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DEGENERATIONS IN THE MODULE VARIETIES OF GENERALIZED STANDARD AUSLANDER–REITEN COMPONENTS

BY

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1. Introduction and the main results. Throughout the paper K denotes a fixed algebraically closed field. By an *algebra* we mean an associative finite-dimensional K-algebra with an identity, and by an A-*module* a finite-dimensional (unital) right A-module. We shall denote by $\operatorname{mod} A$ the category of A-modules, by Γ_A the Auslander–Reiten quiver of A, and by τ_A the Auslander–Reiten translation $D\operatorname{Tr}$ in Γ_A.

For an algebra A with basis $a_1 = 1, \ldots, a_n$, we have the structure constants c_{ijk} defined by $a_i a_j = \sum c_{ijk} a_k$. The affine variety $\operatorname{mod}_A(d)$ of d-dimensional A-modules consists of n-tuples $m = (m_1, \ldots, m_n)$ of $d \times d$-matrices with coefficients in K such that m_1 is the identity matrix and $m_i m_j = \sum m_k c_{ijk}$ holds for all indices i and j. The general linear group $\operatorname{Gl}_d(K)$ acts on $\operatorname{mod}_A(d)$ by conjugation, and the orbits correspond to the isomorphism classes of d-dimensional modules (see [15]). We identify a d-dimensional A-module M with the point of $\operatorname{mod}_A(d)$ corresponding to it. We denote by $\mathcal{O}(M)$ the $\operatorname{Gl}_d(K)$-orbit of a module M in $\operatorname{mod}_A(d)$. Then one says that a module N in $\operatorname{mod}_A(d)$ is a *degeneration* of a module M in $\operatorname{mod}_A(d)$ if N belongs to the Zariski closure $\overline{\mathcal{O}(M)}$ of $\mathcal{O}(M)$ in $\operatorname{mod}_A(d)$, and we denote this fact by $M \le_{\deg} N$. Thus $\le_{\deg}$ is a partial order on the set of isomorphism classes of A-modules of a given dimension. It is not clear how to characterize $\le_{\deg}$ in terms of representation theory.

There has been important work by S. Abeasis and A. del Fra [1], K. Bongartz [11]–[13], and Ch. Riedtmann [18] connecting $\le_{\deg}$ with other partial orders $\le_{\mathrm{ext}}$, $\le_{\mathrm{virt}}$ and $\le$ on the isomorphism classes in $\operatorname{mod}_A(d)$. They are defined in terms of representation theory as follows:

- $M \le_{\mathrm{ext}} N :\Leftrightarrow$ there are modules M_i, U_i, V_i and short exact sequences $0 \to U_i \to M_i \to V_i \to 0$ in $\operatorname{mod} A$ such that $M = M_1$, $M_{i+1} = U_i \oplus V_i$, $1 \le i \le s$, and $N = M_{s+1}$ for some natural number s.
- $M \le_{\mathrm{virt}} N :\Leftrightarrow M \oplus X \le_{\deg} N \oplus X$ for some A-module X.

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• $M \le N :\Leftrightarrow [X,M] \le [X,N]$ holds for all modules X.

Here and later on we abbreviate $\dim_K \operatorname{Hom}_A(X,Y)$ by $[X,Y]$. Then for modules M and N in $\operatorname{mod}_A(d)$ the following implications hold:

$$M \le_{\mathrm{ext}} N \Rightarrow M \le_{\mathrm{deg}} N \Rightarrow M \le_{\mathrm{virt}} N \Rightarrow M \le N$$

(see [11], [18]). Unfortunately the reverse implications are not true in general, and it would be interesting to find out when they are. This is the case for representations of Dynkin quivers and Kronecker modules [11]. It was shown recently in [13] that $\le_{\mathrm{deg}}$ and $\le$ coincide for representations of extended Dynkin quivers. For a module M in $\operatorname{mod} A$, we shall denote by $[M]$ the image of M in the Grothendieck group $K_0(A)$ of A. Thus $[M]=[N]$ if and only if M and N have the same simple composition factors including the multiplicities. Observe that, if M and N have the same dimension and $M \le N$, then $[M]=[N]$.

We are interested in degenerations of modules whose indecomposable direct summands belong to a connected component $\mathcal{C}$ of the Auslander–Reiten quiver Γ_A of an algebra A. Namely, we may ask when $M \le_{\mathrm{deg}} N$ for M and N from the additive category $\operatorname{add}(\mathcal{C})$ of $\mathcal{C}$ with $[M]=[N]$. Then the following partial order on the isomorphism classes in $\operatorname{add}(\mathcal{C})$ occurs naturally [25]:

• $M \le_{\mathcal{C}} N :\Leftrightarrow [X,M] \le [X,N]$ for all modules X in $\operatorname{add}(\mathcal{C})$.

Clearly, for M and N in $\operatorname{add}(\mathcal{C})$, $M \le N$ implies $M \le_{\mathcal{C}} N$.

In the representation theory of algebras an important role is played by generalized standard Auslander–Reiten components. Recall that following A. Skowroński [22] a connected component $\mathcal{C}$ in Γ_A is called *generalized standard* if $\operatorname{rad}^\infty(X,Y)=0$ for all modules X and Y from $\mathcal{C}$, where $\operatorname{rad}^\infty(X,Y)$ denotes the intersection of all powers $\operatorname{rad}^i(X,Y)$, $i \ge 1$, of the radical $\operatorname{rad}(X,Y)$. The Auslander–Reiten quiver Γ_A of any algebra A of finite representation type is generalized standard. Examples of infinite generalized standard components are the preprojective components, preinjective components, the connected components of tilted algebras, and tubes over tame tilted algebras and tubular algebras (see [19]). It was shown in [20] that any generalized standard component without oriented cycles is a glueing of finitely many preprojective and preinjective components. The structure of arbitrary generalized standard components is not known. In general we know only by [22] that if $\mathcal{C}$ is a generalized standard component in Γ_A, then all but finitely many τ_A-orbits in $\mathcal{C}$ are periodic. It is known that $\le_{\mathrm{ext}}$ and $\le_\Gamma$ coincide in the case when Γ is preprojective (preinjective) [11] or a generalized standard quasi-tube [25]. Moreover, there are generalized standard components (see [18], [25]) for which $\le_{\mathrm{ext}}$ and $\le_{\mathrm{deg}}$ do not coincide. But the question whether $M \le_\Gamma N$ implies $M \le_{\mathrm{deg}} N$ for M and N from the additive category of a generalized standard component Γ is still an open problem. The situation is not even clear in the case of finite representation type, although it is known that then the orders $\leq$ and $\leq_{\text{virt}}$ coincide [18]. Our first main result is as follows.

THEOREM 1. *Let A be an algebra, Γ a generalized standard component in Γ_A, and M, N modules in* $\operatorname{add}(\Gamma)$ *with* $[M] = [N]$. *Then $M \leq_{\text{virt}} N$ if and only if $M \leq_\Gamma N$.*

In the study of simply connected algebras of polynomial growth, a natural generalization of the notion of a tube appeared, called a coil, and then a more general concept of a multicoil, which is a glueing of a finite number of coils by directed parts (see [3], [4], [5]). By abuse of language we consider a directed Auslander–Reiten component as a (trivial) multicoil. One of the important results proved in [24] (see also [23]) says that a strongly simply connected algebra A is of polynomial growth if and only if every component of Γ_A is a generalized standard multicoil. Our second main result is as follows.

THEOREM 2. *Let A be an algebra, Γ a generalized standard multicoil in Γ_A, and M, N modules in* $\operatorname{add}(\Gamma)$ *with* $[M] = [N]$. *Then $M \leq_{\text{deg}} N$ if and only if $M \leq_\Gamma N$.*

The paper is organized as follows. In Section 2 we fix the notation, recall the relevant definitions and facts, and prove some preliminary results on modules which we apply in our investigations. Section 3 is devoted to the shape of arbitrary generalized standard components. In Section 4 we prove some results concerning dimension functions on the generalized standard components, playing a fundamental role in the proofs of our main results. Sections 5 and 6 are devoted to the proofs of Theorems 1 and 2, respectively.

For basic background on the topics considered here we refer to [4], [5], [8], [11], [19], [21], [22]. Main results of the paper were announced at the Conference on "Tame Algebras and Deformations" in Luminy (18–22 March 1996).

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2. Preliminary results

2.1. Throughout the paper A denotes a fixed finite-dimensional associative K-algebra with an identity over an algebraically closed field K. We denote by $\operatorname{mod} A$ the category of finite-dimensional right A-modules, by $\operatorname{ind} A$ the full subcategory of $\operatorname{mod} A$ formed by indecomposable modules, by $\operatorname{rad}(\operatorname{mod} A)$ the Jacobson radical of $\operatorname{mod} A$, and by $\operatorname{rad}^\infty(\operatorname{mod} A)$ the intersection of all powers $\operatorname{rad}^i(\operatorname{mod} A)$, $i \geq 1$, of $\operatorname{rad}(\operatorname{mod} A)$. By an A-module we mean an object from $\operatorname{mod} A$. Further, we denote by Γ_A the Auslander–Reiten quiver of A and by $\tau = \tau_A$ and $\tau^- = \tau_A^-$ the Auslander–Reiten translations $D\operatorname{Tr}$ and $\operatorname{Tr} D$, respectively. We identify the vertices of Γ_A with the corresponding indecomposable modules. For M in $\operatorname{mod} A$ we denote by $[M]$ the image of M in the Grothendieck group $K_0(A)$. Further, for X, Y from $\operatorname{mod} A$ we abbreviate $\dim_K \operatorname{Hom}_A(X,Y)$ by $[X,Y]$. For a family $\mathcal{F}$ of A-modules, we denote by $\operatorname{add}(\mathcal{F})$ the additive category given by $\mathcal{F}$, that is, the full subcategory of $\operatorname{mod} A$ formed by all modules isomorphic to the direct sums of modules from $\mathcal{F}$. Finally, for a quiver Γ, we denote by $(\Gamma)_0$ the set of all vertices of Γ.

