COLLOQUIUM MATHEMATICUM

VOL. 72 1997 NO. 2

THE IDEMPOTENT LIFTING THEOREM FOR ALMOST COMPLETELY DECOMPOSABLE ABELIAN GROUPS

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1. Introduction. Let X be an almost completely decomposable group, $\mathrm{T}_{\mathrm{cr}}(X)$ its critical typeset, A its regulator and e an integer such that $eX \subset A$ (see [Mad95] or [MV94] for definitions). Then the groups $(A(\tau)+eA)/eA$, where $\tau \in \mathrm{T}_{\mathrm{cr}}(X)$, and the groups eX/eA are distinguished subgroups of the finite $\mathbb{Z}/e\mathbb{Z}$-module $\overline{A} = A/eA$. This is the $\mathbb{Z}/e\mathbb{Z}$-(*anti-*)*representation* of X. The representation maps are those endomorphisms of $\overline{A}$ which map the distinguished subgroups into themselves, i.e.,

$$\mathrm{TypEnd}_X(\overline{A}) = \left\{\xi \in \mathrm{End}(\overline{A}) : \frac{A(\tau)+eA}{eA}\xi \subset \frac{A(\tau)+eA}{eA},\ \frac{eX}{eA}\xi \subset \frac{eX}{eA}\right\}.$$

In [MV94] this approach was used successfully to study, up to near-isomorphism, the almost completely decomposable groups with common regulator and regulator quotient. In the present paper we will use a modification of the same approach in order to study direct decompositions of the group X.

Consider an almost completely decomposable X and a fully invariant completely decomposable subgroup A that is fully invariant and has finite index in X. The regulator of X is an example of such a group. Every endomorphism of X induces an endomorphism of A and further an endomorphism of $A^\circ(\tau) = A(\tau)/A^{\#}(\tau)$. Assume that $eX \le A \le X$ for some integer e, and let $^{-} : A \to \overline{A}$ be the natural epimorphism. Then an endomorphism of A induces an endomorphism of $\overline{A} = A/eA$ and of $\overline{A}^\circ(\tau) = \overline{A(\tau)}/\overline{A^\sharp(\tau)}$ for every critical type τ. We identify the type τ with a rational group which represents τ and set $e_\tau = |\tau/e\tau|$.

Our main tool is the following theorem.

THEOREM 1.1 (Idempotent Lifting Theorem). *Let X be an almost completely decomposable group. Suppose that A is a completely decomposable*

1991 *Mathematics Subject Classification*: 20K15, 20K25.

fully invariant subgroup of X such that $eX \leq A$ and A contains no nonzero e-divisible subgroup. Let $\{\psi_i : i \in I\}$ be a complete set of orthogonal idempotents of $\operatorname{TypEnd}_X \overline{A}$ such that for each of the induced maps $\psi_{i\tau} \in \operatorname{End} \overline{A}^\circ(\tau)$, the image $\overline{A}^\circ(\tau)\psi_{i\tau}$ is a free $\mathbb{Z}/e_\tau\mathbb{Z}$-submodule of $\overline{A}^\circ(\tau) = \overline{A(\tau)/A^\sharp(\tau)}$. Then there is a complete family of idempotents $\phi_i \in \operatorname{End} X$ such that $\overline{\phi_i} = \psi_i$.

The first application of the Idempotent Lifting Theorem is a proof of a theorem of Dave Arnold in the special case of almost completely decomposable groups. Arnold proved the following theorem ([Arn82, 12.9, p. 144]).

THEOREM 1.2 (Arnold's Theorem). *If X and Y are nearly isomorphic torsion-free abelian groups of finite rank and $X = X_1 \oplus X_2$, then $Y = Y_1 \oplus Y_2$ with Y_i nearly isomorphic to X_i for $i = 1, 2$.*

Arnold's Theorem is important in the theory of almost completely decomposable groups since a number of subclasses of these groups can be classified up to near-isomorphism (see [Mad95, Section 7]). Arnold's Theorem says that the decomposition properties of nearly isomorphic groups are much alike. In particular, two near-isomorphic groups of finite rank are either both indecomposable or both decomposable. Therefore, essential decomposition properties are coded into any complete set of near-isomorphism invariants. For an example see [BM94] (or [Mad95, Section 8]), where decomposition properties are reduced to a factorization problem of numerical near-isomorphism invariants.

Arnold's Theorem is deep and rather difficult to prove. It is therefore desirable to have a proof using the tools of the theory of almost completely decomposable groups and standard facts about torsion-free groups. In [Sch95] such a proof is presented but it is irreparably flawed.

Our second application is a short proof of a recent theorem of Faticoni and Schultz [FS96, Theorem 3.5]. We follow the terminology of [AF92] and call a decomposition *indecomposable* if its direct summands are indecomposable.

THEOREM 1.3 (The Faticoni–Schultz Theorem). *The indecomposable decompositions of an almost completely decomposable group with prime power regulating index are unique up to near-isomorphism.*

This result significantly improves the prospects for understanding the idiosyncratic decompositions of almost completely decomposable groups. Faticoni–Schultz derive their result by utilizing the so-called near-endomorphism ring of an almost completely decomposable group whose regulating index is a power of some prime p. This is simply the endomorphism ring localized at p. It is shown that this localization is a semi-perfect ring. Then Arnold's Theorem is used along with an Azumaya–Krull–Schmidt theorem. In our approach we only require properties of artinian (actually finite) rings, the Idempotent Lifting Theorem, and an Azumaya–Krull–Schmidt Theorem.

