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SOME NONEXISTENCE THEOREMS ON STABLE MINIMAL SUBMANIFOLDS

BY

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We prove that there exist no stable minimal submanifolds in some n-dimensional ellipsoids, which generalizes J. Simons' result about the unit sphere and gives a partial answer to Lawson–Simons' conjecture.

1. Introduction. In [S], J. Simons proved that there exist no stable minimal submanifolds in the n-dimensional unit sphere S^n. In this paper, we establish the following general results.

THEOREM 1. *Let N^n be an n-dimensional compact hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. If the sectional curvature $\overline{K}$ of N^n satisfies*

$$1/2 < \overline{K} \le 1, \tag{1}$$

then there exist no stable m-dimensional minimal submanifolds in N^n for each m with $1 \le m \le n-1$.

Remark 1. If N^n is an n-dimensional unit hypersphere S^n in $\mathbb{R}^{n+1}$, then the sectional curvature $\overline{K}$ of S^n is 1, and from Theorem 1 we deduce that there exist no stable m-dimensional minimal submanifolds in S^n for each m with $1 \le m \le n-1$, which was proved by Simons [S].

THEOREM 2. *Let N^n be an n-dimensional $(n \ge 4)$ compact submanifold in an $(n+p)$-dimensional Euclidean space $\mathbb{R}^{n+p}$. Let R and H denote the normalized scalar curvature and the mean curvature functions of N^n, respectively. If R satisfies the following pointwise $n(n-2)/(n-1)^2$-pinching condition:*

$$\frac{n(n-2)}{(n-1)^2}H^2 < R \le H^2, \tag{2}$$

then there exist no stable m-dimensional minimal submanifolds in N^n for each m with $2 \le m \le n-2$.

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Corollary 1. *Let N^n be an n-dimensional $(n \geq 4)$ compact hypersurface in $\mathbb{R}^{n+1}$. If all the principal curvatures k_a of N^n satisfy*

$$0 < k_a < \sqrt{\frac{1}{n(n-1)}} \sum_{b=1}^{n} k_b, \qquad 1 \leq a \leq n, \tag{3}$$

then there exists no m-dimensional minimal submanifold in N^n for each m with $2 \leq m \leq n-2$.

As direct applications of Theorem 1 and Corollary 1, we have

Proposition 1. *Let N^n be the following n-dimensional $(n \geq 4)$ ellipsoid in $\mathbb{R}^{n+1}$:*

$$N^n : \quad \frac{x_1^2}{a_1^2} + \ldots + \frac{x_{n+1}^2}{a_{n+1}^2} = 1, \qquad 0 < a_1 \leq a_2 \leq \ldots \leq a_{n+1}, \tag{4}$$

(1) *If $1 \leq a_{n+1} < \sqrt[3]{2}$ and $a_1 \geq \sqrt{a_{n+1}}$, then there exist no stable m-dimensional minimal submanifolds of N^n for each m with $1 \leq m \leq n-1$.*

(2) *If $a_{n+1}/a_1 < \sqrt[6]{n/(n-1)}$, then there exist no stable m-dimensional minimal submanifolds of N^n for each m with $2 \leq m \leq n-2$.*

Remark 2. It can be proved in a similar way that the above results all keep valid for *stable m-currents* on N^n (for concepts of *stable current*, see Lawson–Simons [LS]). For example, we can state the counterpart of Theorem 1 as follows:

Theorem 1′. *Let N^n be an n-dimensional compact hypersurface in the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$. If the sectional curvature $\overline{K}$ of N^n satisfies*

$$1/2 < \overline{K} \leq 1, \tag{5}$$

then there exist no stable m-currents on N^n for each m with $1 \leq m \leq n-1$.

Remark 3. Let N^n be an n-dimensional compact hypersurface in $\mathbb{R}^{n+1}$ and suppose that every principal curvature k_a of N^n satisfies $\sqrt{\delta} < k_a \leq 1$ $(a = 1, \ldots, n)$. H. Mori [M] and Y. Ohnita [O] proved the conclusion of Theorem 1′ under the stronger conditions $\delta > n/(n+1)$ and $\delta > 1/2$, respectively. Our Theorem 1′ also gives a partial answer to the following Lawson–Simons' conjecture:

Conjecture ([LS]). Let N^n be a compact n-dimensional connected Riemannian manifold with the sectional curvature $\overline{K}$ satisfying

$$1/4 < \overline{K} \leq 1. \tag{6}$$

Then there exist no stable m-currents on N^n for each m with $1 \leq m \leq n-1$.

