COLLOQUIUM MATHEMATICUM

VOL. 72 1997 NO. 2

SIDON SETS AND RIESZ SETS FOR SOME MEASURE ALGEBRAS ON THE DISK

BY

OLIVIER GEBUHRER (STRASBOURG) AND
ALAN L. SCHWARTZ (ST. LOUIS, MISSOURI)

Sidon sets for the disk polynomial measure algebra (the continuous disk polynomial hypergroup) are described completely in terms of classical Sidon sets for the circle; an analogue of the F. and M. Riesz theorem is also proved.

1. Introduction. Many of the ideas and methods of classical Fourier analysis on the circle and the real line have been interesting and fruitful when studied in other contexts where at least some of the useful structures from the classical cases persist. Two sorts of examples are when the circle or line is replaced by a more general group, or when the expansion of a function in terms of the exponential functions $\{e^{ik\theta}\}_{k=-\infty}^{\infty}$ is replaced by expansions in terms of some other system of functions.

A particular case of the latter is the subject of this article. Here we will direct our attention to functions and measures on the unit disk. The role that is classically played by the exponential functions is played here by a system of complex-valued polynomials called the *disk polynomials* which are orthogonal on the unit disk. There is actually a continuum of such systems with a distinct one for each non-negative value of a parameter α. When α is an integer n, the disk polynomials are essentially the spherical functions of the Gelfand pair $(U(n+2), U(n+1))$. In this case the geometric and algebraic structure of the groups leads in a natural way to a pair of dual convolution measure algebras (one for measures on the unit disk, and the other for bivariate sequences). In fact, these measure algebras can be interpolated to obtain a distinct pair of dual convolution measure algebras for each non-negative α, even though the algebraic and geometric structures vanish for

1991 *Mathematics Subject Classification*: Primary 33C50, 43A62; Secondary 42C10.

Key words and phrases: Sidon sets, Riesz sets, disk polynomials, hypergroups, bivariate polynomials.

Both authors were supported by the NATO Collaborative Research Grants Programme, Grant No. CRG 921244, and the second author was also supported by the National Science Foundation, Grant No. DMS 9404316.

non-integral α. This has proved to be an interesting system in which to do both harmonic analysis and probability: see, for example, [AT74, Kan76, Kan85, BG91, BG92, CS92, CS95, HK93].

This article is devoted to two issues. First we define Sidon sets in this context, and we are able to describe them entirely in terms of the Sidon sets for the circle (Theorem 1). Second, we prove a version of the F. and M. Riesz theorem (Theorem 3) which describes a new class of Riesz sets that includes those discovered earlier by Kanjin [Kan76, Thm. 7].

The question naturally arises of generalizing our results as far as possible. In particular Sidon sets can be defined on any compact commutative hypergroup, but there are two striking contrasts between the situation in a group and in a hypergroup. The first is that in general the Plancherel measure for a compact commutative hypergroup is not proportional to the counting measure, and the second is that the dual of a compact commutative hypergroup is not necessarily a hypergroup. These observations do not raise difficulties in this article because the dual object of the disk polynomial hypergroup is, in fact, a hypergroup. This observation plays a crucial role in Lemmas 3.1 and 4.1. In a forthcoming article, we will show that generalizations of our Sidon set results can be obtained without requiring that the dual object be a hypergroup or even have a convolution structure of any kind.

The rest of the article is organized as follows: Section 2 contains the definitions and notations required to describe the measure algebras $D(\alpha)$ on the disk, Section 3 contains some properties of these measure algebras, Section 4 contains the discussion of Sidon sets, and Section 5 is devoted to Riesz sets.

2. Some measure algebras on the disk

2.1. *Definitions and notations.* We employ the usual notations of $\mathbb{R}$ and $\mathbb{C}$ for the real and complex numbers. We also require the closed and open unit disks and circle in $\mathbb{C}$ given by $\mathbb{D} = \{z \in \mathbb{C} : |z| \leq 1\}$, $\mathbb{D}^0 = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Let $\mathbb{N} = \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{N}_0^2 = \mathbb{N}_0 \times \mathbb{N}_0$. If $\mathbf{n} = (n_1, n_2) \in \mathbb{N}_0^2$ we write $\|\mathbf{n}\| = n_1 + n_2$, and $n_1 \wedge n_2 = \min(n_1, n_2)$.