2. Preliminaries. As usual, we refer to [Fuc73] and [Arn82] for general background. For background on almost completely decomposable groups we rely on the survey [Mad95], which contains references to the original sources. A *type* τ is considered to be an isomorphism class of rank-one groups, and sometimes is identified with a representative of the class. In particular, if e is a positive integer, then $e\tau = \tau$ makes sense—it means that the groups of the class τ are e-divisible.

Let X be an almost completely decomposable group and A a completely decomposable subgroup of X such that $eX \le A$ for some positive integer e. In our context, e-divisible subgroups of A are a harmless nuisance, and we begin by showing that A may be assumed *e-reduced* for most purposes.

LEMMA 2.1. *Let $eX \le A \le X$, where A is completely decomposable. Let D be the largest e-divisible subgroup of A. If $A = \bigoplus_{\varrho\in \mathrm{T}_{\mathrm{cr}}(A)} A_\varrho$ is a homogeneous decomposition of A, then $D = \bigoplus\{A_\varrho : e\varrho = \varrho\}$ and D is at the same time the largest e-divisible subgroup of eA and X. Set $B = \bigoplus\{A_\varrho : e\varrho \ne \varrho\}$ and $Y = B_*$, the purification of B in X. Then the following hold*:

(1) *$A = D \oplus B$ and $X = D \oplus Y$.*

(2) *Suppose that $X = X_1 \oplus X_2$. Then $D = (D\cap X_1) \oplus (D \cap X_2)$ and $X_i = (D\cap X_i)\oplus Y_i$, $i = 1, 2$, for some e-reduced groups Y_i with $Y \cong Y_1 \oplus Y_2$.*

We leave the easy verification to the reader.

We now summarize the concepts and facts that we will need. They are just reformulations of results in [MV94] ([Mad95, Section 5]).

DEFINITION 2.2. (1) For any torsion-free group G, the *type subgroups* are denoted by $G(\tau)$, $G^*(\tau) = \sum_{\varrho>\tau} G(\varrho)$, and $G^\sharp(\tau) = G^*(\tau)_*$.

(2) Let A be a completely decomposable group. If $A = \bigoplus_\varrho A_\varrho$ is the decomposition of A into homogeneous components, then the *critical typeset* of A is by definition $\mathrm{T}_{\mathrm{cr}}(A) = \{\varrho : A_\varrho \ne 0\}$.

(3) The map $^{-} : A \to A/eA = \overline{A}$ denotes the natural epimorphism as well as the induced map $^{-} : \operatorname{End} A \to \operatorname{End} \overline{A}$.

(4) Define $\overline{e} : X \to \overline{A}$ by $\overline{e} = e \circ {}^{-}$ and, by abuse of notation, set $\overline{X} = X\overline{e} = eX/eA \le \overline{A}$.

(5) The ring $\operatorname{TypEnd}\overline{A} = \{\eta \in \operatorname{End}\overline{A} : (\forall\tau \in \mathrm{T}_{\mathrm{cr}}(A))\ \overline{A(\tau)}\eta \subset \overline{A(\tau)}\}$ is the *ring of type endomorphisms* of $\overline{A}$. The *group of type automorphisms*, $\operatorname{TypAut}\overline{A}$, is the unit group of $\operatorname{TypEnd}\overline{A}$.

(6) Recall that the automorphism group $\operatorname{Aut}(\tau)$ of the rational group τ is generated multiplicatively by -1 and the primes p with $p\tau = \tau$. Given a

positive integer e, let $e_\tau = |\tau/e\tau|$. Let $\overline{\mathrm{Aut}(\tau)}$ denote the image of $\mathrm{Aut}(\tau)$ in $\mathbb{Z}/e_\tau\mathbb{Z} \cong \mathrm{End}(\tau/e\tau)$.

LEMMA 2.3. *Let X be an almost completely decomposable group and let A be a fully invariant completely decomposable subgroup satisfying $eX \le A \le X$ for some positive integer e.*

(1) *The restriction map embeds* $\mathrm{End}\, X$ *in* $\mathrm{End}\, A$ *and justifies the identification* $\mathrm{End}\, X = \{\alpha \in \mathrm{End}\, A : \overline{X}\overline{\alpha} \subset \overline{X}\}$. *Further*, $\mathrm{TypEnd}_X \overline{A} = \{\eta \in \mathrm{TypEnd}\, \overline{A} : \overline{X}\eta \subset \overline{X}\}$ *is the* type endomorphism ring *of X.*

(2) *There are exact sequences of rings and ring homomorphisms*

$$0 \to e\,\mathrm{End}\, A \to \mathrm{End}\, A \xrightarrow{\ \bar{}\ } \mathrm{TypEnd}\, \overline{A} \to 0$$

and

$$0 \to e\,\mathrm{End}\, A \to \mathrm{End}\, X \xrightarrow{\ \bar{}\ } \mathrm{TypEnd}_X\, \overline{A} \to 0.$$

(3) *The quotient $A^\circ(\tau) = A(\tau)/A^\sharp(\tau)$ is τ-homogeneous completely decomposable, and $\overline{A}^\circ(\tau) = \overline{A(\tau)}/\overline{A^\sharp(\tau)}$ is a free $\mathbb{Z}/e_\tau\mathbb{Z}$-module, where $e_\tau = |\tau/e\tau|$.*

(4) *Let $\xi \in \mathrm{TypAut}\, \overline{A}$. Then, for each $\tau \in \mathrm{T_{cr}}(A)$, the map ξ induces an automorphism ξ_τ of the free $\mathbb{Z}/e_\tau\mathbb{Z}$-module $\overline{A}^\circ(\tau) = \overline{A(\tau)}/\overline{A^\sharp(\tau)}$. As in vector spaces, an endomorphism η of $\overline{A}^\circ(\tau)$ has a matrix representation with respect to some basis and a well-defined determinant* $\det(\eta) \in \mathbb{Z}/e_\tau\mathbb{Z}$.