2.2. Following [18], for M, N from $\operatorname{mod} A$, we set $M \le N$ if and only if $[X,M] \le [X,N]$ for all A-modules X. The fact that $\le$ is a partial order on the isomorphism classes of A-modules follows from a result by M. Auslander (see [6], [10]). M. Auslander and I. Reiten have shown in [7] that, if $[M] = [N]$, then for all nonprojective A-modules X and all noninjective modules Y the following formulas hold:

$$[X,M] - [M,\tau X] = [X,N] - [N,\tau X],$$
$$[M,Y] - [\tau^- Y, M] = [N,Y] - [\tau^- Y, N].$$

Hence, if $[M] = [N]$, then $M \le N$ if and only if $[M,X] \le [N,X]$ for all A-modules X.

2.3. Let M and N be A-modules with $[M] = [N]$ and

$$\Sigma: \quad 0 \to D \to E \to F \to 0$$

an exact sequence in $\operatorname{mod} A$. Following [18] we define the additive functions $\delta_{M,N}$, $\delta'_{M,N}$, δ_Σ and δ'_Σ on A-modules X as follows:

$$\delta_{M,N}(X) = [N,X] - [M,X], \qquad \delta'_{M,N}(X) = [X,N] - [X,M],$$
$$\delta_\Sigma(X) = \delta_{E,D\oplus F}(X) = [D \oplus F, X] - [E,X],$$
$$\delta'_\Sigma(X) = \delta'_{E,D\oplus F}(X) = [X, D \oplus F] - [X,E].$$

From the Auslander–Reiten formulas (2.2) we get the following very useful equalities:

$$\delta_{M,N}(X) = \delta'_{M,N}(\tau^- X), \qquad \delta_{M,N}(\tau X) = \delta'_{M,N}(X)$$

and

$$\delta_\Sigma(X) = \delta'_\Sigma(\tau^- X), \qquad \delta_\Sigma(\tau X) = \delta'_\Sigma(X)$$

for all A-modules X. Observe also that $\delta_{M,N}(I) = 0$ for any injective A-module I, and $\delta'_{M,N}(P) = 0$ for any projective A-module P. In particular, we see that the following conditions are equivalent:

(1) $M \le N$,
(2) $\delta_{M,N}(X) \ge 0$ for all $X \in \operatorname{ind} A$,
(3) $\delta'_{M,N}(X) \ge 0$ for all $X \in \operatorname{ind} A$.

2.4. For an A-module M and an indecomposable A-module Z, we denote by $\mu(M,Z)$ the multiplicity of Z as a direct summand of M. For a noninjective indecomposable A-module U we denote by $\Sigma(U)$ an Auslander–Reiten sequence

$$\Sigma(U): \quad 0 \to U \to E(U) \to \tau^{-}U \to 0.$$

We need the following lemmas.

LEMMA 2.5. *Let M, N be A-modules with $[M]=[N]$ and U an indecomposable A-module. Then*

(i) *If U is noninjective, then $\delta_{\Sigma(U)}(M)=\mu(M,U)$ and*
$$\mu(N,U)-\mu(M,U)=\delta'_{M,N}(U)-\delta'_{M,N}(E(U))+\delta'_{M,N}(\tau^{-}U).$$

(ii) *If U is injective, then $[U,M]-[U/\mathrm{soc}(U),M]=\mu(M,U)$ and*
$$\mu(N,U)-\mu(M,U)=\delta'_{M,N}(U)-\delta'_{M,N}(U/\mathrm{soc}(U)).$$

(iii) *If U is nonprojective, then $\delta'_{\Sigma(\tau U)}(M)=\mu(M,U)$ and*
$$\mu(N,U)-\mu(M,U)=\delta_{M,N}(U)-\delta_{M,N}(E(\tau U))+\delta_{M,N}(\tau U).$$

(iv) *If U is projective, then $[M,U]-[M,\mathrm{rad}\ U]=\mu(M,U)$ and*
$$\mu(N,U)-\mu(M,U)=\delta_{M,N}(U)-\delta_{M,N}(\mathrm{rad}\ U).$$

Proof. (i) The Auslander–Reiten sequence $\Sigma(U)$ induces an exact sequence

$$0 \to \mathrm{Hom}_A(\tau^{-}U,M) \to \mathrm{Hom}_A(E(U),M) \to \mathrm{rad}(U,M) \to 0,$$

and hence we get

$$\begin{aligned}\delta_{\Sigma(U)}(M) &= [U\oplus\tau^{-}U,M]-[E(U),M]\\ &= [U,M]-\dim_K \mathrm{rad}(U,M)=\mu(M,U).\end{aligned}$$

Similarly we have

$$[U\oplus\tau^{-}U,N]-[E(U),N]=\mu(N,U)$$

and consequently

$$\begin{aligned}&\mu(N,U)-\mu(M,U)\\ &\quad=([U\oplus\tau^{-}U,N]-[U\oplus\tau^{-}U,M])-([E(U),N]-[E(U),M])\\ &\quad=\delta'_{M,N}(U)-\delta'_{M,N}(E(U))+\delta'_{M,N}(\tau^{-}U).\end{aligned}$$

(ii) Since $\mathrm{Hom}_A(U/\mathrm{soc}(U),M)\simeq\mathrm{rad}(U,M)$ as K-vector spaces, we have

$$[U,M]-[U/\mathrm{soc}(U),M]=\mu(M,U).$$

Similarly we have

$$[U,N]-[U/\mathrm{soc}(U),N]=\mu(N,U)$$

and consequently

$$\begin{aligned}
\mu(N,U)-\mu(M,U)&\\
&=([U,N]-[U,M])-([U/\mathrm{soc}(U),N]-[U/\mathrm{soc}(U),M])\\
&=\delta'_{M,N}(U)-\delta'_{M,N}(U/\mathrm{soc}(U)).
\end{aligned}$$

We obtain (iii) and (iv) by duality.

LEMMA 2.6. *Let Γ be a generalized standard component of Γ_A, M and N be two modules in* $\mathrm{add}(\Gamma)$ *with $[M]=[N]$ and assume that there are modules U_i, V_i in Γ for all $i\ge 1$ such that all V_i are pairwise nonisomorphic. Then*

(i) *If there exists in Γ a sectional path $V_1\to V_2\to V_3\to\ldots$ and meshes*

$$\begin{tikzcd} & U_i \arrow[dr] & \\ V_i \arrow[ur]\arrow[dr] & & U_{i+1} \\ & V_{i+1} \arrow[ur] & \end{tikzcd}$$

for all $i\ge 1$, then

$$[V_1,M]-[U_1,M]=\sum_{i\ge 1}\mu(M,V_i)$$

and

$$\delta'_{M,N}(V_1)-\delta'_{M,N}(U_1)=\sum_{i\ge 1}(\mu(N,V_i)-\mu(M,V_i)).$$

(ii) *If there exists in Γ a sectional path $\ldots\to V_3\to V_2\to V_1$ and meshes*

$$\begin{tikzcd} & U_i \arrow[dr] & \\ U_{i+1} \arrow[ur]\arrow[dr] & & V_i \\ & V_{i+1} \arrow[ur] & \end{tikzcd}$$

for all $i\ge 1$, then

$$[M,V_1]-[M,U_1]=\sum_{i\ge 1}\mu(M,V_i)$$

and

$$\delta_{M,N}(V_1)-\delta_{M,N}(U_1)=\sum_{i\ge 1}(\mu(N,V_i)-\mu(M,V_i)).$$

Proof. (i) By assumption there are irreducible maps $h_1:V_1\to U_1$ and $f_i:V_i\to V_{i+1}$ for all $i\ge 1$. By induction we define irreducible maps $g_i:U_i\to U_{i+1}$ and $h_{i+1}:V_{i+1}\to U_{i+1}$ for all $i\ge 1$ as follows. Assume that a map $h_i:V_i\to U_i$ is defined for some $i\ge 1$. Then $\left[\begin{smallmatrix} f_i\\ h_i\end{smallmatrix}\right]:V_i\to V_{i+1}\oplus U_i$ is a left minimal almost split morphism. Thus there exist irreducible maps $h_{i+1}:V_{i+1}\to U_{i+1}$ and $g_i:U_i\to U_{i+1}$ such that $[h_{i+1},-g_i]\circ\left[\begin{smallmatrix} f_i\\ h_i\end{smallmatrix}\right]=0$, so

$g_ih_i = h_{i+1}f_i$. Hence we have maps f_i, g_i, h_i such that $g_ih_i = h_{i+1}f_i$ for all $i\geq 1$. Since Γ is generalized standard, for all indecomposable modules X and Y in Γ, any nonzero morphism in $\mathrm{rad}(X,Y)$ is a linear combination of the composites of irreducible morphisms between indecomposable modules in Γ. Clearly, in order to prove the formula $[V_1,M]-[U_1,M]=\sum_{i\geq 1}\mu(M,V_i)$, we may assume that M is an indecomposable module in Γ. First, observe that the induced map $\mathrm{Hom}_A(h_1,M):\mathrm{Hom}_A(U_1,M)\to\mathrm{Hom}_A(V_1,M)$ is a monomorphism. Indeed, take a nonzero map w in $\mathrm{Hom}_A(U_1,M)$. Then there exists $r\geq 0$ such that $w\in\mathrm{rad}^r(U_1,M)\backslash\mathrm{rad}^{r+1}(U_1,M)$. Applying now the dual of Corollary 1.6 in [16] we see that $h_1:V_1\to U_1$ is of infinite right degree, and consequently $wh_1\in\mathrm{rad}^{r+1}(V_1,M)\setminus\mathrm{rad}^{r+2}(V_1,M)$. In particular, $wh_1\neq 0$ and we are done. Further, we know that any irreducible map $V_i\to W$ with W indecomposable is of the form $\alpha f_i+\varphi$, $\varphi\in\mathrm{rad}^2(V_i,V_{i+1})$, or $\alpha h_i+\psi$, $\psi\in\mathrm{rad}^2(V_i,U_i)$, for some $0\neq\alpha\in K$. Hence, if $M\not\simeq V_i$, for any $i\geq 1$, then using the equalities $g_ih_i=h_{i+1}f_i$ we see that the map $\mathrm{Hom}_A(h_1,M)$ is an isomorphism. Then