We are greatly indebted to P. F. Leung's papers [L1, L2] which motivated us to do this work.

2. Basic formulas and notations. In this paper, we shall make use of the following convention on the ranges of indices:

$$1 \le A, B, C, \ldots \le n+p; \quad 1 \le a, b, c, \ldots \le n; \quad n+1 \le \mu, \nu, \ldots \le n+p;$$
$$1 \le i, j, k \ldots \le m; \quad m+1 \le \alpha, \beta, \gamma \ldots \le n.$$

Let M^m and N^n be Riemannian manifolds of dimension m and dimension n, respectively. Let M^m be an m-dimensional compact minimal submanifold of N^n, $n > m$. For any normal variation vector field $U = \sum_\alpha u_\alpha e_\alpha$ of M^m, the second variation of the volume is given by (see [S])

$$I(U,U) = \int_{M^m} \Big[\sum_{\alpha,i} u_{\alpha i}^2 - \sum_{\alpha,\beta} (\sigma_{\alpha\beta} + \overline{R}_{\alpha\beta} u_\alpha u_\beta) \Big] \, dv, \tag{7}$$

where $u_{\alpha i}$ are the covariant derivatives of u_α,

$$\sigma_{\alpha\beta} = \sum_{i,j} h_{ij}^\alpha h_{ij}^\beta, \tag{8}$$

$$\overline{R}_{\alpha\beta} = \sum_i \overline{R}_{\alpha i \beta i}, \tag{9}$$

and h_{ij}^α are the components of the second fundamental form h of M^m in N^n.

Now let $x : N^n \to \mathbb{R}^{n+p}$ be an n-dimensional submanifold in the $(n+p)$-dimensional Euclidean space $\mathbb{R}^{n+p}$. We choose a local field of orthonormal frames $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+p}$ in $\mathbb{R}^{n+p}$ such that, restricted to N^n, the vectors $e_1, \ldots, e_n$ are tangent to N^n. Their dual coframe fields are $\omega_1, \ldots, \omega_n, \omega_{n+1}, \ldots, \omega_{n+p}$. Then we have

$$dx = \sum_a \omega_a e_a, \tag{10}$$

$$de_a = \sum_b \omega_{ab} e_b + \sum_{\mu,b} B_{ab}^\mu \omega_b e_\mu, \tag{11}$$

$$de_\mu = -\sum_{a,b} B_{ab}^\mu \omega_b e_a + \sum_\nu \omega_{\mu\nu} e_\nu, \tag{12}$$

and the second fundamental form of N^n in $\mathbb{R}^{n+p}$ is

$$B = \sum_{a,b,\mu} B_{ab}^\mu \omega_a \otimes \omega_b \otimes e_\mu. \tag{13}$$

The Gauss equation of N^n in $\mathbb{R}^{n+p}$ is

$$n(n-1)R = n^2 H^2 - S, \tag{14}$$

where R, H and S are the normalized scalar curvature, the mean curvature and the length square of the second fundamental form of N^n in $\mathbb{R}^{n+p}$, respectively.

3. An m-dimensional minimal submanifold in N^n. Let M^m be an m-dimensional minimal submanifold in N^n, and N^n be an n-dimensional submanifold in $\mathbb{R}^{n+p}$. In this case we can choose a local orthonormal basis $e_1, \dots, e_m, e_{m+1}, \dots, e_n, e_{n+1}, \dots, e_{n+p}$ in $\mathbb{R}^{n+p}$ such that, restricted to M^m, the vectors $e_1, \dots, e_m$ are tangent to M^m, $e_1, \dots, e_n$ are tangent to N^n, $e_{n+1}, \dots, e_{n+p}$ are normal to N^n. Their dual coframe fields are $\omega_1, \dots, \omega_m$, $\omega_{m+1}, \dots, \omega_n, \omega_{n+1}, \dots, \omega_{n+p}$. From (10)–(12), restricted to M^m, we have