If X is a locally compact Hausdorff space, $C(X)$ denotes the complex-valued continuous functions on X and $C_0(X)$ the members of $C(X)$ which vanish at ∞; both spaces are given the uniform norm $\|\cdot\|_\infty$. $M(X)$ is the Banach space of complex-valued Borel measures on X endowed with the total variation norm $\|\cdot\|$, and $M_1(X)$ denotes the probability measures on X (non-negative members of $M(X)$ with unit total variation).

Let α be a fixed non-negative real number, and let $m_\alpha \in M_1(\mathbb{D})$ be given by

$$dm_\alpha(x,y) = \frac{\alpha+1}{\pi}(1-x^2-y^2)^\alpha\,dx\,dy,$$

and let $L^p = L^p(\mathbb{D}, m_\alpha)$ with the usual norm $\|\cdot\|_p$.

The *disk polynomials* are a family of polynomials in two variables obtained by orthogonalizing 1, z, $\overline{z}$, z^2, $z\overline{z}$, $\overline{z}^2, \ldots$ with respect to m_α where α is a fixed non-negative real number. They are given explicitly in terms of the Jacobi polynomials by

$$R^{(\alpha)}_{\mathbf{n}}(z) = R^{(\alpha)}_{\mathbf{n}}(x,y) = R^{(\alpha)}_{\mathbf{n}}(re^{i\theta}) = e^{i(n_1-n_2)\theta} r^{|n_1-n_2|} R^{(\alpha,|n_1-n_2|)}_{n_1\wedge n_2}(2r^2-1),$$

where $z = x+iy = re^{i\theta}$, $\mathbf{n} = (n_1,n_2)$, and $R^{(\alpha,\beta)}_n(x) = P^{(\alpha,\beta)}_n(x)/P^{(\alpha,\beta)}_n(1)$. $R^{(\alpha)}_{\mathbf{n}}$ has degree $\|\mathbf{n}\|$, and $|R^{(\alpha)}_{\mathbf{n}}(z)| \le 1$ for all $x \in \mathbb{D}$. We define (cf. [Sze67, §(4.3.3)])

$$h^{(\alpha)}_{\mathbf{n}} = \Big(\int_{\mathbb{D}} \big|R^{(\alpha)}_{\mathbf{n}}(z)\big|^2 dm_\alpha(z)\Big)^{-1} = \frac{n_1+n_2+\alpha+1}{\alpha+1}\binom{n_1+\alpha}{n_1}\binom{n_2+\alpha}{n_2}.$$

Note that if $\mathbf{m} = (n_2,n_1)$, then

$$h^{(\alpha)}_{\mathbf{n}} = h^{(\alpha)}_{\mathbf{m}} \quad\text{and}\quad R^{(\alpha)}_{\mathbf{m}}(z) = \overline{R^{(\alpha)}_{\mathbf{n}}(z)} = R^{(\alpha)}_{\mathbf{n}}(\overline{z}). \tag{2.1}$$

We also introduce the following Banach spaces of complex-valued functions on $\mathbb{N}_0^2$:

$$\ell^p = \Big\{\phi : \|\phi\|_p = \Big(\sum_{\mathbf{n}\in\mathbb{N}_0^2} |\phi(\mathbf{n})|^p h^{(\alpha)}_{\mathbf{n}}\Big)^{1/p} < \infty\Big\} \qquad (1 \le p < \infty),$$

$$\ell^\infty = \{\phi : \|\phi\|_\infty = \sup_{\mathbf{n}\in\mathbb{N}_0^2} |\phi(\mathbf{n})| < \infty\}.$$

For $\alpha > 0$ and $\mu \in M(\mathbb{D})$ define the Fourier–Stieltjes coefficients of μ by

$$\widehat{\mu}(\mathbf{n}) = \widehat{\mu}^{(\alpha)}(\mathbf{n}) = \int_{\mathbb{D}} R^{(\alpha)}_{\mathbf{n}}(\overline{z})\,d\mu(z).$$

If $f \in L^1$, define

$$\widehat{f}(\mathbf{n}) = \widehat{f}^{(\alpha)}(\mathbf{n}) = \int_{\mathbb{D}} f(z) R^{(\alpha)}_{\mathbf{n}}(\overline{z})\,dm_\alpha(z).$$