$$[V_1,M]-[U_1,M]=0=\sum_{i\geq 1}\mu(M,V_i).$$

Assume $M=V_j$ for some $j\geq 1$. Then we get

$$\mathrm{Hom}_A(V_1,M)=\mathrm{im}\,\mathrm{Hom}_A(h_1,M)+Kf_{j-1}\dots f_1$$

where, in case $j=1$, f_0 is the identity map $V_1\to V_1$. Moreover, by [9], $f_{j-1}\dots f_1$ does not belong to $\mathrm{im}\,\mathrm{Hom}_A(h_1,M)$, because $\tau^- V_i=U_{i+1}\not\simeq V_{i+2}$ for any $i\geq 1$. Therefore, we get

$$[V_1,M]-[U_1,M]=1=\mu(M,V_j)=\sum_{i\geq 1}\mu(M,V_i)$$

because the modules $V_1,V_2,\dots$ are pairwise nonisomorphic. Moreover, we have

$$\begin{aligned}\delta'_{M,N}(V_1)-\delta'_{M,N}(U_1)&=([V_1,N]-[U_1,N])-([V_1,M]-[U_1,M])\\&=\sum_{i\geq 1}\mu(N,V_i)-\sum_{i\geq 1}\mu(M,V_i)\\&=\sum_{i\geq 1}(\mu(N,V_i)-\mu(M,V_i)).\end{aligned}$$

The proof of (ii) is dual.

2.7. Let Γ be a connected component of Γ_A. For modules M and N in $\mathrm{add}(\Gamma)$ we set

$$M\leq_\Gamma N\Leftrightarrow [X,M]\leq[X,N]\text{ for all modules }X\in\mathrm{add}(\Gamma).$$

Clearly, $M\leq N$ implies $M\leq_\Gamma N$. By [25], $\leq_\Gamma$ is a partial order on the isomorphism classes of modules in $\mathrm{add}(\Gamma)$ having the same dimension vectors.

Corollary. *Let M and N be two modules in* $\operatorname{add}(\Gamma)$ *such that* $[M] = [N]$. *Then* $M \simeq N$ *if and only if* $M \leq_\Gamma N$ *and* $N \leq_\Gamma M$.

Moreover, *if* M *and* N *belong to* $\operatorname{add}(\Gamma)$ *and* $[M] = [N]$ *then the following conditions are equivalent* (*see* (2.3)):

(1) $M \leq_\Gamma N$.
(2) $\delta_{M,N}(X) \geq 0$ *for all modules* X *in* Γ.
(3) $\delta'_{M,N}(X) \geq 0$ *for all modules* X *in* Γ.

2.8. Following I. Assem and A. Skowroński ([3], [4]) a translation quiver $\mathcal{C}$ is said to be a *coil* if there exists a sequence of translation quivers $\Gamma_0, \Gamma_1, \ldots, \Gamma_m = \mathcal{C}$ such that Γ_0 is a stable tube and, for each $0 \leq i < m$, Γ_{i+1} is obtained from Γ_i by an admissible operation of type (ad 1), (ad 1*), (ad 2), (ad 2*), (ad 3) or (ad 3*). A coil $\mathcal{C}$ is said to be *proper* [4, (3.3)] if each of its vertices belongs to an oriented cycle in $\mathcal{C}$. Finally, a translation quiver Γ is said to be a *multicoil* if Γ contains a full translation subquiver Γ' such that Γ' is a disjoint union of (proper) coils and no vertex in $\Gamma \setminus \Gamma'$ belongs to an oriented cycle of Γ. For more details on coils and multicoils we refer the reader to [4].

We end this section with the following lemma.

Lemma 2.9. *Let $\mathcal{C}$ be a proper coil. Then there exist in $\mathcal{C}$ pairwise different vertices U_i, $i \geq 1$, and pairwise different vertices V_j, $j \geq 0$, such that any oriented cycle in $\mathcal{C}$ contains some vertex U_i, and one of the following conditions is satisfied*:

(i) *In $\mathcal{C}$ there are meshes*

$$\begin{tikzcd}[column sep=small] V_0 \arrow[dr] & & U_1 \\ & V_1 \arrow[ur] & \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd}[column sep=small] & U_i \arrow[dr] & \\ V_i \arrow[ur] \arrow[dr] & & U_{i+1} \\ & V_{i+1} \arrow[ur] & \end{tikzcd} \quad \textit{for } i \geq 1.$$

(ii) *In $\mathcal{C}$ there are meshes*

$$\begin{tikzcd}[column sep=small] U_1 \arrow[dr] & & V_0 \\ & V_1 \arrow[ur] & \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd}[column sep=small] & U_i \arrow[dr] & \\ U_{i+1} \arrow[ur] \arrow[dr] & & V_i \\ & V_{i+1} \arrow[ur] & \end{tikzcd} \quad \textit{for } i \geq 1.$$

Proof. First, we prove the existence of modules U_i and V_j satisfying one of the conditions (i) or (ii). If $\mathcal{C}$ is a stable tube, $U_1 \to U_2 \to U_3 \to \ldots$ an infinite sectional path in $\mathcal{C}$ with U_1 lying on the mouth, and $V_j = \tau U_{j+1}$ for all $j \geq 0$, then clearly the condition (i) is satisfied. Assume that $\mathcal{C}$ is not a stable tube. Then $\mathcal{C}$ is obtained from a coil $\mathcal{C}'$ by an admissible operation.

Assume that this operation is one of the types (ad 1), (ad 2) or (ad 3). Then $\mathcal{C}$ admits a full translation subquiver of the form

$$\begin{tikzcd}[column sep=small] X'_s \arrow[dr] & & \tau^- X'_s \arrow[dr] & & \\ & X'_{s+1} \arrow[ur] \arrow[dr] & & \tau^- X'_{s+1} \arrow[dr] & \\ & & X'_{s+2} \arrow[ur] \arrow[dr] & & \ddots \\ & & & \ddots & \end{tikzcd}$$

where $s = 0$ or X'_{s-1} is injective (see [4]). Then, for $U_i = \tau^- X'_{s+i-1}$, $i \geq 1$, and $V_j = X'_{s+j}$, $j \geq 0$, the condition (i) is satisfied. Dually, if $\mathcal{C}$ is obtained from $\mathcal{C}'$ by an admissible operation of type (ad 1^*), (ad 2^*) or (ad 3^*), then there are U_i and V_j satisfying (ii). Assume now that we have in $\mathcal{C}$ vertices U_i, $i \geq 1$, and V_j, $j \geq 0$, satisfying (i). We claim that any oriented cycle in $\mathcal{C}$ contains at least one vertex U_i. First, observe that for each vertex X in $\mathcal{C}$, there exists exactly one infinite sectional path in $\mathcal{C}$ with source X. Moreover, if two infinite sectional paths

$$X = X_1 \to X_2 \to X_3 \to \dots \quad \text{and} \quad Y = Y_1 \to Y_2 \to Y_3 \to \dots$$

have a common vertex $X_k = Z = Y_l$, then $X_{k+i} = Y_{l+i}$ for any $i \geq 1$. If this is the case, we say that X is *equivalent* to Y. This divides the set of all vertices of $\mathcal{C}$ into (disjoint) equivalence classes $A_1, A_2, \dots, A_p$. We may assume that, for each $1 \leq i \leq p$, any arrow in $\mathcal{C}$ with source in A_i has a target in A_i or A_{i+1} (where $A_{p+1} = A_1$). Observe then that each oriented cycle in $\mathcal{C}$ has at least one vertex from any set A_k, $1 \leq k \leq p$. Moreover, by the property (i), if it contains a vertex from the set A_k containing U_1, then it contains a vertex U_i, for some $i \geq 1$. This shows our claim. The proof is similar if the modules U_i and V_j satisfy the condition (ii).

3. Shape of generalized standard components

3.1. We shall recall some definitions introduced in [17]. Let (Γ, τ) be a translation quiver. A vertex x of Γ is said to be *left stable* if $\tau^n x$ is defined for all $n \geq 0$, *right stable* if $\tau^n X$ is defined for all $n \leq 0$, and *stable* if it is both left and right stable. We denote by ${}_l\Gamma$ (${}_r\Gamma$, ${}_s\Gamma$) a full translation subquiver of Γ consisting all left stable (respectively, right stable, stable) vertices of Γ. The connected components of ${}_l\Gamma$ (${}_r\Gamma$, ${}_s\Gamma$) are called *left stable* (respectively, *right stable*, *stable*) *components* of Γ.

A connected full subquiver Δ of Γ is said to be a *section* in Γ if it has the following properties:

(S1) There is no oriented cycle in Δ.
(S2) The subquiver Δ meets each τ-orbit in Γ exactly once.

(S3) Each path in Γ with end-points in Δ lies completely in Δ.

Observe that for any section Δ in Γ and an integer n, if $\tau^n\delta$ is defined for all $\delta \in \Delta$, then the quiver $\tau^n\Delta$, with the set of vertices $(\tau^n\Delta)_0 = \{\tau^n\delta : \delta \in \Delta_0\}$ and arrows $\tau^n x \to \tau^n y$ for all arrows $x \to y$ in Δ, is also a section in Γ.