$$dx = \sum_i \omega_i e_i, \tag{15}$$

$$de_i = \sum_j \omega_{ij} e_j + \sum_{\alpha,j} h^\alpha_{ij} \omega_j e_\alpha + \sum_{\mu,j} B^\mu_{ij} \omega_j e_\mu, \tag{16}$$

$$de_\alpha = -\sum_{i,j} h^\alpha_{ij} \omega_i e_j + \sum_\beta \omega_{\alpha\beta} e_\beta + \sum_{\mu,j} B^\mu_{\alpha j} \omega_j e_\mu, \tag{17}$$

$$de_\mu = -\sum_{i,j} B^\mu_{ij} \omega_i e_j - \sum_{\alpha,j} B^\mu_{\alpha j} \omega_j e_\alpha + \sum_\nu \omega_{\mu\nu} e_\nu, \tag{18}$$

where $h = \sum_{i,j,\alpha} h^\alpha_{ij} \omega_i \otimes \omega_j \otimes e_\alpha$ is the second fundamental form of M^m in N^n and $\sum_i h^\alpha_{ii} = 0$ for any α, since M^m is a minimal submanifold in N^n.

We choose the following normal variation vector field of M^m in N^n:

$$U = \sum_\alpha u_\alpha e_\alpha, \qquad u_\alpha = \langle \Lambda, e_\alpha \rangle, \tag{19}$$

where Λ is a constant vector in $\mathbb{R}^{n+p}$.

Using (15)–(18), a straightforward computation shows

$$u_{\alpha i} = -\sum_k h^\alpha_{ki} u_k + \sum_\mu B^\mu_{\alpha i} u_\mu, \tag{20}$$

$$\sum_{\alpha,i} u^2_{\alpha i} = \sum_{\alpha,i} \Big[\sum_{j,k} h^\alpha_{ki} h^\alpha_{ij} u_k u_j + \sum_{\mu,\nu} B^\mu_{\alpha i} B^\nu_{\alpha i} u_\mu u_\nu - 2 \sum_{\mu,k} h^\alpha_{ki} B^\mu_{\alpha i} u_k u_\mu \Big], \tag{21}$$

where

$$u_j = \langle \Lambda, e_j \rangle, \qquad u_\mu = \langle \Lambda, e_\mu \rangle. \tag{22}$$

Let $E_1, \dots, E_{n+p}$ be a fixed orthonormal basis of $\mathbb{R}^{n+p}$, and $U_A = \sum_\alpha \langle E_A, e_\alpha \rangle e_\alpha$. Since

$$\sum_{A=1}^{n+p} \langle E_A, v \rangle \langle E_B, w \rangle = \langle v, w \rangle \tag{23}$$

for any vectors v, w in $\mathbb{R}^{n+p}$, putting (21) into (7) and using (22) and (23), we obtain

$$\begin{aligned}
\text{trace}(I) &\equiv \sum_{A=1}^{n+p} I(U_A, U_A) \\
&= -\int_{M^m} \Big[-\sum_{\alpha,k,\mu} (B^\mu_{\alpha k})^2 + \sum_{\alpha} \overline{R}_{\alpha\alpha} \Big]\, dv \\
&= -\int_{M^m} \sum_{\alpha,k} \Big[-\sum_{\mu} (B^\mu_{\alpha k})^2 + \overline{R}_{\alpha k \alpha k} \Big]\, dv \\
&= -\int_{M^m} \Big[-\sum_{\alpha,\mu,k} B^\mu_{\alpha\alpha} B^\mu_{kk} + 2\sum_{\alpha,k} \overline{R}_{\alpha k \alpha k} \Big]\, dv \\
&= \int_{M^m} \Big[2\sum_{\mu,\alpha,k} (B^\mu_{\alpha k})^2 - \sum_{\mu,\alpha,k} B^\mu_{\alpha\alpha} B^\mu_{kk} \Big]\, dv.
\end{aligned} \tag{24}$$

Thus we obtain

PROPOSITION 2. *Let N^n be an n-dimensional compact submanifold in $\mathbb{R}^{n+p}$. Let M^m be an m-dimensional compact minimal submanifold of N^n. If*

$$\text{trace}(I) = \int_{M^m} \Big[2\sum_{\mu,\alpha,k} (B^\mu_{\alpha k})^2 - \sum_{\mu,\alpha,k} B^\mu_{\alpha\alpha} B^\mu_{kk} \Big]\, dv < 0, \tag{25}$$

then M^m is not a stable minimal submanifold of N^n.