Thus, if f is a polynomial in two variables, then $\operatorname{supp}(\widehat{f})$ is finite and

$$f(z) = \sum_{\mathbf{n}\in\mathbb{N}_0^2} \widehat{f}(\mathbf{n}) h^{(\alpha)}_{\mathbf{n}} R^{(\alpha)}_{\mathbf{n}}(z).$$

Indeed, a Plancherel formula holds: if $f \in L^2$, then $\|f\|_2 = \|\widehat{f}\|_2$, or more explicitly

$$\int_{\mathbb{D}} |f|^2\, dm_a = \sum_{\mathbf{n}\in\mathbb{N}_0^2} |\widehat{f}(\mathbf{n})|^2 h_{\mathbf{n}}^{(\alpha)}. \tag{2.2}$$

Consequently, $\widehat{f} \in C_0(\mathbb{N}_0^2)$ since $h_{\mathbf{n}}^{(\alpha)} \geq 1$ for all $\mathbf{n}$. Since L^2 is dense in L^1,

$$\lim_{\|\mathbf{n}\|\to\infty} \widehat{f}(\mathbf{n}) = 0 \qquad (f \in L^1). \tag{2.3}$$

2.2. *The disk measure algebras* $D(\alpha)$. The disk polynomials interest us because they satisfy two important kinds of formulas. The first type are product formulas [Koo72]:

$$R_{\mathbf{n}}^{(\alpha)}(z)R_{\mathbf{n}}^{(\alpha)}(\zeta) = \frac{\alpha}{\alpha+1}\int_{\mathbb{D}} R_{\mathbf{n}}^{(\alpha)}(z\zeta + (1-|z|^2)^{1/2}(1-|\zeta|^2)^{1/2}\xi)\,\frac{dm_\alpha(\xi)}{1-|\xi|^2} \qquad (\alpha > 0),$$

$$R_{\mathbf{n}}^{(0)}(z)R_{\mathbf{n}}^{(0)}(\zeta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} R_{\mathbf{n}}^{(0)}(z\zeta + (1-|z|^2)^{1/2}(1-|\zeta|^2)^{1/2}e^{i\theta})\, d\theta$$

(the second formula is obtained as a limit when $\alpha \to 0$), and the second type is the linearization formula [Koo78, Cor. 5.2]:

$$R_{\mathbf{m}}^{(\alpha)}(z)R_{\mathbf{n}}^{(\alpha)}(z) = \sum_{\mathbf{k}\in\mathbb{N}_0^2} C_\alpha(\mathbf{k},\mathbf{n},\mathbf{m}) h_{\mathbf{k}}^{(\alpha)} R_{\mathbf{k}}^{(\alpha)}(z) \qquad (\alpha \geq 0) \tag{2.4}$$

where

$$C_\alpha(\mathbf{k},\mathbf{n},\mathbf{m}) = \int_{\mathbb{D}} R_{\mathbf{k}}^{(\alpha)}(\overline{z})R_{\mathbf{m}}^{(\alpha)}(z)R_{\mathbf{n}}^{(\alpha)}(z)\, dm_\alpha(z) \geq 0, \tag{2.5}$$

so from orthogonality, it follows that

$$C_\alpha(\mathbf{k},\mathbf{n},\mathbf{m}) = 0 \quad \text{unless} \quad |\,\|\mathbf{m}\| - \|\mathbf{n}\|\,| \leq \|\mathbf{k}\| \leq \|\mathbf{m}\| + \|\mathbf{n}\|. \tag{2.6}$$

Now setting $z = 1$ in (2.4) and then using (2.1) we obtain

$$\begin{aligned} \sum_{\mathbf{k}\in\mathbb{N}_0^2} C_\alpha(\mathbf{k},\mathbf{m},\mathbf{n}) h_{\mathbf{k}}^{(\alpha)} &= \sum_{\mathbf{m}\in\mathbb{N}_0^2} C_\alpha(\mathbf{k},\mathbf{m},\mathbf{n}) h_{\mathbf{m}}^{(\alpha)} \\ &= \sum_{\mathbf{n}\in\mathbb{N}_0^2} C_\alpha(\mathbf{k},\mathbf{m},\mathbf{n}) h_{\mathbf{n}}^{(\alpha)} = 1. \end{aligned} \tag{2.7}$$

