mathbb{R}$. Hence the first of the three integrals above is bounded by a constant provided that $c\ge C$.

Since $\widetilde{L}$ is elliptic on G we have

$$\|\widetilde{X}^2 f\|_{L^2(G)} \le C(\|\widetilde{L}f\|_{L^2(G)} + \|f\|_{L^2(G)}).$$

This inequality remains valid in every unitary representation, so

$$\|x^2e^{2t}f\|_{L^2(G_0)} \le C(\|H_xf\|_{L^2(G_0)} + \|f\|_{L^2(G_0)})$$

and

$$\begin{aligned}\|e^{t/2}f\|_{L^2(G_0)} &\le \|e^{2t}f\|_{L^2(G_0)}^{1/4}\|f\|_{L^2(G_0)}^{3/4} = |x|^{-1/2}\|x^2e^{2t}f\|_{L^2(G_0)}^{1/4}\|f\|_{L^2(G_0)}^{3/4} \\ &\le C|x|^{-1/2}(\|H_xf\|_{L^2(G_0)} + \|f\|_{L^2(G_0)})^{1/4}\|f\|_{L^2(G_0)}^{3/4}.\end{aligned}$$

We have

$$\begin{aligned}&\|e^{t/2}\exp(-(1+is)H_x)\delta_0\|_{L^2(G_0)}^2 \\ &\quad\le C|x|^{-1}(\|H_x\exp(-(1+is)H_x)\delta_0\|_{L^2(G_0)} \\ &\qquad + \|\exp(-(1+is)H_x)\delta_0\|_{L^2(G_0)})^{1/2}\|\exp(-(1+is)H_x)\delta_0\|_{L^2(G_0)}^{3/2} \\ &\quad\le C|x|^{-1}(\|H_x\exp(-(1+is)H_x)\delta_0\|_{L^2(G_0)}^2 + \|\exp(-(1+is)H_x)\delta_0\|_{L^2(G_0)}^2) \\ &\quad= C|x|^{-1}(\|H_x\exp(-H_x)\delta_0\|_{L^2(G_0)}^2 + \|\exp(-H_x)\delta_0\|_{L^2(G_0)}^2) \\ &\quad\le C|x|^{-1}\|\exp(-(1/2)H_x)\delta_0\|_{L^2(G_0)}^2\end{aligned}$$

so the second integral is bounded by

$$2C\int_{\exp(-c|s|^2)}^{1} x^{-1}\,dx = C_1|s|^2.$$

If $|x| \ge 1$, then

$$\int\limits_{|x|\ge 1} \frac{1}{|x|} \|\exp(-(1/2)H_x)\delta_0\|^2_{L^2(G_0)} \le \int\limits_{-\infty}^{\infty} \|\exp(-(1/2)H_x)\delta_0\|^2_{L^2(G_0)} = \|p_{1/2}\|^2_{L^2(G)}$$

so the third integral is bounded by a constant. Consequently,

$$\|e^{t/2} p_{1+is}\|^2_{L^2} \le C(1+|s|^2),$$

which ends the proof.

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