(5) (The Krapf–Mutzbauer Lifting Theorem) *Let $\xi \in \mathrm{TypAut}\, \overline{A}$. Then $\xi \in \overline{\mathrm{Aut}\, A}$ if and only if $\xi \in \mathrm{TypAut}\, \overline{A}$ and $\det \xi_\tau \in \overline{\mathrm{Aut}(\tau)}$ for each $\tau \in \mathrm{T_{cr}}(A)$.*

In order to see 2.3(1) and (2), consider the endomorphisms of A and X as linear transformations ϕ of the common divisible hull $\mathbb{Q}A = \mathbb{Q}X$ with $A\phi \subset A$ and $X\phi \subset X$ respectively. Since A is fully invariant in X, we have $\mathrm{End}\, X \subset \mathrm{End}\, A$ and, in fact, $\mathrm{End}\, X = \{\phi \in \mathrm{End}\, A : eX\phi \subset eX\} = \{\phi \in \mathrm{End}\, A : \overline{X}\phi \subset \overline{X}\}$. This last description has the advantage that it involves only the endomorphism ring of the completely decomposable group A.

3. Categories of summands. When considering direct decompositions of an almost completely decomposable group X, the $\mathbb{Z}/e\mathbb{Z}$-representations of X and those of its direct summands must be considered simultaneously. Since the regulator of a direct sum need not be the direct sum of the regulators of the summands, the representation approach used for classification in [BM94], [MV94], [KM84] breaks down. However, it suffices to work with any completely decomposable fully invariant subgroup of finite index. The regulator is such a group, so that existence is assured. The following trivial observation makes things work.

LEMMA 3.1. *Let X be an almost completely decomposable group, and A a completely decomposable fully invariant subgroup of X satisfying $eX \le A \le X$ for some positive integer e. If $X = Y \oplus Z$, then $A \cap Y$ is a completely decomposable fully invariant subgroup of Y satisfying $eY \le A \cap Y \le Y$.*

Proof. Since A is fully invariant, we have $A = A \cap Y \oplus A \cap Z$, and as a summand of a completely decomposable group, $A \cap Y$ is itself completely decomposable. Since every endomorphism of Y extends to an endomorphism of X, it is clear that $A \cap Y$ is fully invariant in Y. ∎

We now fix the notation that will be employed for the remainder of this section.

NOTATION. In this section X denotes a fixed almost completely decomposable group, A a fixed fully invariant completely decomposable subgroup such that $eX \le A \le X$ for some positive integer e. Let Y be a direct summand of X. Setting $A_Y = Y \cap A$, we have $eY \le A_Y \le Y$ and there is the corresponding $\mathbb{Z}/e\mathbb{Z}$-representation of Y. In particular, according to previous definitions $\overline{Y} = eY/eA_Y$.

Since

$$\overline{Y} = \frac{eY}{eA_Y} = \frac{eY}{e(A \cap Y)} = \frac{eY}{eY \cap eA} \cong \frac{eY + eA}{eA} \le \frac{eX + eA}{eA} = \overline{X}$$

and since the type subgroups of a direct sum are the direct sums of the type subgroups of the summands, we can embed the induced $\mathbb{Z}/e\mathbb{Z}$-representation of Y in the $\mathbb{Z}/e\mathbb{Z}$-representation of X. In this fashion we can study the representations of X and of its direct summands in their interaction. A more precise statement is the following:

PROPOSITION 3.2. *Let Y be a summand of X and $i_Y \in \operatorname{End} X$ an idempotent with $Y = Xi_Y$. Then*

$$\operatorname{TypEnd}_Y \overline{A_Y} \to \overline{i_Y}(\operatorname{TypEnd}_X \overline{A})\overline{i_Y} : \quad \eta \mapsto \overline{i_Y}\,\eta\,\overline{i_Y},$$

is a ring isomorphism.

Proof. Let $i_Z = 1 - i_Y$ and $Z = Xi_Z$, so that $X = Y \oplus Z$. Since A is fully invariant in X, there is a corresponding decomposition $A = A_Y \oplus A_Z$ of A, where $A_Y = A \cap Y$ and $A_Z = A \cap Z$, and a corresponding decomposition $\overline{A} = \overline{A_Y} \oplus \overline{A_Z}$. It is easily seen that the idempotents $\overline{i_Y}$ and $\overline{i_Z}$ are the projections belonging to the last decomposition. Hence ([AF92, 5.9, p. 71])

$$\operatorname{End} \overline{A_Y} \to \overline{i_Y}(\operatorname{End} \overline{A})\overline{i_Y} : \quad \xi \mapsto \overline{i_Y}\,\xi\,\overline{i_Y}$$

is an isomorphism. Furthermore, since $\overline{A(\tau)} = \overline{A_Y(\tau)} \oplus \overline{A_Z(\tau)}$, this isomorphism restricts to an isomorphism

$$\operatorname{TypEnd} \overline{A_Y} \to \overline{i_Y}(\operatorname{TypEnd} \overline{A})\overline{i_Y}$$

and finally, since $\overline{X} = \overline{Y} \oplus \overline{Z}$, to an isomorphism

$$\operatorname{TypEnd}_Y \overline{A_Y} \to \overline{i_Y}(\operatorname{TypEnd}_X \overline{A})\overline{i_Y}. \ \blacksquare$$

We now introduce suitable categories of summands.

DEFINITION 3.3. Let X be an almost completely decomposable group, A a fully invariant completely decomposable subgroup of X and e a positive integer such that $eX \subset A$. Assume that X is e-reduced.