These formulas give rise to two Banach algebras. The product formula gives rise to a product (called a convolution) on $M(\mathbb{D})$. If $\mu, \nu \in M(\mathbb{D})$, $\mu *_\alpha \nu$ is defined by its action on $f \in C(\mathbb{D})$:

$$\int_{\mathbb{D}} f\, d(\mu *_\alpha \nu)$$
$$= \frac{\alpha}{\alpha+1} \int_{\mathbb{D}}\int_{\mathbb{D}}\int_{\mathbb{D}} f(z\zeta + (1-|z|^2)^{1/2}(1-|\zeta|^2)^{1/2}\xi)\, \frac{dm_\alpha(\xi)}{1-|\xi|^2}\, d\mu(z)\, d\nu(\zeta), \qquad (\alpha > 0)$$

and

$$\int_{\mathbb{D}} f\, d(\mu *_0 \nu) = \frac{1}{2\pi}\int_{\mathbb{D}}\int_{\mathbb{D}}\int_{-\pi}^{\pi} f(z\zeta + (1-|z|^2)^{1/2}(1-|\zeta|^2)^{1/2}e^{i\theta})\, d\theta\, d\mu(z)\, d\nu(\zeta).$$

Thus, if $\mu, \nu \in M(\mathbb{D})$, and $\alpha \ge 0$, $(\mu *_\alpha \nu)\widehat{\ } = \widehat{\mu}\widehat{\nu}$. We denote this Banach algebra of measures $D(\alpha)$. Each $f \in L^1$ can be identified with $\sigma_f \in M(\mathbb{D})$ by setting $\sigma_f = f\, dm_\alpha$. With this identification L^1 is a closed ideal in $D(\alpha)$, and if $f, g \in L^1$, $h \in L^\infty$, and $\mu \in M(\mathbb{D})$ we have

$$\|f *_\alpha g\|_1 \le \|f\|_1 \cdot \|g\|_1, \qquad \|f *_\alpha h\|_\infty \le \|f\|_1 \cdot \|h\|_\infty,$$
$$\|\mu *_\alpha f\|_1 \le \|\mu\| \cdot \|f\|_1.$$

Remark. $M_1(\mathbb{D})$ is a semigroup with respect to $*_\alpha$ provided $\alpha \ge 0$; this property cannot be extended to $\alpha < 0$ (see [BH95, p. 142]).

The second convolution is most conveniently defined on ℓ^1. If $\phi, \psi \in \ell^1$, define

$$(\phi \star_\alpha \psi)(\mathbf{k}) = \sum_{\mathbf{m},\mathbf{n}\in\mathbb{N}_0^2} \phi(\mathbf{m}) h_{\mathbf{n}}^{(\alpha)} \psi(\mathbf{n}) h_{\mathbf{n}}^{(\alpha)} C_\alpha(\mathbf{k},\mathbf{m},\mathbf{n}). \tag{2.8}$$

A consequence of this is

$$\Big[\sum_{\mathbf{n}\in\mathbb{N}_0^2} \phi(\mathbf{n}) h_{\mathbf{n}}^{(\alpha)} R_{\mathbf{n}}^{(\alpha)}\Big] \cdot \Big[\sum_{\mathbf{m}\in\mathbb{N}_0^2} \psi(\mathbf{m}) h_{\mathbf{m}}^{(\alpha)} R_{\mathbf{m}}^{(\alpha)}\Big] = \sum_{\mathbf{k}\in\mathbb{N}_0^2} (\phi \star_\alpha \psi)(\mathbf{k})\, h_{\mathbf{k}}^{(\alpha)} R_{\mathbf{k}}^{(\alpha)}. \tag{2.9}$$

Remark. A convolution can also be defined on $M(\mathbb{N}_0^2)$ by the formula

$$(\mu \star_\alpha \nu)(\{\mathbf{k}\}) = \sum_{\mathbf{m},\mathbf{n}\in\mathbb{N}_0^2} \mu(\{\mathbf{m}\})\nu(\{\mathbf{n}\}) C_\alpha(\mathbf{k},\mathbf{m},\mathbf{n});$$

$D(\alpha)$ and $(M(\mathbb{N}_0^2), \star_\alpha)$ form a pair of dual Banach measure algebras; indeed, this is one of the rare situations when both structures are hypergroups.