Let Δ be a section in Γ. If Δ consists of left stable vertices, then we denote by $\mathcal{L}(\Delta)$ the full subquiver of Γ with the set of vertices

$$\mathcal{L}(\Delta)_0 = \bigcup_{n\ge 0} (\tau^n\Delta)_0 = \{\tau^n\delta : n \ge 0,\ \delta \in \Delta\}.$$

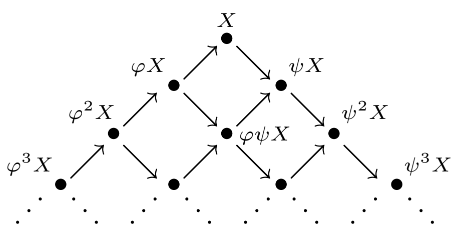
If Δ consists of right stable vertices, then we denote by $\mathcal{R}(\Delta)$ the full subquiver of Γ with the set of vertices

$$\mathcal{R}(\Delta)_0 = \bigcup_{n\le 0} (\tau^n\Delta)_0 = \{\tau^n\delta : n \le 0,\ \delta \in \Delta\}.$$

Immediately from the above definition, the arrows in $\mathcal{L}(\Delta)$ are of the form $\tau^n x \to \tau^n y$, $\tau^{n+1}y \to \tau^n x$ for any $n \ge 0$ and arrows $x \to y$ in Δ. Dually, the arrows in $\mathcal{R}(\Delta)$ are of the form $\tau^n x \to \tau^n y$, $\tau^n y \to \tau^{n-1}x$ for any $n \le 0$ and arrows $x \to y$ in Δ.

A translation subquiver $\mathcal{L}(\Delta)$ ($\mathcal{R}(\Delta)$) of Γ is called a *proper left part* in Γ (respectively, a *proper right part* in Γ) if Δ is finite and the subquiver $\mathcal{L}(\Delta)$ (respectively, $\mathcal{R}(\Delta)$) is closed under predecessors (respectively, successors) in Γ. Of course, if $\mathcal{L}(\Delta)$ ($\mathcal{R}(\Delta)$) is a proper left (respectively, right) part in Γ, then for any $N \ge 0$, $\mathcal{L}(\tau^N\Delta)$ (respectively, $\mathcal{R}(\tau^{-N}\Delta)$) is also a proper left (respectively, right) part in Γ and it is a cofinite subquiver of $\mathcal{L}(\Delta)$ (respectively, $\mathcal{R}(\Delta)$).

3.2. Let $\mathcal{T}(X)$ be a translation quiver



with the set of vertices

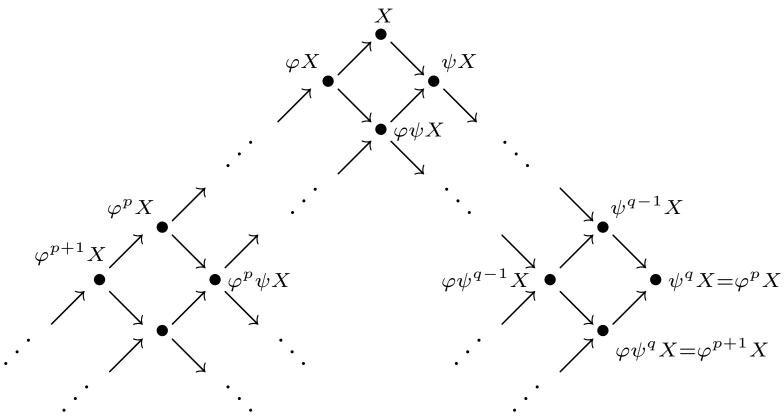
$$\mathcal{T}(X)_0 = \{\varphi^i\psi^j X : i, j \ge 0\}$$

and arrows

$$\varphi^{i+1}\psi^j X \to \varphi^i\psi^j X, \qquad \varphi^i\psi^j X \to \varphi^i\psi^{j+1} X,$$

where $\tau(\varphi^i\psi^{j+1}X) = \varphi^{i+1}\psi^j X$ for all $i, j \ge 0$. For convenience we set $\varphi^i\psi^0 X = \varphi^i X$, $\varphi^0\psi^j X = \psi^j X$, $\varphi^0\psi^0 X = X$.

For $p, q > 0$ let $\mathcal{T}(X, p, q)$ be the quiver obtained from $\mathcal{T}(X)$ by identifying the vertices $\varphi^{i+p}\psi^j X$ and $\varphi^i\psi^{j+q}X$ for all $i, j \geq 0$. Observe that $\{\varphi^i\psi^j X : i \geq 0, 0 \leq j < q\}$ is a complete set of pairwise different vertices in $\mathcal{T}(X, p, q)$.



A subquiver of a translation quiver Γ is called a *proper subtube* if it is of the form $\mathcal{T}(X, p, q)$ and for any $i, j \geq 0$ there is a mesh in Γ

$$\begin{tikzcd}
 & \varphi^i\psi^j X \arrow[dr] & \\
\varphi^{i+1}\psi^j X \arrow[ur] \arrow[dr] & & \varphi^i\psi^{j+1} X \\
 & \varphi^{i+1}\psi^{j+1} X \arrow[ur] &
\end{tikzcd}$$

We can see that if $\mathcal{T}(X, p, q)$ is a proper subtube in Γ and $Y = \varphi^k\psi^l X$, then $\mathcal{T}(Y, p, q)$ (where $\varphi^i\psi^j Y = \varphi^{i+k}\psi^{j+l}X$) is a cofinite subquiver of $\mathcal{T}(X, p, q)$ and it is a proper subtube in Γ.

LEMMA 3.3. *Let Γ be a connected component of Γ_A and Γ' be a left stable component in Γ without τ-periodic modules and consisting of finitely many τ-orbits. Then there exists a subquiver $\mathcal{C}$ of Γ' such that $\mathcal{C}$ is a proper left part or a proper subtube in Γ and the following conditions are satisfied*:

(i) *For any Y in Γ' there is an integer N such that $\tau^n Y$ belongs to $\mathcal{C}$ for all $n \geq N$.*

(ii) *For any Y in Γ' there is an integer N' such that $\tau^n Y$ does not belong to $\mathcal{C}$ for all $n \leq N'$.*

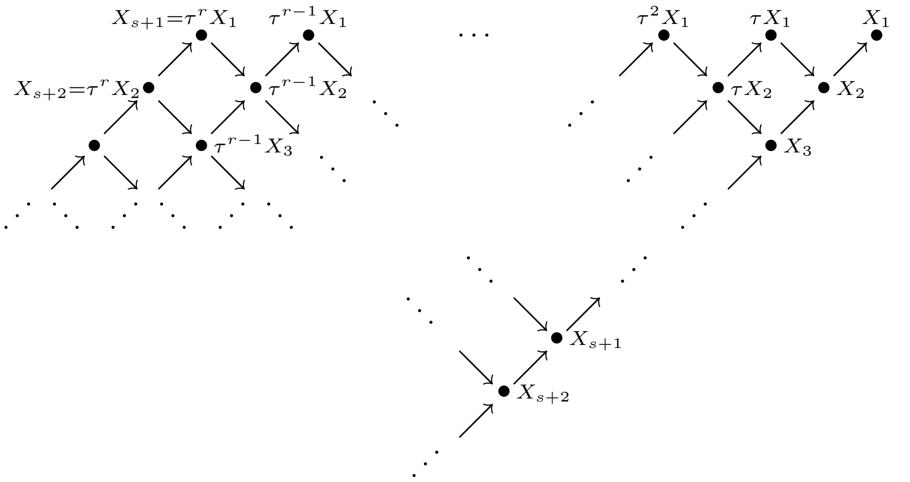
Proof. Let Y be any module in Γ'. Since $Y, \tau Y, \tau^2 Y, \ldots$ are pairwise nonisomorphic and there are at most finitely many projective modules in Γ, there is $m \geq 0$ such that for $n \geq m$ the vertex $\tau^n Y$ has no immediate projective predecessor.

Now, let Z be an immediate predecessor of $\tau^k Y$, for some $k \geq 0$, such that Z does not belong to Γ'. Then Z is not left stable and consequently there is $l \geq 0$ such that $\tau^l Z \neq 0$ is an immediate projective predecessor of $\tau^{k+l} Y$. Thus $k + l < m$, and so $k < m$. Hence, for $n \geq m$, the immediate predecessors of $\tau^n Y$ belong to Γ'.

Let $\mathcal{S}$ be any complete set of representatives of the τ-orbits in Γ'. Since $\mathcal{S}$ is a finite set, there is $N \geq 0$ such that for all $n \geq N$ and $Z \in \mathcal{S}$ the immediate predecessors of $\tau^n Z$ belong to Γ'.

Assume that Γ' does not contain an oriented cycle. By [17, Theorem 3.4] there exists a section Δ in Γ'. By the condition (S2) in the definition of a section, Δ_0 is a complete set of representatives of τ-orbits in Γ', so Δ_0 is finite. Thus there is $N \geq 0$ such that $\mathcal{L}(\tau^N \Delta)$ has no immediate predecessor which does not belong to Γ'. Since $\tau^N \Delta$ is also a section in Γ', we may assume that $N = 0$. Hence $\mathcal{L}(\Delta)$ is closed under predecessors in Γ. This implies that $\mathcal{L}(\Delta)$ is a proper left part in Γ and of course the conditions (i) and (ii) are satisfied for $\mathcal{C} = \mathcal{L}(\Delta)$.