4. The proof of Theorem 1. Let N^n be an n-dimensional hypersurface in $\mathbb{R}^{n+1}$ and M^m be an m-dimensional compact minimal submanifold in N^n. At a given point $p \in M^m$ in N^n, we can choose a local orthonormal frame field $e^*_1, \dots, e^*_n, \vec{n}$ in $\mathbb{R}^{n+1}$ such that $e^*_1, \dots, e^*_n$ are tangent to N^n and at $p \in M^m$,

$$B^*_{ab} = \langle B(e^*_a, e^*_b), \vec{n} \rangle = k_a \delta_{ab}, \quad 1 \le a, b \le n, \tag{26}$$

where the k_a are the principal curvatures of N^n in $\mathbb{R}^{n+1}$.

Since M^m is an m-dimensional compact minimal submanifold in N^n, at a given point $p \in M^m$ in N^n, we can also choose a local orthonormal frame field $e_1, \dots, e_m, e_{m+1}, \dots, e_n$ in N^n such that $e_1, \dots, e_m$ are tangent to M^m. Noting that $e_1, \dots, e_n$ and $e^*_1, \dots, e^*_n$ are two local orthonormal frame fields in a neighborhood of $p \in M^m$, we can set

$$e_i = \sum_{b=1}^n A^b_i e^*_b, \quad 1 \le i \le m, \tag{27}$$

$$e_\alpha = \sum_{b=1}^n A^b_\alpha e^*_b, \quad m+1 \le \alpha \le n, \tag{28}$$

where $(A_a^b) \in SO(n)$, i.e.

$$\sum_{a=1}^{n} A_b^a A_c^a = \delta_{bc}, \quad \sum_{a=1}^{n} A_a^b A_a^c = \delta^{bc}. \tag{29}$$

It is a direct verification that at $p \in M^m$, by use of (26)–(29) and (1),

$$\begin{aligned}\sum_{\alpha,k} B_{\alpha\alpha}B_{kk} &= \sum_{\alpha,k} \langle B(e_\alpha, e_\alpha), B(e_k, e_k)\rangle \\ &= \sum_{\alpha,k,a,b,c,d} A_\alpha^a A_\alpha^b A_k^c A_k^d \langle B(e_a^*, e_b^*), B(e_c^*, e_d^*)\rangle \\ &= \sum_{\alpha,k,a,c} k_a k_c (A_\alpha^a)^2 (A_k^c)^2 \\ &= \sum_{a,c} \overline{R}_{acac} (A_\alpha^a)^2 (A_k^c)^2 \\ &\leq \sum_{a,c,\alpha,k} (A_\alpha^a)^2 (A_k^c)^2 = m(n-m),\end{aligned} \tag{30}$$

where $\overline{R}_{acac} = k_a k_c$ is the sectional curvature of N^n. From (1), we also have

$$-2\sum_{\alpha,k} \overline{R}_{\alpha k\alpha k} < -2 \cdot \frac{1}{2} m(n-m) = -m(n-m). \tag{31}$$

Putting (30) and (31) into (24), we obtain trace$(I) < 0$. From Proposition 2, we infer that M^m is not a stable minimal submanifold of N^n.

5. The proof of Theorem 2. We first establish the following algebraic lemma in order to prove our Theorem 2:

LEMMA 1. *Let*

$$1 \leq a, b \leq n; \quad 1 \leq i, j \leq m; \quad m+1 \leq \alpha, \beta \leq n,$$

and consider the symmetric $n \times n$ *matrix*

$$\begin{bmatrix} T_{ij} & T_{i\alpha} \\ T_{\beta j} & T_{\beta\alpha} \end{bmatrix}$$

such that

$$\sum_{i=1}^{m} T_{ii} + \sum_{\alpha=m+1}^{n} T_{\alpha\alpha} = D, \quad \sum_{a,b=1}^{n} T_{ab}^2 = S. \tag{32}$$

Then:

(1) *If* $m = 1$ *or* $m = n-1$, *we have*

$$\Big(\sum_i T_{ii}\Big)^2 - D\sum_i T_{ii} + 2\sum_{i,\alpha} (T_{i\alpha})^2 \leq S + \frac{n-5}{2} D^2. \tag{33}$$

(2) *If* $2 \le m \le n-2$, *we have*

$$\Big(\sum_i T_{ii}\Big)^2 - D\sum_i T_{ii} + 2\sum_{i,\alpha}(T_{i\alpha})^2 \le \frac{m(n-m)}{n}S + \frac{|(2m-n)D|}{n^2}\sqrt{m(n-m)(Sn-D^2)} - \frac{2m(n-m)D^2}{n^2}. \tag{34}$$

Proof. We apply the Lagrange multiplier method to the problem (cf. P. F. Leung [L1, L2])

$$\Big(\sum_i X_{ii}\Big)^2 - D\sum_i X_{ii} + 2\sum_{i,\alpha}(X_{i\alpha})^2 = \max! \tag{35}$$

subject to the constraints

$$\sum_i X_{ii} + \sum_\alpha X_{\alpha\alpha} = D \tag{36}$$

and

$$\sum_i (X_{ii})^2 + \sum_\alpha (X_{\alpha\alpha})^2 + 2\sum_{i<j}(X_{ij})^2 + 2\sum_{\alpha<\beta}(X_{\alpha\beta})^2 + 2\sum_{i,\alpha}(X_{i\alpha})^2 = S, \tag{37}$$

where $S = \sum_{a,b}(T_{ab})^2$ and the X_{ab} form a symmetric $n \times n$ matrix

$$\begin{bmatrix} X_{ij} & X_{i\alpha} \\ X_{\beta j} & X_{\beta\alpha} \end{bmatrix}.$$

We consider the function

$$\begin{aligned} f = {} & \Big(\sum_i X_{ii}\Big)^2 - D\sum_i X_{ii} + 2\sum_{i,\alpha}(X_{i\alpha})^2 \\ & + \lambda\Big(\sum_i X_{ii} + \sum_\alpha X_{\alpha\alpha} - D\Big) + \mu\Big[\sum_i (X_{ii})^2 + \sum_\alpha (X_{\alpha\alpha})^2 \\ & + 2\sum_{i<j}(X_{ij})^2 + 2\sum_{\alpha<\beta}(X_{\alpha\beta})^2 + 2\sum_{i,\alpha}(X_{i\alpha})^2 - S\Big], \end{aligned}$$

where λ, μ are the Lagrange multipliers.

Differentiating with respect to each variable and equating to zero, we obtain

$$2\sum_j X_{jj} - D + \lambda + 2\mu X_{ii} = 0, \tag{38}$$

$$\lambda + 2\mu X_{\alpha\alpha} = 0, \tag{39}$$

$$4X_{i\alpha} + 4\mu X_{i\alpha} = 0, \tag{40}$$

$$4\mu X_{ij} = 0, \quad i < j, \tag{41}$$

$$4\mu X_{\alpha\beta} = 0, \quad \alpha < \beta. \tag{42}$$

Hence (with the numbers standing for the corresponding left hand sides)

$$\sum_i X_{ii}(38) + \sum_\alpha X_{\alpha\alpha}(39) + \sum_{i,\alpha} X_{i\alpha}(40) + \sum_{i<j} X_{ij}(41) + \sum_{\alpha<\beta} X_{\alpha\beta}(42) = 0$$

gives

$$2\Big(\sum_i X_{ii}\Big)^2 - D\sum_i X_{ii} + 4\sum_{i,\alpha}(X_{i\alpha})^2 = -(\lambda D + 2\mu S). \tag{43}$$

(1) Case $\mu = 0$. It is easy to see in this case

$$\Big(\sum_i X_{ii}\Big)^2 - D\sum_i X_{ii} + 2\sum_{i,\alpha}(X_{i\alpha})^2 = -\frac{D^2}{4}. \tag{44}$$

(2) Case $\mu = -1$. First we suppose $m(n-m) > n$, and putting $X_{\alpha\alpha} = \lambda/2$, $\sum_i X_{ii} = D - (n-m)\lambda/2$ into (38), we have

$$\begin{gathered} \lambda = \frac{(m-2)D}{m(n-m)-n}, \quad X_{ii} = \frac{(n-m-2)D}{2[m(n-m)-n]}, \\ X_{\alpha\alpha} = \frac{(m-2)D}{2[m(n-m)-n]}, \end{gathered} \tag{45}$$

and

$$\Big(\sum_i X_{ii}\Big)^2 - D\sum_i X_{ii} + 2\sum_{i,\alpha}(X_{i\alpha})^2 = S - \frac{m(n-m)-4}{4[m(n-m)-n]}D^2 \tag{46}$$

is another critical value.