Let $\mathcal{X}$ be the category whose objects are the direct summands $Y, Z, \ldots$ of X and whose morphisms are the ordinary group homomorphisms $\operatorname{Hom}_{\mathcal{X}}(Y, Z) = \operatorname{Hom}(Y, Z)$.

Let $\overline{\mathcal{X}}$ be the category whose objects are the groups $\overline{Y}, \overline{Z}, \ldots$ for $Y, Z, \ldots \in \mathcal{X}$ and whose morphisms are

$$\operatorname{Hom}_{\overline{\mathcal{X}}}(\overline{Y}, \overline{Z}) = \{\phi \in \operatorname{Hom}(\overline{Y}, \overline{Z}) : \phi = \xi\restriction_{\overline{Y}} \text{ for some } \xi \in \operatorname{TypEnd}_X \overline{A}\}.$$

Note that $\overline{Y} \cong_{\overline{\mathcal{X}}} \overline{Z}$ if and only if there exist maps $\xi, \eta \in \operatorname{TypEnd}_X \overline{A}$ such that $(\xi\restriction_{\overline{Y}})(\eta\restriction_{\overline{Z}}) = 1_{\overline{Y}}$ and $(\eta\restriction_{\overline{Z}})(\xi\restriction_{\overline{Y}}) = 1_{\overline{Z}}$. Also note that for a summand Y of X, we have $eY \le Y \cap A$ and hence two summands Y, Z of X are nearly isomorphic ($Y \cong_{\mathrm{n}} Z$) if and only if there is an embedding $\phi : Y \to Z$ such that $Y\phi$ has finite index in Z and $[Z : Y\phi]$ is relatively prime to e.

The following lemma connects near-isomorphism in $\mathcal{X}$ with isomorphism in $\overline{\mathcal{X}}$, denoted by $\cong_{\overline{\mathcal{X}}}$.

LEMMA 3.4. *Let X be e-reduced and Y, Z be direct summands of X. Then*

$$\overline{Y} \cong_{\overline{\mathcal{X}}} \overline{Z} \quad \textit{if and only if} \quad Y \cong_{\mathrm{n}} Z.$$

Proof. (a) Suppose first that $\overline{Y} \cong_{\overline{\mathcal{X}}} \overline{Z}$. Then, by definition, there exist $\xi, \eta \in \operatorname{TypEnd}_X \overline{A}$ such that $\xi : \overline{Y} \to \overline{Z}$ and $\eta : \overline{Z} \to \overline{Y}$ are isomorphisms. Let i_Y, i_Z be idempotents in $\operatorname{End} X$ with $Xi_Y = Y$ and $Xi_Z = Z$, and, using 2.3(2), let $\xi_0, \eta_0 \in \operatorname{End} X$ be preimages of ξ, η, so that $\overline{\xi_0} = \xi$ and $\overline{\eta_0} = \eta$. Consider the map $\phi = i_Y \xi_0 i_Z : Y \to Z$. Let $y \in Y$ and suppose that $y\phi = 0$. Then $0 = \overline{y}\overline{\phi} = \overline{y}\overline{i_Y}\xi\overline{i_Z} = \overline{y}\xi$. Since ξ is injective on $\overline{Y}$ it follows that $\overline{y} = 0$. Thus $\operatorname{Ker}\phi \subset eA$ and $\operatorname{Ker}\phi = \operatorname{Ker}\phi \cap eX = e\operatorname{Ker}\phi$. Since X is e-reduced, $\operatorname{Ker}\phi = 0$ and ϕ is injective on Y. Further, $\overline{Y}\,\overline{\phi} = \overline{X}\,\overline{\phi} = \overline{X}\,\overline{i_Z} = \overline{Z}$, which means that $Z \subset Y\phi + eA$ and so $Z = Y\phi + eZ$. Hence $Z/Y\phi = (Y\phi + eZ)/Y\phi = e(Z/Y\phi)$ is e-divisible. By symmetry, the map $\psi = i_Z \eta i_Y$ is injective, hence $\phi\psi : Y \to Y$ is injective and, by [Arn82, 6.1, p. 59], $Y/Y\phi\psi$ is finite. It follows that $Z/Y\phi \cong Z\psi/Y\phi\psi$ is a finite e-divisible abelian group, so that $[Z : Y\phi]$ is relatively prime to e. This shows that $Y \cong_{\mathrm{n}} Z$.

(b) Suppose that $Y \cong_{\mathrm{n}} Z$. Let $\phi : Y \to Z$ be a monomorphism such that $[Z : Y\phi]$ is relatively prime to e. Choose $\psi \in \operatorname{End} X$ extending ϕ. Then

$\overline{\psi} \in \operatorname{TypEnd}_X \overline{A}$ and

$$\overline{Y}\,\overline{\psi} = \left(\frac{eY+eA}{eA}\right)\overline{\psi} = \frac{eY\phi+eA}{eA} \le \frac{eZ+eA}{eA} = \overline{Z}.$$

The group $\overline{Z}/\overline{Y\psi} \cong (eZ+eA)/(eY\phi+eA)$ is $[Z : Y\phi]$-bounded and e-bounded, so zero, and thus $\overline{Y}\,\overline{\psi} = \overline{Z}$. Set $d = [Z : Y\phi]$ and define $\phi' : Z \to Y$ by $\phi' = d\phi^{-1}$. Then ϕ' is a monomorphism and $[Y : Z\phi'] = [Y\phi : Z\phi'\phi] = [Y\phi : dZ]$, which divides $[Z : dZ]$, so $[Y : Z\phi']$ is relatively prime to e. Choosing $\psi' \in \operatorname{End} X$ extending ϕ', it follows as before that $\overline{Z\psi'} = \overline{Y}$. Since $\overline{Y}$ and $\overline{Z}$ are both finite, the map $\overline{\psi}$ is injective on $\overline{Y}$, and $\overline{\psi}$ maps $\overline{Y}$ isomorphically to $\overline{Z}$. ■