Assume now that Γ' contains an oriented cycle. By [17, Section 2] there exists a sectional path in $\Gamma' : \ldots \to X_{k+1} \to X_k \to \ldots \to X_1$ and numbers $r > s > 0$ such that $\{X_1, \ldots, X_s\}$ is a complete set of representatives of τ-orbits in Γ' and for all $a \geq 0$, $1 \leq b < s$, $X_{as+b} = \tau^{ar} X_b$ (so $X_{c+s} = \tau^r X_c$ for all $c \geq 1$) and Γ' contains a full subquiver of the form



where the vertices $\tau^r X_i$ and X_{i+s} coincide for all $i \geq 1$. By the remark at the beginning of our proof, there exists $N \geq 0$ such that for any $n \geq N$ and $1 \leq i \leq s$ the immediate predecessors of $\tau^n X_i$ belong to Γ'. Let $Y_i = \tau^N X_i$ for all $i \geq 1$. The sectional path $\ldots \to Y_{k+1} \to Y_k \to \ldots \to Y_1$ satisfies the same conditions as the sectional path $\ldots \to X_{k+1} \to X_k \to \ldots \to X_1$, so without loss of generality we may assume that $N = 0$. Let $X = \tau^r X_1 = X_{s+1}$, $p = r - s$, $q = r$. We set, for $i, j \geq 0$, $\varphi^i \psi^j X = \tau^{r-j} X_{i+j+1}$. It is easy to see that for $i, j \geq 0$ we have $\varphi^{i+p} \psi^j X = \varphi^i \psi^{j+q} X$, and therefore $\mathcal{T}(X, p, q)$

is a subquiver of Γ'. Since $\{\varphi^i\psi^j X : \ i \geq 0, \ 0 \leq j < q\}$ is a complete set of pairwise nonisomorphic modules of $\mathcal{T}(X,p,q)$, for any $i,j \geq 0$ there are numbers $k \geq 0$, $0 \leq l < q$ such that $\varphi^i\psi^j X = \varphi^k\psi^l X = \tau^{r-l}X_{k+l+1}$. Thus $\mathcal{T}(X,p,q)_0 \subseteq \{\tau^n X_i : \ n \geq 0, \ 1 \leq i \leq s\}$. By the above remarks, $\varphi^i\psi^{j+1}X$ has no immediate predecessors which do not belong to Γ'. This implies that for any $i,j \geq 0$ we have in Γ a mesh

$$\begin{tikzcd}[column sep=small] & \varphi^i\psi^j X \arrow[dr] & \\ \varphi^{i+1}\psi^j X \arrow[ur] \arrow[dr] & & \varphi^i\psi^{j+1}X \\ & \varphi^{i+1}\psi^{j+1}X \arrow[ur] & \end{tikzcd}$$

and $\mathcal{T}(X,p,q)$ is a full subquiver of Γ'. Therefore $\mathcal{T}(X,p,q)$ is a full subquiver of Γ and moreover, $\mathcal{T}(X,p,q)$ is a proper subtube of Γ. For any $a \geq 0$, $0 \leq b < r$, $1 \leq c < s$ we have also

$$\begin{aligned}\tau^{ar+b}X_{r+c} &= \tau^b X_{r+c+as} = \tau^{r-(r-b)}X_{(c+as+b-1)+(r-b)+1} \\ &= \varphi^{(c+as+b-1)}\psi^{(r-b)}X.\end{aligned}$$

Thus for any numbers $n \geq 0$ and $r+1 \leq k \leq r+s$ the vertex $\tau^n X_k$ belongs to $\mathcal{T}(X,p,q)$. Since $\{X_{r+1},\ldots,X_{r+s}\}$ is a complete set of representatives of the τ-orbits in Γ', the condition (i) for $\mathcal{C} = \mathcal{T}(X,p,q)$ holds. The condition (2) also holds, because $\mathcal{T}(X,p,q)_0 \subseteq \{\tau^n X_i : \ n \geq 0, \ 1 \leq i \leq s\}$ and the vertices $X_1,\ldots,X_s$ belong to pairwise different τ-orbits. This finishes the proof of our lemma.

Dually we obtain the following

LEMMA 3.4. *Let Γ be a connected component of Γ_A and Γ' be a right stable component in Γ without τ-periodic modules and consisting of finitely many τ-orbits. Then there exists a subquiver $\mathcal{C}$ of Γ' such that $\mathcal{C}$ is a proper right part or a proper subtube in Γ and the following conditions are satisfied*:

(i) *For any Y in Γ' there is an integer N such that $\tau^n Y$ belongs to $\mathcal{C}$ for all $n \leq N$.*

(ii) *For any Y in Γ' there is an integer N' such that $\tau^n Y$ does not belong to $\mathcal{C}$ for all $n \geq N'$.*

LEMMA 3.5. *Let Γ be a connected component of Γ_A and Γ' be an infinite stable component in Γ containing a τ-periodic module. Then Γ' consists of τ-periodic modules and there exists a proper subtube in Γ which is a cofinite subquiver of Γ'.*

Proof. Since Γ' is a connected and locally finite quiver containing a τ-periodic module and consisting of τ-stable modules, each vertex in Γ' is τ-periodic. By the Happel–Preiser–Ringel theorem [14], Γ' is then a stable tube of rank r, for some $r \geq 1$. Thus there is a sectional path $\ldots \to X_{k+1} \to$

$X_k \to \ldots \to X_1$ in Γ such that $(\Gamma')_0 = \{\tau^j X_i : \ i \geq 1, \ 0 \leq j < r\}$. Let Y in Γ be an immediate predecessor or successor of a module from Γ', such that Y does not belong to Γ'. Since Γ is locally finite, Y belongs to the τ-orbit with projective and injective modules. There are at most finitely many vertices Y in Γ which belong to the τ-orbit with projective and injective modules. Thus there exists $N \geq 1$ such that for any $n \geq N$ and integer j, the immediate predecessors and successors of $\tau^j X_n$ in Γ belong to Γ'. Let $X = X_N$ and, for any $i, j \geq 0$, set $\varphi^i \psi^j X = \tau^{-j} X_{N+i+j}$. Then $\mathcal{T}(X, r, r)$ is a proper subtube in Γ. Since the vertices of $\Gamma' \setminus \mathcal{T}(X, r, r)$ belong to the set $\{\tau^j X_i : 1 \leq i < N + r, 0 \leq j < r\}$, $\mathcal{T}(X, r, r)$ is a cofinite subquiver of Γ'.

THEOREM 3.6. *Let Γ be a generalized standard component of Γ_A and $\mathcal{S}$ be a finite subset of vertices of Γ. Then there exists a finite family Γ_i, $i \in I$, of pairwise disjoint translation subquivers in Γ such that*

(i) *$\Gamma \setminus \bigcup_{i \in I} \Gamma_i$ is finite and contains $\mathcal{S}$.*

(ii) *Each Γ_i is a proper left part of Γ, a proper right part of Γ, or a proper subtube of Γ.*

Proof. Let $\{\Gamma''_1, \ldots, \Gamma''_s\}$ ($\{\Gamma''_{s+1}, \ldots, \Gamma''_t\}$) be a complete set of left stable (respectively, right stable) components of Γ without τ-periodic modules. Let $\{\Gamma''_{t+1}, \ldots, \Gamma''_h\}$ be a complete set of infinite stable components of Γ containing a τ-periodic module. By [22, Theorem 2.3], Γ admits at most finitely many nonperiodic τ-orbits. Thus, for any $1 \leq k \leq t$, the component Γ''_k consists of finitely many τ-orbits. For any $1 \leq k \leq h$, let Γ'_k be a subquiver of Γ''_k which satisfies the conditions of one of Lemmas 3.3, 3.4 or 3.5, respectively. We set $I = \{1, \ldots, h\}$. Since Γ contains at most finitely many stable components, all but finitely many τ-periodic modules in Γ belong to $\bigcup_{t<k\leq h} \Gamma''_k$. By Lemma 3.5, at most finitely many τ-periodic modules do not belong to $\bigcup_{t<k\leq h} \Gamma'_k \subseteq \bigcup_{k \in I} \Gamma'_k$. If X is a left stable and nonperiodic vertex in Γ, then X belongs to Γ''_k for some $1 \leq k \leq s$. By Lemma 3.3(i) there is a number N_1 such that $\tau^n X$ belongs to Γ'_k for all $n \geq N_1$. Dually, if X is a right stable and nonperiodic vertex in Γ, then X belongs to Γ''_k for some $s < k \leq t$ and there is a number N_2 such that $\tau^n X$ belongs to Γ'_k for all $n \leq N_2$. Therefore, for any nonperiodic τ-orbit, all but finitely many of its modules belong to $\bigcup_{1\leq k\leq t} \Gamma'_k \subseteq \bigcup_{k \in I} \Gamma'_k$. Since Γ contains at most finitely many nonperiodic τ-orbits, all but finitely many nonperiodic modules belong to $\bigcup_{k \in I} \Gamma'_k$. Thus $\Gamma \setminus \bigcup_{k \in I} \Gamma'_k$ consists of at most finitely many vertices.

We claim that, for any $i, j \in I$, if $i \neq j$ then $\Gamma'_i \cap \Gamma'_j$ contains at most finitely many vertices. The components Γ_k for $t<k\leq h$ are pairwise disjoint and since they contain only τ-periodic modules, they are disjoint from Γ''_l for all $1 \leq l \leq t$. Hence $\Gamma'_k \cap \Gamma'_l = \emptyset$ for all $t < k \leq h$ and $l \in I \setminus \{k\}$. Further, the left stable components $\Gamma''_1, \ldots, \Gamma''_s$ are pairwise disjoint, which implies that

$\Gamma'_1, \ldots, \Gamma'_s$ are pairwise disjoint. Dually $\Gamma'_{s+1}, \ldots, \Gamma'_t$ are pairwise disjoint.

It remains to consider the case when $1 \le k \le s$ and $s < l \le t$. If X is a common vertex of Γ'_k and Γ'_l then X is a stable module. By Lemmas 3.3(ii) and 3.4(ii), each τ-orbit contains at most finitely many vertices belonging to $\Gamma'_k \cap \Gamma'_l$. Since there are at most finitely many nonperiodic τ-orbits in Γ, $\Gamma'_k \cap \Gamma'_l$ is a finite quiver.

For any $k \in I$ let $\mathcal{S}_k$ be the set of all vertices of Γ'_k which belong to $\mathcal{S}$ or belong to Γ'_l for some $l \in I \setminus \{k\}$. By the above considerations $\mathcal{S}_k$ is a finite set. If $\Gamma'_k = \mathcal{T}(X,p,q)$ then there exists $N \ge 0$ such that $\varphi^{i+N}\psi^j X \notin \mathcal{S}_k$ for any $i, j \ge 0$. In this case, we set $\Gamma_k = \mathcal{T}(Y,p,q)$, where $Y = \varphi^N X$. Then Γ_k is a proper subtube in Γ and $\Gamma'_k \setminus \Gamma_k$ is a finite quiver. If $\Gamma'_k = \mathcal{L}(\Delta)$ is a proper left part in Γ then there exists $N \ge 0$ such that $\mathcal{L}(\tau^N\Delta)$ does not contain vertices from $\mathcal{S}_k$. In this case, we set $\Gamma_k = \mathcal{L}(\tau^N\Delta)$. Then Γ_k is a proper left part in Γ and $\Gamma'_k \setminus \Gamma_k$ is a finite quiver. We proceed dually if $\Gamma'_k = \mathcal{R}(\Delta)$ is a proper right part in Γ. Hence,

$$\Gamma \setminus \bigcup_{k\in I} \Gamma_k \subseteq \Big(\Gamma \setminus \bigcup_{k\in I} \Gamma'_k\Big) \cup \bigcup_{k\in I} (\Gamma'_k \setminus \Gamma_k)$$

is a finite quiver. Of course, the subquivers Γ_k are pairwise disjoint and do not contain any vertices from $\mathcal{S}$. This finishes our proof.