Now suppose $m(n-m) = n$, i.e. $n = 4, m = 2$. If $\mu = -1$, then

$$X_{ii} = \frac{1}{2}(D-\lambda), \qquad X_{\alpha\alpha} = \frac{\lambda}{2}, \tag{47}$$

$$\Big(\sum_i X_{ii}\Big)^2 - D\sum_i X_{ii} + 2\sum_{i,\alpha}(X_{i\alpha})^2 = S - \frac{D^2}{2}, \tag{48}$$

that is, equality holds in (34) in this case.

(3) Case $\mu \neq 0, -1$. Let $X = \sum_i X_{ii}$. Then

$$X_{\alpha\alpha} = -\frac{\lambda}{2\mu}, \qquad 2\mu(X-D) = (n-m)\lambda, \tag{49}$$

$$\lambda = D - 2\Big(1 + \frac{\mu}{m}\Big)X. \tag{50}$$

Substituting (50) into the second formula of (49), we get

$$\mu = \frac{m(n-m)(D-2X)}{2(nX-mD)}, \qquad \frac{\lambda}{\mu} = \frac{2}{n-m}(X-D). \tag{51}$$

From (43), we have

$$\frac{X(D-2X)}{\mu} = \frac{\lambda}{\mu}D + 2S. \tag{52}$$

Putting (51) into (52), we get

$$X^2 - \frac{2mD}{n}X - \left(\frac{m(n-m)}{n}S - \frac{m}{n}D^2\right) = 0,$$

that is,

$$X = \frac{m}{n}D \pm \sqrt{\frac{m(n-m)}{n}\left(S - \frac{D^2}{n}\right)}. \tag{53}$$

The critical value is

$$\Big(\sum_i X_{ii}\Big)^2 - D\sum_i X_{ii} + 2\sum_{i,\alpha}(X_{i\alpha})^2 \tag{54}$$
$$= \frac{m(n-m)}{n}S + \frac{|(2m-n)D|}{n^2}\sqrt{m(n-m)(Sn-D^2)} - \frac{2m(n-m)D^2}{n^2}.$$

Hence, the critical values are

$$-\frac{D^2}{4}, \quad S - \frac{m(n-m)-4}{4[m(n-m)-n]}D^2,$$
$$\frac{m(n-m)}{n}S + \frac{|(2m-n)D|}{n^2}\sqrt{m(n-m)(Sn-D^2)} - \frac{2m(n-m)D^2}{n^2}.$$

It can be verified directly by calculation that if $m = 1$ or $m = n-1$, then $m(n-m) = n-1$ and the maximum is $S + \frac{n-5}{4}D^2$; if $2 \le m \le n-2$, the maximum is (cf. [L1])

$$\frac{m(n-m)}{n}S + \frac{|(2m-n)D|}{n^2}\sqrt{m(n-m)(Sn-D^2)} - \frac{2m(n-m)D^2}{n^2}.$$

This completes the proof of Lemma 1.

PROPOSITION 3. *Let N^n be an n-dimensional ($n \ge 4$) compact submanifold in $\mathbb{R}^{n+p}$. Let S be the length square of the second fundamental form. If*

$$S < 2nH^2 - |(2m-n)H|\sqrt{\frac{n}{m(n-m)}(S_H - nH^2)}, \tag{55}$$

then there exist no stable m-dimensional minimal submanifolds of N^n for each m with $2 \le m \le n-2$, where S_H is the length square of the second fundamental form in the direction of the mean curvature vector of N^n.