4. Lifting idempotents. We begin by lifting type-automorphisms.

LEMMA 4.1. *Let A be e-reduced and $\xi \in \operatorname{TypAut} \overline{A}$. If $\eta \in \operatorname{End} A$ is any map with $\overline{\eta} = \xi$, then η is injective and $[A : A\eta]$ is relatively prime to e.*

Proof. (1) η is injective. In fact, $\operatorname{Ker}\eta \subset eA$ since $a\eta = 0$ implies $\overline{a}\xi = 0$, so $\overline{a} = 0$, i.e. $a \in eA$. On the other hand, $\operatorname{Ker}\eta$ is pure in A, so $\operatorname{Ker}\eta = eA \cap \operatorname{Ker}\eta = e\operatorname{Ker}\eta$ is e-divisible and hence $\operatorname{Ker}\eta = 0$.

(2) $\gcd([A : A\eta], e) = 1$. Since $\overline{\eta} = \xi$ is surjective, it is true that $A\eta + eA = A$, i.e. $A/A\eta$ is e-divisible and the claim follows since $A/A\eta$ is a finite group by [Arn82, 6.1, p. 59]. ■

If ϕ is an idempotent of $\operatorname{End} A$, then $\bar{\phi}$ is an idempotent of $\operatorname{TypEnd} \overline{A}$. However, an idempotent of $\operatorname{TypEnd} \overline{A}$ need not lift to an idempotent of $\operatorname{End} A$ as the following trivial example shows.

EXAMPLE 4.2. Let $A = \mathbb{Z}$, $e = 6$. Then $\overline{A} \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$ but A is not decomposable.

The question therefore is to describe the idempotents of $\operatorname{TypEnd} \overline{A}$ which are induced by idempotents of $\operatorname{End} A$. It turns out that an obviously necessary condition is also sufficient. The condition is as follows:

LEMMA 4.3. *Let A be a completely decomposable group. Any $\psi \in \operatorname{TypEnd} \overline{A}$ induces, for each $\tau \in \mathrm{T}_{\mathrm{cr}}(A)$, a map $\psi_\tau \in \operatorname{End} \overline{A}^\circ(\tau)$, where $\overline{A}^\circ(\tau) = \overline{A(\tau)}/\overline{A^\sharp(\tau)}$ is a free $\mathbb{Z}/e_\tau\mathbb{Z}$-module. If $\phi \in \operatorname{End} A$ is an idempotent, then $\overline{A}^\circ(\tau)\bar{\phi}_\tau$ is a free $\mathbb{Z}/e_\tau\mathbb{Z}$-submodule of $\overline{A}^\circ(\tau)$.*

Proof. We have $A = A\phi \oplus A(1-\phi)$. Decomposing both $A\phi$ and $A(1-\phi)$ into homogeneous components we obtain a decomposition $A = \bigoplus_{\varrho \in \mathrm{T}_{\mathrm{cr}}(A)} (A'_\varrho \oplus A''_\varrho)$, where $A\phi = \bigoplus_{\varrho \in \mathrm{T}_{\mathrm{cr}}(A)} A'_\varrho$ is the decomposition into homogeneous components of $A\phi$ and $A(1-\phi) = \bigoplus_{\varrho \in \mathrm{T}_{\mathrm{cr}}(A)} A''_\varrho$ is the decomposition into homogeneous components of $A(1-\phi)$. Now $\overline{A}^\circ(\tau) \cong \overline{A'_\tau} \oplus \overline{A''_\tau}$ and $\overline{A}^\circ(\tau)\bar{\phi}_\tau \cong \overline{A'_\tau}$ is a free $\mathbb{Z}/e_\tau\mathbb{Z}$-module. ■

A family of *orthogonal idempotents* $\{\psi_i\}$ is a set of idempotents (of some ring) such that $\psi_i\psi_j = 0$ whenever $i \neq j$. Given a decomposition of a module, the projections onto the summands interpreted as endomorphisms form an orthogonal family of idempotents whose sum is the identity. Conversely, an orthogonal family with sum 1 determines a decomposition in which the summands are the images of the idempotents. Recall that a family of orthogonal idempotents is *complete* if the sum of its members is the identity.

We will show next that any idempotent of $\operatorname{TypEnd}\overline{A}$ which satisfies the necessary condition in 4.3 lifts to an idempotent of $\operatorname{End} A$. The proof is accomplished in steps starting with the homogeneous case. The following lemma is essentially Lemma 1.4 in [KM84].

LEMMA 4.4. *Let A be a τ-homogeneous completely decomposable group, e a positive integer and $\overline{A} = A/eA$. If ψ_i is a complete family of orthogonal idempotents of* $\operatorname{TypEnd}\overline{A} = \operatorname{End}\overline{A}$ *such that for each i, $\overline{A}\psi_i$ is a free $\mathbb{Z}/e_\tau\mathbb{Z}$-submodule of $\overline{A}$, then there is a complete family of orthogonal idempotents $\phi_i \in \operatorname{End} A$ such that $\overline{\phi_i} = \psi_i$.*

In other words, any decomposition of $\overline{A}$ into a direct sum of free submodules lifts to a decomposition of A.