4. Dimension functions on generalized standard components

LEMMA 4.1. *Let Γ be a generalized standard component of Γ_A, $\mathcal{T}(X,p,q)$ be a proper subtube in Γ and assume that M and N are two modules in* $\operatorname{add}(\Gamma \setminus \mathcal{T}(X,p,q))$ *with $[M] = [N]$. Then*

(i) $[\psi^q X] > [X]$ (*the vector $[\psi^q X] - [X]$ is nonzero and has nonnegative coordinates*).

(ii) *There is a number n such that $\delta_{M,N}(\varphi^i\psi^j X) = n$ for all $i \ge 1$ and $j \ge 0$.*

P r o o f.(i) Since $\mathcal{T}(X,p,q)$ is a proper subtube in Γ, there are Auslander–Reiten sequences

$$0 \to \varphi^{i+1}\psi^j X \to \varphi^i\psi^j X \oplus \varphi^{i+1}\psi^{j+1} X \to \varphi^i\psi^{j+1} X \to 0$$

for all $i, j \ge 0$. Applying now [2, Corollary 2.2] we get exact sequences

$$0 \to \varphi^{rp} X \to \varphi^{rp}\psi^q X \oplus X \to \psi^q X \to 0$$

for all $r \ge 1$. Since $\varphi^{rp}X = \psi^{rq}X$, $\varphi^{rp}\psi^q X = \psi^{(r+1)q}X$, it follows that $[\psi^{(r+1)q}X] - [\psi^{rq}X] = [\psi^q X] - [X]$ for all $r \ge 1$. By induction we obtain $[\psi^{rq}X] = r([\psi^q X] - [X]) + [X]$. Thus $[\psi^q X] \ge [X]$. But the equality $[\psi^q X] = [X]$ implies that the pairwise nonisomorphic modules $X, \psi^q X, \psi^{2q} X, \ldots$ have the same dimension vectors, which is false by [22, Corollary 2.7]. Hence $[\psi^q X] > [X]$.

(ii) There exists a sectional path

$$\varphi^i\psi^{j+1}X \to \varphi^i\psi^{j+2}X \to \varphi^i\psi^{j+3}X \to \ldots$$

and the meshes

$$\begin{array}{ccccc} & & \varphi^{i-1}\psi^{j+s}X & & \\ & \nearrow & & \searrow & \\ \varphi^i\psi^{j+s}X & & & & \varphi^{i-1}\psi^{j+1+s}X \\ & \searrow & & \nearrow & \\ & & \varphi^i\psi^{j+1+s}X & & \end{array}$$

for all $s \geq 1$. By Lemma 2.6(i) we obtain

$$\begin{aligned} &\delta'_{M,N}(\varphi^i\psi^{j+1}X) - \delta'_{M,N}(\varphi^{i-1}\psi^{j+1}X) \\ &\qquad = \sum_{s\geq 1}(\mu(N,\varphi^i\psi^{j+s}X) - \mu(M,\varphi^i\psi^{j+s}X)) = 0, \end{aligned}$$

because $\mathcal{T}(X,p,q)$ does not contain any direct summands of $M \oplus N$. Thus $\delta_{M,N}(\tau(\varphi^i\psi^{j+1}X)) = \delta_{M,N}(\tau(\varphi^{i-1}\psi^{j+1}X))$, which implies

$$\delta_{M,N}(\varphi^{i+1}\psi^j X) = \delta_{M,N}(\varphi^i\psi^j X) \quad \text{ for all } i \geq 1,\ j \geq 0.$$

For any numbers $i \geq 1$ and $j \geq 0$ there exists also a sectional path

$$\ldots \to \varphi^{i+2}\psi^{j+1}X \to \varphi^{i+1}\psi^{j+1}X \to \varphi^i\psi^{j+1}X$$

and the meshes

$$\begin{array}{ccccc} & & \varphi^{i+s}\psi^{j}X & & \\ & \nearrow & & \searrow & \\ \varphi^{i+s+1}\psi^{j}X & & & & \varphi^{i+s}\psi^{j+1}X \\ & \searrow & & \nearrow & \\ & & \varphi^{i+s+1}\psi^{j+1}X & & \end{array} \qquad \text{for all } s \geq 0.$$

In a similar way, by Lemma 2.6(ii), we obtain

$$\delta_{M,N}(\varphi^i\psi^{j+1}X) = \delta_{M,N}(\varphi^i\psi^j X) \quad \text{ for all } i \geq 1,\ j \geq 0.$$

Therefore, there exists a number n such that $\delta_{M,N}(\varphi^i\psi^j X) = n$ for all $i \geq 1$ and $j \geq 0$.

Now we prove a statement which is a generalization of [25, Lemma 5.2].

PROPOSITION 4.2. *Let Γ be a generalized standard component of Γ_A and assume that M and N are two modules in* $\mathrm{add}(\Gamma)$ *with $[M] = [N]$ and $M \leq_\Gamma N$. Then $\delta_{M,N}(X) = 0$ and $\delta'_{M,N}(X) = 0$ for all but finitely many modules X in Γ and all modules X in $\Gamma_A \setminus \Gamma$.*

Proof. Let $\mathcal{S}$ be the set of all indecomposable direct summands of $M \oplus N$. Let $\{\Gamma_k\}_{k\in I}$ be the family of subquivers of Γ satisfying the conditions of Theorem 3.6 for the set $\mathcal{S}$.

Let J be the subset of I formed by all k such that $\Gamma_k = \mathcal{T}(X_k, p_k, q_k)$ is a proper subtube in Γ. We set $\mathcal{S}_k = \{X_k, \psi X_k, \dots, \psi^{q_k - 1} X_k\}$ for all $k \in J$. Then $\mathcal{S}_k$ is the set of all vertices of $\mathcal{T}(X_k, p_k, q_k) \setminus \mathcal{T}(\varphi X_k, p_k, q_k)$. Let

$$\mathcal{F}' = \Big(\Gamma \setminus \bigcup_{k\in I} \Gamma_k\Big)_0 \cup \bigcup_{k \in J} \mathcal{S}_k \quad \text{and} \quad \mathcal{F} = \{W \in \mathcal{F}' :\ \delta_{M,N}(W) > 0\}.$$

By Theorem 3.6, the sets $\mathcal{F}'$ and $\mathcal{F}$ are finite. Moreover, $\mathcal{F}$ has no injective modules, so for any $X \in \mathcal{F}$ we have an Auslander–Reiten sequence

$$\Sigma(X): \quad 0 \to X \to E(X) \to \tau^- X \to 0.$$

Let

$$N' = \Big(\bigoplus_{X\in\mathcal{F}} E(X)^{\delta_{M,N}(X)}\Big) \oplus N, \qquad M' = \Big(\bigoplus_{X\in\mathcal{F}} (X \oplus \tau^- X)^{\delta_{M,N}(X)}\Big) \oplus M.$$

Then the modules M', N' belong to $\operatorname{add}(\Gamma)$, $[M'] = [N']$ and $\delta_{M',N'} = \delta_{M,N} - \sum_{X\in\mathcal{F}} (\delta_{M,N}(X) \cdot \delta_{\Sigma(X)})$. By Lemma 2.5(i)

$$\delta_{M',N'}(Y) = \delta_{M,N}(Y) - \sum_{X\in\mathcal{F}} (\delta_{M,N}(X) \cdot \mu(Y, X)).$$

Hence, $\delta_{M'N'}(Y) = 0$ for $Y \in \mathcal{F}$ and $\delta_{M',N'}(Y) = \delta_{M,N}(Y)$ for the remaining $Y \in \Gamma_A$. Consequently, we obtain $M' \le_\Gamma N'$. By definition of $\mathcal{F}$ we have $\delta_{M',N'}(X) = 0$ for all $X \in \mathcal{F}'$. Observe that if $\delta_{M,N}(X) = 0$ then $\delta_{M',N'}(X) = 0$ for any X in Γ_A. Let k be any element in $I \setminus J$, X be any module in Γ_k and Y be any indecomposable direct summand of $M \oplus N$. Of course Y does not belong to Γ_k. Assume that Γ_k is a proper left part in Γ. Then Y is not a predecessor of X in Γ. Since Γ is generalized standard, we have $[Y, X] = 0$, and hence

$$\delta_{M,N}(X) = [N, X] - [M, X] = 0 - 0 = 0.$$

Assume now that Γ_k is a proper right part in Γ. Then $\tau^- X$ belongs to Γ_k, and Y is not a successor of $\tau^- X$ in Γ. Since Γ is generalized standard, $[\tau^- X, Y] = 0$, and we get

$$\delta_{M,N}(X) = \delta'_{M,N}(\tau^- X) = [\tau^- X, N] - [\tau^- X, M] = 0 - 0 = 0.$$

Hence $\delta_{M,N}(X) = 0$, which implies that $\delta_{M',N'}(X) = 0$ for any X in Γ_k, where $k \in I \setminus J$.