Proof. We choose a local orthonormal frame field $e_1, \dots, e_{n+p}$ in $\mathbb{R}^{n+p}$ with $e_1, \dots, e_n$ tangent to N^n and $e_{n+1}, \dots, e_{n+p}$ normal to N^n. Let e_{n+1}

be parallel to the mean curvature vector $\overrightarrow{H}$ and

$$B(X,Y)=\sum_{\mu=n+1}^{n+p} B^{\mu}(X,Y)e_{\mu}, \tag{56}$$

then

$$\sum_a B^{n+1}(e_a,e_a)=nH,\quad \sum_a B^{\mu}(e_a,e_a)=0,\quad n+2\le\mu\le n+p. \tag{57}$$

Moreover,

$$\begin{aligned}(58)\quad &\sum_{i,\alpha}[2\|B(e_i,e_\alpha)\|^2-\langle B(e_i,e_i),B(e_\alpha,e_\alpha)\rangle]\\ &=\Big(\sum_i B^{n+1}(e_i,e_i)\Big)^2+2\sum_{i,\alpha}(B^{n+1}(e_i,e_\alpha))^2-nH\sum_i B^{n+1}(e_i,e_i)\\ &\quad+\sum_{\mu=n+2}^{n+p}\Big[\Big(\sum_i B^{\mu}(e_i,e_i)\Big)^2+2\sum_{i,\alpha}(B^{\mu}(e_i,e_\alpha))^2\Big].\end{aligned}$$

For each symmetric $n\times n$-matrix $(B^{n+1}(e_a,e_b))$ and $(B^{\mu}(e_a,e_b))$, $1\le a,b\le n$, $n+1\le\mu\le n+p$, applying Lemma 1, we have

$$\begin{aligned}(59)\quad &\Big(\sum_i B^{n+1}(e_i,e_i)\Big)^2+2\sum_{i,\alpha}(B^{n+1}(e_i,e_\alpha))^2-nH\sum_i B^{n+1}(e_i,e_i)\\ &\le\frac{m(n-m)}{n}S_H+|(2m-n)H|\sqrt{\frac{m(n-m)}{n}(S_H-nH^2)}-2m(n-m)H^2\end{aligned}$$

and

$$\Big(\sum_i B^{\mu}(e_i,e_i)\Big)^2+2\sum_{i,\alpha}(B^{\mu}(e_i,e_\alpha))^2\le\frac{m(n-m)}{n}\sum_{a,b}(B^{\mu}(e_a,e_b))^2. \tag{60}$$

Combining (58), (59) with (60), from assumption (55) we get

$$\begin{aligned}(61)\quad &\sum_{i,\alpha}[2\|B(e_i,e_\alpha)\|^2-\langle B(e_i,e_i),B(e_\alpha,e_\alpha)\rangle]\\ &\le\frac{m(n-m)}{n}S-2m(n-m)H^2+|(2m-n)H|\sqrt{\frac{m(n-m)}{n}(S_H-nH^2)}<0.\end{aligned}$$

This completes the proof of Proposition 3.

P r o o f o f T h e o r e m 2. Let N^n be an n-dimensional ($n\ge 4$) compact submanifold in $\mathbb{R}^{n+p}$. By the Gauss equation (14) and the fact that $S\ge nH^2$, we know that condition (2) is equivalent to

$$S<\frac{n^2H^2}{n-1}. \tag{62}$$

But (62) is equivalent to

$$\sqrt{S-nH^2} < \sqrt{\frac{n}{n-1}}|H| = \frac{1}{2}\sqrt{\frac{n}{n-1}}n|H| - \frac{1}{2}(n-2)\sqrt{\frac{n}{n-1}}|H|. \tag{63}$$

Now (63) is equivalent to

$$\left(\sqrt{S-nH^2} + \frac{1}{2}(n-2)\sqrt{\frac{n}{n-1}}|H|\right)^2 < \left(\frac{1}{2}\sqrt{\frac{n}{n-1}}n|H|\right)^2, \tag{64}$$

that is,

$$S < 2nH^2 - (n-2)\sqrt{\frac{n}{n-1}}|H|\sqrt{S-nH^2}. \tag{65}$$

Since $|2m-n|\sqrt{n/(m(n-m))} \le (n-2)\sqrt{n/(n-1)}$ and $S_H \le S$, we see that (65) implies (55) for each m with $2 \le m \le n-2$. Therefore, Theorem 2 follows from Proposition 3 directly.