Proof. Write $A = A_1 \oplus \ldots \oplus A_n$, where the A_i are isomorphic rational groups. Then

$$\overline{A} = \overline{A_1} \oplus \ldots \oplus \overline{A_n},$$

the $\overline{A_i}$ are all isomorphic to $\mathbb{Z}/e_\tau\mathbb{Z}$, and $\overline{A}$ is a free $\mathbb{Z}/e_\tau\mathbb{Z}$-module. The idea of the proof is to compare any other decomposition of $\overline{A}$ into free submodules with this particular one which lifts, and then show that the other decomposition lifts as well.

Suppose that $\{\psi_i : i \in I\} \subset \operatorname{End}\overline{A}$ is a complete family of orthogonal idempotents such that each summand $\overline{A}\psi_i$ of $\overline{A}$ is a free $\mathbb{Z}/e_\tau\mathbb{Z}$-submodule of $\overline{A}$. Then there is a partition $\{1, \ldots, n\} = \bigcup_{i\in I} S_i$ such that $\overline{A}\psi_i \cong \overline{B_i}$, where $B_i = \sum_{j\in S_i} A_j$. Hence there is an automorphism $\widetilde{\xi}$ of $\overline{A}$ such that $\overline{B_i}\widetilde{\xi} = \overline{A}\psi_i$ for all i. Let $\pi_i : A \to B_i$ be the projections belonging to the decomposition $A = \bigoplus_{i\in I} B_i$. By checking the action on each summand $\overline{A}\psi_j$ it follows easily that $\psi_i = \widetilde{\xi}^{-1}\overline{\pi_i}\widetilde{\xi}$.

Suppose for the moment that $\widetilde{\xi}$ lifts to an automorphism α of A, i.e. $\overline{\alpha} = \widetilde{\xi}$. Then $\alpha^{-1}\pi_i\alpha$ are idempotents of $\operatorname{End} A$ which induce the ψ_i. In this case the claim is established.

It is easy to replace $\widetilde{\xi}$ by an automorphism ξ which does lift and still maps $\overline{B_i}$ onto $\overline{A}\psi_i$. In fact, let $u = \det\widetilde{\xi}$ and let $\widetilde{u}$ be the automorphism of $\overline{A}$ which is multiplication by u^{-1} on $\overline{A_1}$ and is the identity on all other $\overline{A_i}$. Let $\xi = \widetilde{u}\widetilde{\xi}$. Then $\det\xi = 1$, so ξ lifts to an automorphism of A by the Krapf–Mutzbauer Theorem 2.3(5), thereby producing the desired idempotents of $\operatorname{End} A$. ∎

We can now prove the general case.

THEOREM 4.5. *Let A be a completely decomposable group and let e be a positive integer and $\overline{A} = A/eA$. Let $\{\psi_i : i \in I\}$ be a complete set of orthogonal idempotents of* $\operatorname{TypEnd} \overline{A}$ *such that for each of the induced maps $\psi_{i\tau} \in \operatorname{End} \overline{A}^\circ(\tau)$, the image $\overline{A}^\circ(\tau)\psi_{i\tau}$ is a free $\mathbb{Z}/e_\tau\mathbb{Z}$-submodule of $\overline{A}^\circ(\tau) = \overline{A(\tau)}/\overline{A^\sharp(\tau)}$. Then there is a complete family of idempotents $\phi_i \in$* $\operatorname{End} A$ *such that $\overline{\phi_i} = \psi_i$.*

Proof. We use induction on the depth of critical types to show that the specified idempotents lift to idempotent endomorphisms of A. Let $\{\psi_i\}$ be a complete family of orthogonal idempotents of $\operatorname{TypEnd} \overline{A}$ satisfying the hypotheses of the theorem. The depth of a type in $\mathrm{T}_{\mathrm{cr}}(A)$ is the longest path from the type to a maximal critical type, so that maximal types have depth zero. By $\operatorname{depth}(\mathrm{T}_{\mathrm{cr}}(A))$ we mean the largest of the depths of critical types.

If $\operatorname{depth}(\mathrm{T}_{\mathrm{cr}}(A)) = 0$, then $\mathrm{T}_{\mathrm{cr}}(A)$ is an anti-chain, the $A(\tau)$ are the unique homogeneous components of A, so $\overline{A}^\circ(\tau) = \overline{A(\tau)}$ and, by 4.4, every complete orthogonal family $\psi_{i\tau}$ is induced by some complete family of orthogonal idempotents $\phi_{i\tau} \in \operatorname{End} A(\tau)$. Then $\phi_i = \bigoplus_\varrho \phi_{i\varrho}$ is an idempotent which induces ψ_i.

Now let $d = \operatorname{depth}(\mathrm{T}_{\mathrm{cr}}(A))$ be arbitrary. Write $A = A_M \oplus A^1$, where A_M is a direct sum of rank-one groups of types minimal in $\mathrm{T}_{\mathrm{cr}}(A)$ and $A^1 = \sum_{\sigma \in \mathrm{T}_{\mathrm{cr}}(A)} A^\sharp(\sigma)$. Then $\overline{A^1}$ is invariant under all type-endomorphisms of $\overline{A}$ and the ψ_i restrict to idempotents ψ_i^1 of $\overline{A^1}$ which satisfy all the hypotheses. Since $\operatorname{depth}(\mathrm{T}_{\mathrm{cr}}(A^1)) \leq d-1$, by induction hypothesis the family $\{\psi_i^1\}$ lifts to a family $\{\phi_i^1\}$ of idempotents of A^1. We now have $\overline{A} = \bigoplus_i \overline{A}\psi_i = \overline{A_M} \oplus \bigoplus_i \overline{A^1}\psi_i$. By the modular law there are groups $K_i \leq \overline{A}\psi_i$ such that $\overline{A} = (\bigoplus_i K_i) \oplus \overline{A^1}$. Then $K = \bigoplus_i K_i$ is invariant under each ψ_i, $\overline{A} = K \oplus \overline{A^1}$ and $K \cong \overline{A_M}$. Applying 4.4 to each homogeneous component of K, we obtain a subgroup L of A such that $A = L \oplus A^1$ and $\overline{L} = K$. The restrictions ψ_i^0 of ψ_i to K lift to idempotents ϕ_i^0 of $\operatorname{End} L$ since $\operatorname{depth}(\mathrm{T}_{\mathrm{cr}}(L)) = 0$, and the idempotents $\phi_i = \phi_i^0 \oplus \phi_i^1$ lift ψ_i. ∎