Let k belong to J and X be any module in $\mathcal{T}(\varphi X_k, p_k, q_k)$. Then $X \notin \mathcal{F}$ and $\delta_{M',N'}(X) = \delta_{M,N}(X)$. By Lemma 4.1(ii) there exist numbers n_k, for all $k \in J$, such that $\delta_{M',N'}(X) = \delta_{M,N}(X) = n_k$ for all $X \in \mathcal{T}(\varphi X_k, p_k, q_k)$. Since $M \le_\Gamma N$, we get $n_k \ge 0$. Further, by the above considerations, $\delta_{M',N'}(X) = 0$ for any X in $(\Gamma \setminus \bigcup_{k\in J} \mathcal{T}(\varphi X_k, p_k, q_k))_0 = \mathcal{F}' \cup \bigcup_{k \in I\setminus J} (\Gamma_k)_0$. We claim that $n_k = 0$ for any $k \in J$. Observe that for any Y in Γ, if $\delta_{M',N'}(Y) > 0$, then the mesh starting at Y is contained in a proper subtube Γ_k, for some $k \in J$. Let X be a module in $\Gamma \setminus \bigcup_{k\in J} \Gamma_k$. Then the mesh starting at X is not contained in Γ_k for any $k \in J$. The same is true for the meshes starting at Y, where Y is an immediate predecessor of X or $Y = \tau X$, if X is not projective. Thus $\delta_{M',N'}(Y) = 0$ if $Y = X$ or Y is an immediate predecessor of X or $Y = \tau X$. By Lemma 2.5(iii) and (iv), $\mu(N', X) - \mu(M', X) = 0$. Assume now that X belongs to Γ_k for some $k \in J$, so $X = \varphi^i \psi^j X_k$ for some $i, j \geq 0$. We shall compute $\ell_X = \mu(N', X) - \mu(M', X)$. Put $p = p_k$ and $q = q_k$. Assume that $j \geq 1$. Since Γ_k is a proper subtube in Γ, there is an Auslander–Reiten sequence

$$0 \to \varphi^{i+1}\psi^{j-1}X_k \to \varphi^i\psi^{j-1}X_k \oplus \varphi^{i+1}\psi^j X_k \to \varphi^i\psi^j X_k \to 0.$$

By Lemma 2.5(iii) we get

$$\begin{aligned}\ell_X &= \delta_{M',N'}(\varphi^{i+1}\psi^{j-1}X_k) - \delta_{M',N'}(\varphi^{i+1}\psi^j X_k)\\ &\quad - \delta_{M',N'}(\varphi^i\psi^{j-1}X_k) + \delta_{M',N'}(\varphi^i\psi^j X_k)\\ &= n_k - n_k - \delta_{M',N'}(\varphi^i\psi^{j-1}X_k) + \delta_{M',N'}(\varphi^i\psi^j X_k)\\ &= \delta_{M',N'}(\varphi^i\psi^j X_k) - \delta_{M',N'}(\varphi^i\psi^{j-1}X_k).\end{aligned}$$

If, moreover, $i \geq 1$, which is equivalent to $X \in \mathcal{T}(\varphi\psi X_k, p, q)$, then $\ell_X = n_k - n_k = 0$.

Assume now that $X \in \mathcal{T}(X_k, p, q) \setminus \mathcal{T}(\varphi\psi X_k, p, q)$, so

$$X \in \{X_k, \varphi X_k, \varphi^2 X_k, \ldots, \varphi^p X_k = \psi^q X_k, \psi^{q-1}X_k, \ldots \psi X_k\}.$$

If $X = \psi^j X_k$ for $1 \leq j \leq q-1$, then

$$\ell_X = \delta_{M',N'}(\psi^j X_k) - \delta_{M',N'}(\psi^{j-1}X_k) = 0 - 0 = 0,$$

because the modules $\psi^j X_k$ and $\psi^{j-1}X_k$ belong to $\mathcal{S}_k \subseteq \mathcal{F}'$. If $X = \psi^q X_k$, then

$$\ell_X = \delta_{M',N'}(\psi^q X_k) - \delta_{M',N'}(\psi^{q-1}X_k) = n_k - 0 = n_k.$$

Let now $X = \varphi^i X_k$ for some $0 \leq i \leq p-1$. We claim that $\ell_X = \delta_{M',N'}(\varphi^i X_k) - \delta_{M',N'}(\varphi^{i+1}X_k)$. Assume that X is projective. Then $\mathrm{rad}(X) = \varphi^{i+1}X_k \oplus E$ and no indecomposable direct summand of E belongs to Γ_k. Then any indecomposable direct summand Y of E is injective or the mesh starting at Y is not contained in Γ_ℓ, for any $\ell \in J$. Thus $\delta_{M',N'}(Y) = 0$, and consequently $\delta_{M',N'}(E) = 0$. By Lemma 2.5(iv) we have

$$\begin{aligned}\ell_X &= \delta_{M',N'}(\varphi^i X_k) - \delta_{M',N'}(\varphi^{i+1}X_k) - \delta_{M',N'}(E)\\ &= \delta_{M',N'}(\varphi^i X_k) - \delta_{M',N'}(\varphi^{i+1}X_k).\end{aligned}$$

Assume that X is not projective. Then there is an Auslander–Reiten sequence

$$0 \to \tau(\varphi^i X_k) \to E \oplus \varphi^{i+1}X_k \to \varphi^i X_k \to 0.$$

The indecomposable direct summands of E and $\tau(\varphi^i X_k)$ do not belong to the quiver $\mathcal{T}(\varphi X_k, p_k, q_k)$. As above, we get $\delta_{M',N'}(E) = 0$ and

$\delta_{M',N'}(\tau(\varphi^i X_k)) = 0$. By Lemma 2.5(iii) we obtain

$$\begin{aligned}\ell_X &= \delta_{M',N'}(\varphi^i X_k) - \delta_{M',N'}(\varphi^{i+1} X_k) - \delta_{M',N'}(E) + \delta_{M',N'}(\tau(\varphi^i X_k))\\ &= \delta_{M',N'}(\varphi^i X_k) - \delta_{M',N'}(\varphi^{i+1} X_k).\end{aligned}$$

If $X = \varphi^i X_k$, where $1 \le i \le p-1$, then $\ell_X = n_k - n_k = 0$. But, if $X = X_k$, then $\ell_X = 0 - n_k = -n_k$, because $X_k \in \mathcal{F}'$. Thus

$$M' = \bigoplus_{k\in J} (X_k)^{n_k} \oplus W \quad \text{and} \quad N' = \bigoplus_{k\in J} (\psi^{q_k} X_k)^{n_k} \oplus W,$$

where W is the greatest common direct summand of M' and N'. By Lemma 4.1(i), $[\psi^{q_k} X_k] - [X_k] > 0$. Since $0 = [N'] - [M'] = \sum_{k\in J} n_k([\psi^{q_k} X_k] - [X_k])$ and $n_k \ge 0$, we have $n_k = 0$ for all $k \in J$. Thus $\delta_{M',N'}(X) = 0$ for all X in Γ. This implies that $M' \le_\Gamma N'$ and $N' \le_\Gamma M'$. Consequently, $M' = N'$. Hence $\delta_{M,N}(X) = \delta_{M',N'}(X) = 0$ for all indecomposable modules X which do not belong to $\mathcal{F}'$. Since $\mathcal{F}' \subseteq \Gamma_0$, $\delta'_{M,N}(X) = \delta_{M,N}(\tau X)$ and $\mathcal{F}$ is a finite set, we have $\delta_{M,N}(X) = 0$ and $\delta'_{M,N}(X) = 0$ for all but finitely many X in Γ and all X in $\Gamma_A \setminus \Gamma$. This finishes our proof.

The following proposition shows the convexity of the degenerations of modules from the additive categories of generalized standard components.

PROPOSITION 4.3. *Let A be an algebra, and Γ a generalized standard component in Γ_A. Assume that M, N, V are A-modules such that $[M] = [V] = [N]$, $M \le_{\deg} V \le_{\deg} N$ and N belongs to* add(Γ). *Then V belongs to* add(Γ).

Proof. By Proposition 4.2, $\delta_{M,N}(X) = 0$ for all X in $\Gamma_A \setminus \Gamma$. This implies $\delta_{M,V}(X) = 0$ for all X in $\Gamma_A \setminus \Gamma$. Applying Lemma 2.5, we get $\mu(V, X) = \mu(V, X) - \mu(M, X) = 0$ for all X in $\Gamma_A \setminus \Gamma$. Hence V belongs to add(Γ).

5. Proof of Theorem 1. Let M, N be modules in add(Γ) with $[M] = [N]$. Clearly, $M \le_{\mathrm{virt}} N$ implies $M \le_\Gamma N$. Assume that $\delta_{M,N}(X) \ge 0$ for all modules X in Γ. By Proposition 4.2, $\delta_{M,N}(X) = 0$ for all X in $\Gamma_A \setminus \Gamma$. This implies $M \le N$. We shall prove that $M \le_{\mathrm{virt}} N$ applying arguments similar to those in [18, Section 2]. We set $\mathcal{F} = \{X \in \Gamma_A : \delta_{M,N}(X) > 0\}$. By Proposition 4.2, $\mathcal{F}$ is a finite subset of Γ without injective modules. There exist Auslander–Reiten sequences

$$\Sigma(X): \quad 0 \to X \to E(X) \to \tau^- X \to 0$$

for all $X \in \mathcal{F}$. Thus we have the exact sequence

$$\Sigma: \quad 0 \to U \to W \oplus M \to V \oplus M \to 0$$

where $U = \bigoplus_{X\in\mathcal{F}} X^{\delta(X)}$, $W = \bigoplus_{X\in\mathcal{F}} E(X)^{\delta(X)}$, $V = \bigoplus_{X\in\mathcal{F}} (\tau^- X)^{\delta(X)}$, and $\delta = \delta_{M,N}$. Therefore $W \oplus M \le_{\deg} U \oplus V \oplus M$. Take any X in Γ_A.