6. The proof of Corollary 1 and Proposition 1

Proof of Corollary 1. Let N^n be an n-dimensional compact hypersurface in $\mathbb{R}^{n+1}$ and let the principal curvatures be k_a, $1 \le a \le n$. By assumption (3), we have

$$S = \sum_i k_i^2 < \frac{n^2H^2}{n-1}. \tag{66}$$

By the Gauss equation (14) and the fact $S \ge nH^2$, (66) is equivalent to (2). Now Corollary 1 follows from Theorem 2 directly.

Proof of Proposition 1. Let N^n be the following n-dimensional ($n \ge 4$) ellipsoid in $\mathbb{R}^{n+1}$:

$$N^n: \quad \frac{x_1^2}{a_1^2} + \ldots + \frac{x_{n+1}^2}{a_{n+1}^2} = 1, \quad 0 < a_1 \le a_2 \le \ldots \le a_{n+1}.$$

It is not difficult to verify by a direct computation that the maximum and minimum of the principal curvatures are

$$k_{\max} = \frac{a_{n+1}}{a_1^2}, \qquad k_{\min} = \frac{a_1}{a_{n+1}^2},$$

respectively.

(1) If $1 \le a_{n+1} < \sqrt[3]{2}$ and $a_1 \ge \sqrt{a_{n+1}}$, then the sectional curvature $\overline{K}$ of N^n satisfies

$$\frac{1}{2} < \frac{a_1^2}{a_{n+1}^4} = k_{\min}^2 \le \overline{K} \le k_{\max}^2 = \frac{a_{n+1}^2}{a_1^4} \le 1.$$

Thus the conclusion of Proposition 1 follows from Theorem 1.

(2) If $a_{n+1}/a_1 < \sqrt[6]{n/(n-1)}$, then

$$k_a - \sqrt{\frac{1}{n(n-1)}} \sum_{b=1}^{n} k_b \le \frac{a_{n+1}}{a_1^2} - \sqrt{\frac{n}{n-1}} \frac{a_1}{a_{n+1}^2} < 0.$$

Thus the conclusion of Proposition 1 follows from Corollary 1.

7. Some remarks. Let N^n be an n-dimensional compact submanifold in an $(n+p)$-dimensional unit sphere S^{n+p} and B the second fundamental form of N^n. By a reduction as in the proof of (24) (cf. (2.11) of Pan–Shen [PS]) we have

$$\text{(67)} \quad \operatorname{trace}(I) = -\int_{M^m} \Big[-\sum_{\alpha,k,\mu} (B^{\mu}_{\alpha k})^2 + \sum_{\alpha} \overline{R}_{\alpha\alpha} \Big]\, dv = \int_{M^m} \Big[-m(n-m) + 2\sum_{\mu,\alpha,k} (B^{\mu}_{\alpha k})^2 - \sum_{\mu,\alpha,k} B^{\mu}_{\alpha\alpha} B^{\mu}_{kk} \Big]\, dv.$$

We can prove the following counterparts of Theorems 1 and 2 by making use of (67):

THEOREM 3. *Let N^n be an n-dimensional compact hypersurface in an $(n+1)$-dimensional unit sphere S^{n+1}. If the sectional curvature $\overline{K}$ of N^n satisfies*

$$\text{(68)} \quad 1/2 < \overline{K} \le 1,$$

then there exist no stable m-dimensional minimal submanifolds in N^n for each m with $1 \le m \le n-1$.

THEOREM 4. *Let N^n be an n-dimensional $(n \ge 4)$ compact submanifold in an $(n+p)$-dimensional Euclidean sphere S^{n+p}. Let S and H be the length square of the second fundamental form and the mean curvature of N^n, respectively. If*

$$\text{(69)} \quad S < n + \frac{n^3}{2(n-1)} H^2 - \frac{n(n-2)}{2(n-1)} \sqrt{n^2 H^4 + 4(n-1)H^2},$$

then there exist no stable m-dimensional minimal submanifolds in N^n for each m with $2 \le m \le n-2$.

Remark 4. From the main theorem of [L2], we can prove that condition (2) or (69) implies $\operatorname{Ric}(N^n) > 0$.

Remark 5. These conclusions keep valid for *stable currents* (see Lawson–Simons [LS] or Federer–Fleming [FF]).

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