COROLLARY 4.6 (Idempotent Lifting Theorem). *Let X be an almost completely decomposable group. Suppose that A is a fully invariant completely decomposable subgroup such that $eX \leq A$ and X is e-reduced. Let $\{\psi_i : i \in I\}$ be a complete set of orthogonal idempotents of* $\operatorname{TypEnd}_X \overline{A}$ *such that for each of the induced maps $\psi_{i\tau} \in \operatorname{End} \overline{A}^\circ(\tau)$, the image $\overline{A}^\circ(\tau)\psi_{i\tau}$ is a free $\mathbb{Z}/e_\tau\mathbb{Z}$-submodule of $\overline{A}^\circ(\tau) = \overline{A(\tau)}/\overline{A^\sharp(\tau)}$. Then there is a complete family of idempotents $\phi_i \in \operatorname{End} X$ such that $\overline{\phi_i} = \psi_i$.*

P r o o f. The lifting of idempotents is just the lifting of 4.5 applied in the special case of idempotents leaving $\overline{X}$ invariant. ∎

5. Arnold's Theorem. We can now prove Arnold's Theorem 1.2 in the special case where X and Y are almost completely decomposable. The first step is a reduction to the e-reduced case. The routine proof is left to the reader.

LEMMA 5.1. *Let $eX \le A \le X$, where $A = \mathrm{R}(X)$. Let D be the largest e-divisible subgroup of A and $X = D \oplus X'$* (cf. 2.1). *Let Y be another almost completely decomposable group and suppose that $X \cong_{\mathrm{n}} Y$. It may be assumed without loss of generality that $A = \mathrm{R}(Y)$ and $eY \le A \le Y$* ([MV94, 4.6]). *Then*

(1) *$Y = D \oplus Y'$, Y' is e-reduced and $X' \cong_{\mathrm{n}} Y'$.*

(2) *If $X = X_1 \oplus X_2$, then $X_i = (X_i \cap D) \oplus X_i'$ for $i = 1, 2$, $(X_1 \cap D) \oplus (X_2 \cap D) = D$ and $X_1' \oplus X_2' \cong X'$.*

THEOREM 5.2 (Arnold). *If X and Y are nearly isomorphic almost completely decomposable groups of finite rank and if $X = X_1 \oplus X_2$, then $Y = Y_1 \oplus Y_2$ with X_i nearly isomorphic to Y_i, $i = 1, 2$.*

P r o o f. Justified by 5.1 we assume without loss of generality that X and Y are e-reduced. By passing to isomorphic copies if necessary, we may assume further that

$$eX, eY \le A, \qquad \text{where } A = \mathrm{R}(X) = \mathrm{R}(Y).$$

Since $X \cong_{\mathrm{n}} Y$, by [MV94, 4.2, 4.5], there is $\xi \in \mathrm{TypAut}\,\overline{A}$ such that $\overline{X}\xi = \overline{Y}$. By 2.1 there is $\eta \in \mathrm{End}\,A$ such that $\overline{\eta} = \xi$, and since A is e-reduced, any such lifting η is injective with $[A : A\eta]$ relatively prime to e. For the rest, fix an endomorphism η that induces ξ, and fix an endomorphism ζ that induces ξ^{-1}.

Let $\phi \in \mathrm{End}\,X$ be an idempotent with $X\phi = X_1$ and $X(1-\phi) = X_2$. We will obtain an idempotent $\psi \in \mathrm{End}\,Y$ such that $Y\psi \cong_{\mathrm{n}} X\phi$ and $Y(1-\psi) \cong_{\mathrm{n}} X(1-\phi)$. Now $\xi^{-1}\overline{\phi}\xi$ is an idempotent in $\mathrm{TypEnd}_Y\,\overline{A}$. Idempotent Lifting (4.5) applies to produce an idempotent $\psi \in \mathrm{End}\,Y \le \mathrm{End}\,A$ such that $\overline{\psi} = \xi^{-1}\overline{\phi}\xi$.

We show next that $\eta\psi\restriction_{A\phi} : A\phi \to A\psi$ is injective. In fact, assume $a\phi \in A\phi$ such that $a\phi\eta\psi = 0$. Then $0 = \overline{a}\overline{\phi}\xi\overline{\psi} = \overline{a}\overline{\phi}^2\xi = \overline{a}\overline{\phi}\xi$, and hence $\overline{a}\overline{\phi} = 0$. Thus $a\phi \in eA \cap \mathrm{Ker}\,\eta\psi$. Since $K = \mathrm{Ker}(\eta\psi\restriction_{A\phi}) = \mathrm{Ker}\,\eta\psi \cap A\phi$ is pure in A, we have $K \subset eA \cap K = eK$, thus K is e-divisible and hence trivial. By symmetry, $\zeta\phi : A\psi \to A\phi$ is also injective and, by [Arn82, 6.2(d), p. 59], it follows that $A\psi$ and $A\phi$ are quasi-isomorphic, and in fact isomorphic, since both groups are completely decomposable.