Then $[M \oplus U \oplus V] = [N \oplus W]$ and

$$\delta_{M \oplus U \oplus V, N \oplus W}(X) = \delta_{M,N}(X) - \sum_{Y \in \mathcal{F}} \delta_{M,N}(Y) \cdot \delta_{\Sigma(Y)}(X).$$

By Lemma 2.5(i), we have $\delta_{\Sigma(Y)}(X) = \mu(X,Y)$. Hence $\delta_{M \oplus U \oplus V, N \oplus W}(X) = \delta_{M,N}(X) - \delta_{M,N}(X) \cdot 1 = 0$ for any X in $\mathcal{F}$ and $\delta_{M \oplus U \oplus V, N \oplus W}(X) = \delta_{M,N}(X) = 0$ for any indecomposable module $X \notin \mathcal{F}$. Of course, the equalities $\delta_{M \oplus U \oplus V, N \oplus W}(X) = 0$, for any X in Γ_A, imply that $M \oplus U \oplus V = N \oplus W$. Thus $M \oplus W \leq_{\deg} N \oplus W$. Consequently, by definition of the relation $\leq_{\mathrm{virt}}$, we infer that $M \leq_{\mathrm{virt}} N$. This finishes the proof.

6. Proof of Theorem 2. In the proof of Theorem 2 we shall use the following fact.

LEMMA 6.1. *Let Γ be a generalized standard component of Γ_A. Assume that for all modules M, N in* $\mathrm{add}(\Gamma)$ *with $[M] = [N]$ and $M <_\Gamma N$, there exist modules M', N' in* $\mathrm{add}(\Gamma)$ *such that $[M'] = [N']$, $M' <_{\deg} N'$, $\delta_{M',N'}(X) \leq \delta_{M,N}(X)$ for any X in Γ, and one of the following conditions holds*:

(i) $N' = N_1 \oplus N_2 \oplus N_3$, *where $\delta_{M,N}(N_1) = 0$, $\delta'_{M,N}(N_2) = 0$ and N_3 is a direct summand of N.*

(ii) $M' = M_1 \oplus M_2 \oplus M_3$, *where $\delta_{M,N}(M_1) = 0$, $\delta'_{M,N}(M_2) = 0$ and M_3 is a direct summand of M.*

Then the partial orders $\leq_\Gamma$ and $\leq_{\deg}$ coincide on the category of modules of a fixed dimension vector in $\mathrm{add}(\Gamma)$.

Proof. Clearly, $M \leq_{\deg} N$ implies $M \leq_\Gamma N$. In our proof of the reverse implication, we proceed by induction on $\sum_{X \in \Gamma_0} \delta_{M,N}(X) \geq 0$. Observe that by Proposition 4.2, this sum is finite. If $\sum_{X \in \Gamma_0} \delta_{M,N}(X) = 0$ then $\delta_{M,N}(X) = 0$ for all $X \in \Gamma_0$, and so $M \leq_\Gamma N$ and $N \leq_\Gamma M$. Hence, $M \simeq N$, and this implies $M \leq_{\deg} N$.

Assume that $\sum_{X \in \Gamma_0} \delta_{M,N}(X) > 0$. Then $M <_\Gamma N$, and by our assumptions, there exist modules $M', N' \in \mathrm{add}(\Gamma)$ such that $[M'] = [N']$, $M' <_{\deg} N'$ and $\delta_{M',N'}(X) \leq \delta_{M,N}(X)$ for any X in Γ. Assume that $N' = N_1 \oplus N_2 \oplus N_3$, where $\delta_{M,N}(N_1) = \delta'_{M,N}(N_2) = 0$ and $N = N_3 \oplus N_4$ for some module N_4 in $\mathrm{add}(\Gamma)$. Observe that $[M \oplus N_1 \oplus N_2] = [N_4 \oplus M']$ and $\delta_{M \oplus N_1 \oplus N_2, N_4 \oplus M'} = \delta_{M,N} - \delta_{M',N'}$. Thus $M \oplus N_1 \oplus N_2 \leq_\Gamma N_4 \oplus M'$. Moreover,

$$\sum_{X \in \Gamma_0} \delta_{M \oplus N_1 \oplus N_2, N_4 \oplus M'}(X) = \sum_{X \in \Gamma_0} (\delta_{M,N}(X) - \delta_{M',N'}(X)) < \sum_{X \in \Gamma_0} \delta_{M,N}(X),$$

because otherwise $\delta_{M',N'}(X) = 0$ for any X in Γ, which implies $M' \leq_\Gamma N'$, $N' \leq_\Gamma M'$, and consequently $M' \simeq N'$, a contradiction with $M' <_{\deg} N'$.

Therefore, by our inductive assumption, $M \oplus N_1 \oplus N_2 \leq_{\deg} N_4 \oplus M'$. Since $N_4 \oplus M' \leq_{\deg} N_4 \oplus N'$, we have $M \oplus N_1 \oplus N_2 \leq_{\deg} N_4 \oplus N' = N \oplus N_1 \oplus N_2$. The equality $\delta'_{M,N}(N_2) = 0$ implies $[N_2, M \oplus N_1] = [N_2, N \oplus N_1]$. Then by the cancellation theorem for degenerations proved in [10, Corollary 2.5], we have $M \oplus N_1 \leq_{\deg} N \oplus N_1$. The equality $\delta_{M,N}(N_1) = 0$ implies $[M, N_1] = [N, N_1]$. Applying now the dual cancellation theorem for degenerations, we obtain the required relation $M \leq_{\deg} N$.

In a similar way we get $M \leq_{\deg} N$ in the case when condition (ii) holds.

6.2. Proof of Theorem 2. Assume that $M <_{\Gamma} N$ for some modules M, N in $\mathrm{add}(\Gamma)$ with $[M] = [N]$. It suffices to find an exact sequence $0 \to U \to M' \to V \to 0$ such that the modules M' and $N' = U \oplus V$ satisfy the conditions of Lemma 6.1. By Proposition 4.2 the set

$$\mathcal{F} = \{X \in \Gamma_0 : \delta_{M,N}(X) > 0\}$$

is finite, nonempty and without injective modules.

Assume first that there is no cycle $X_0 \to X_1 \to \ldots \to X_c = X_0$ in Γ consisting of modules from $\mathcal{F}$. Then there is a module X in $\mathcal{F}$ such that any immediate predecessor Y of X in Γ does not belong to $\mathcal{F}$. Since X is not injective, there exists an Auslander–Reiten sequence

$$\Sigma(X) : \quad 0 \to X \to E(X) \to \tau^{-}X \to 0.$$

We claim that $\Sigma(X)$ is the required sequence. By Lemma 2.5(i),

$$\delta_{E(X), X\oplus\tau^{-}X}(Y) = \delta_{\Sigma(X)}(Y) = \mu(Y, X) \leq \delta_{M,N}(Y).$$

Let Z be any indecomposable direct summand of $E(X)$. Then $\delta'_{M,N}(Z) = 0$, since Z is either projective or τZ is an immediate predecessor of X and $\delta'_{M,N}(Z) = \delta_{M,N}(\tau Z) = 0$. We set $M_1 = M_3 = 0$ and $M_2 = E(X)$. We see that the condition (ii) in Lemma 6.1 is satisfied, and we are done.

Assume now that there is a cycle $X_0 \to X_1 \to \ldots \to X_c = X_0$, where $X_0, X_1, \ldots, X_{c-1}$ are modules in $\mathcal{F}$. By definition of a multicoil there is a full translation subquiver $\mathcal{C}$ of Γ such that $\mathcal{C}$ is a proper coil and the cycle $X_0 \to X_1 \to \ldots \to X_c = X_0$ is contained in $\mathcal{C}$. Without loss of generality we may assume by Lemma 2.9 that there exist in $\mathcal{C}$ pairwise nonisomorphic modules U_i, for all $i \geq 1$, and modules V_i, for all $i \geq 0$, such that $U_a = X_b$ for some $a \geq 1$ and $0 \leq b < c$ and there are Auslander–Reiten sequences

$$\begin{aligned} \Sigma(U_1) : &\quad 0 \to U_1 \to V_1 \to V_0 \to 0, \\ \Sigma(U_{i+1}) : &\quad 0 \to U_{i+1} \to U_i \oplus V_{i+1} \to V_i \to 0, \qquad \text{for all } i \geq 1. \end{aligned}$$

Since $U_a = X_b$, we have $\delta_{M,N}(U_a) > 0$, and so the set $I = \{i \geq 1 : \delta_{M,N}(U_i) > 0\}$ is nonempty. Since $U_1, U_2, \ldots$ are pairwise nonisomorphic, the set I is finite, by Proposition 4.2. Thus there are numbers $0 \leq l < k$ such that $\delta_{M,N}(U_i) > 0$ for any $l < i \leq k$, $\delta_{M,N}(U_{k+1}) = 0$ and $\delta_{M,N}(U_l) = 0$

provided $l > 0$. We set $U_0 = 0$. Then we have Auslander–Reiten sequences

$$\Sigma(U_i): \quad 0 \to U_i \to U_{i-1} \oplus V_i \to V_{i-1} \to 0 \quad \text{for any } l < i \le k.$$

Applying now [2, Corollary (2.2)] we get an exact sequence

$$\Sigma: \quad 0 \to U_k \to U_l \oplus V_k \to V_l \to 0.$$

We claim that Σ is the required sequence. It is easy to see that

$$\delta_{U_l \oplus V_k, U_k \oplus V_l} = \delta_\Sigma = \sum_{i=l+1}^{k} \delta_{\Sigma(U_i)}.$$

By Lemma 2.5(i) we have $\delta_\Sigma(X) = \sum_{i=l+1}^{k} \mu(X, U_i)$. Thus $\delta_\Sigma(U_i) = 1$ for all $l < i \le k$ and $\delta_\Sigma(X) = 0$ for the remaining indecomposable modules X. Therefore $\delta_{U_l \oplus V_k, U_k \oplus V_l} = \delta_\Sigma \le \delta_{M,N}$.

Observe that $\delta'_{M,N}(V_k) = \delta_{M,N}(\tau V_k) = \delta_{M,N}(U_{k+1}) = 0$. Hence, condition (ii) in Lemma 6.1 is satisfied, if we set $M_1 = U_l$, $M_2 = V_k$, $M_3 = 0$. This finishes our proof.

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