Since $A\phi \geq eX\phi \geq eA\phi$, we conclude that $X\phi$ is quasi-isomorphic with $A\phi$, and similarly, $Y\psi$ is quasi-isomorphic with $A\psi$. Together with the isomorphism $A\psi \cong A\phi$ just shown, we see that $X\phi$ is quasi-isomorphic with $Y\psi$. Therefore, $Y\psi/(X\phi)\eta\psi$ is finite ([Arn82, p. 59]), and e-divisible since $\overline{Y\psi} = \overline{X\phi\eta\psi}$. Finally, the composition

$$\sigma : X\phi \subset A\phi \cong A\psi \xrightarrow{e} Y\psi$$

has the property that $e^2 Y\psi \subset X\phi\sigma$. So given any prime p, one of the two embeddings $\eta\psi$ or σ has a cokernel whose order is prime to p. This shows that $X\phi \cong_{\mathrm{n}} Y\psi$. By symmetry it follows that $X(1-\phi) \cong_{\mathrm{n}} Y(1-\phi)$. ■

6. The Faticoni–Schultz Theorem. In this section it is assumed that e is a power of a prime number p and, as above, A is a fully invariant completely decomposable subgroup with $eX \leq A \leq X$. The key effect of this assumption is to eliminate problems as in Example 4.2. The p-primary assumption guarantees that every direct summand of $\overline{A}^\circ(\tau)$ is a free $\mathbb{Z}/e\mathbb{Z}$-module, so that the hypothesis of the Idempotent Lifting Theorem is always satisfied.

Since p-divisible summands disappear when passing to $\overline{A}$, the unique decomposition problem must be reduced to p-reduced groups. If D is the maximal p-divisible summand of X, then $X = D \oplus X'$ with X' p-reduced and D is a summand of A. If $X = X_1 \oplus \ldots \oplus X_n$ is an indecomposable decomposition of X, then $D = (D \cap X_1) \oplus \ldots \oplus (D \cap X_n)$ since D is fully invariant. If $D \cap X_i \neq 0$, then $D \cap X_i = X_i$ and X_i is a rational group. So without loss of generality $D = X_1 \oplus \ldots \oplus X_k$ and $X' \cong X/D \cong X_{k+1} \oplus \ldots \oplus X_n$. The decomposition of D is unique up to isomorphism. It remains to show uniqueness of decomposition up to near-isomorphism for the p-reduced group X'.

Two lemmas are needed for the proof of the Faticoni–Schultz Theorem.

Lemma 6.1. *If A is e-reduced, then $e\,\mathrm{End}\,A$ contains no idempotent other than* 0.

Proof. Suppose that $i = ej$, $j \in \mathrm{End}\,A$, and $i^2 = i$. Then $j = ej^2$, hence Aj is e-divisible and thus $Aj = 0$, $j = 0$. ■

Lemma 6.2. *Suppose that e is a p-power. Then X is indecomposable if and only if $\mathrm{TypEnd}_X\,\overline{A}$ is a local ring.*

Proof. The group X is indecomposable if and only if $\mathrm{End}\,X$ contains no non-trivial idempotents. By the Idempotent Lifting Theorem and 6.1, $\mathrm{End}\,X$ contains no non-trivial idempotents if and only if $\mathrm{TypEnd}_X\,\overline{A}$ contains no non-trivial idempotents. But $\mathrm{TypEnd}_X\,\overline{A}$ is a finite ring, and so artinian, and it is a well-known theorem of ring theory that an artinian ring contains no non-trivial idempotents if and only if it is local ([AF92, 15.15(h), p. 170, 27.1, p. 301]). ∎

The machinery is now in place for showing that the category $\overline{\mathcal{X}}$ has a Krull–Schmidt theorem for indecomposable decompositions.

PROPOSITION 6.3. *Let e be a p-power and suppose that X is a p-reduced almost completely decomposable group. The category $\overline{\mathcal{X}}$ is pre-additive and idempotents split. Indecomposable objects have local endomorphism rings, and indecomposable decompositions are unique up to $\overline{\mathcal{X}}$-isomorphism.*

Proof. It is clear by definition that the morphism sets are additive abelian groups. To show that idempotents split, let $i \in \mathrm{TypEnd}_X \overline{A}$ be an idempotent. By the Idempotent Lifting Theorem there is $j \in \mathrm{End}\, X$ such that j is idempotent and $\overline{j} = i$. Let $Y = Xj$. Then $\overline{Y} = \overline{X}\,\overline{j} = \overline{X}i \in \overline{\mathcal{X}}$. Let $q : \overline{Y} \to \overline{X}$ be given by $q = 1\restriction_{\overline{Y}}$ and $\pi : \overline{X} \to \overline{Y}$ by $\pi = i\restriction_{\overline{X}}$. Then $q\pi = 0$ and $\pi q = i$, so i splits. The Krull–Schmidt property follows since the proof of Theorem 7.4 in [Arn82], which is stated for additive categories, goes through for a pre-additive category. ∎

COROLLARY 6.4 (The Faticoni–Schultz Theorem). *Let X and Y be p-reduced nearly isomorphic almost completely decomposable groups with p-power regulating index. If $X = \bigoplus_{i=1}^{m} X_i$ and $Y = \bigoplus_{i=1}^{n} Y_i$ are indecomposable decompositions, then $m = n$ and, after relabeling, $X_i \cong_{\mathrm{n}} Y_i$ for $1 \le i \le n$.*

Proof. By Arnold's Theorem we may assume that $X = Y$ and $A = \mathrm{R}(X) = \mathrm{R}(Y)$. Then $\overline{X} = \overline{Y}$, so by 6.3 and 3.4 it follows that $m = n$ and, after relabeling if necessary, $X_i \cong_{\mathrm{n}} Y_i$. ∎

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Received 24 May 1996