It is now possible to define Sidon sets and Riesz sets in this context. A subset E of $\mathbb{N}_0^2$ will be called a *$D(\alpha)$-Sidon set* (or simply a *Sidon set* if there is no ambiguity) if there is a constant B_E such that

$$\|\widehat{f}\|_1 \le B_E \|f\|_\infty$$

for every polynomial f such that $\operatorname{supp}(\widehat{f}) \subset E$. E is a *$D(\alpha)$-Riesz set* if whenever $\mu \in M(\mathbb{D})$ with $\operatorname{supp}(\widehat{\mu}) \subset E$ then μ is absolutely continuous with respect to m_α.

3. Some properties of $D(\alpha)$. We begin with a technical result based on the classical version in Edward book [Edw67, Ex. 2.19]. A similar result holds in compact abelian groups [Rud62, §2.6.8].

LEMMA 3.1. *If E is a finite subset of $\mathbb{N}_0^2$ and $\varepsilon > 0$, there is a polynomial f such that* (i) $0 \le \widehat{f}(\mathbf{n}) \le 1$ *for all* $\mathbf{n} \in \mathbb{N}_0^2$, (ii) $\widehat{f}(\mathbf{n}) = 1$ *for all* $\mathbf{n} \in E$, *and* (iii) $\|f\|_1 \le 1 + \varepsilon$. *In particular, it is possible to choose polynomials* F_N, $N \in \mathbb{N}$, *such that* (i) $0 \le \widehat{F}_N(\mathbf{n}) \le 1$ *for all* $\mathbf{n} \in \mathbb{N}_0^2$, (ii) $\widehat{F}_N(\mathbf{n}) = 1$ *if* $\|\mathbf{n}\| \le N$, *and* (iii) $\|F_N\|_1 \le 2$.

Remark. It is not difficult to show that $\{F_N\}_{N\in\mathbb{N}_0}$ is a polynomial approximate identity analogous to the Fejér kernel of classical Fourier analysis. Indeed, if K_N denotes the Fejér kernel, the functions $F_N = K_{2N} - K_N$ have the properties listed in the lemma.

Proof. Let $r = \max\{\mathbf{n} : \mathbf{n} \in E\}$, let N be a positive integer, and let $A_p = \sum_{\|\mathbf{n}\|\le p} h_{\mathbf{n}}^{(\alpha)}$. Define

$$u = A_N^{-1} \sum_{\|\mathbf{k}\|\le N} h_{\mathbf{k}}^{(\alpha)} R_{\mathbf{k}}^{(\alpha)}, \qquad v = \sum_{\|\mathbf{m}\|\le N+r} h_{\mathbf{m}}^{(\alpha)} R_{\mathbf{m}}^{(\alpha)},$$

and $f = u \cdot v$, so if $\widehat{f}(\mathbf{n})$ is computed by (2.9) and (2.8), (i) becomes an immediate consequence of (2.5) and (2.7). Now if $\|\mathbf{k}\| \le N$ and $\|\mathbf{n}\| \le r$, then $\|\mathbf{n} + \mathbf{k}\| \le N + r$, so (2.7) yields (ii). Let $\varepsilon > 0$, then the Schwarz inequality and the Plancherel formula (2.2) yield $\|f\|_1 \le \|u\|_2 \cdot \|v\|_2 \le (A_{N+r}/A_N)^{1/2}$ which is bounded by $1 + \varepsilon$ if N is sufficiently large. ∎

In the following, we consider $M(\mathbb{T})$ to be a subspace of $M(\mathbb{D})$ and we introduce $M(\mathbb{D}^0) = \{\mu \in M(\mathbb{D}) : |\mu|(\mathbb{T}) = 0\}$. Moreover, if $\mu \in M(\mathbb{T})$, $n \in \mathbb{N}_0$, and $k \in \mathbb{Z}$ then $\widehat{\mu}(n + k, n) = \mathcal{F}\mu(k) = \int_{\mathbb{T}} e^{-ik\theta}\, d\mu(e^{i\theta})$.

Several important and useful properties of $D(\alpha)$ are scattered through [Kan76]; we include them inside the following lemma for convenience.

LEMMA 3.2. (i) *If $\alpha > 0$ and $\mu, \nu \in M(\mathbb{D}_0)$ then $\mu *_\alpha \nu$ is absolutely continuous with respect to m_α.*

(ii) *If $\mu, \nu, \lambda \in M(\mathbb{D}_0)$ then $\mu *_0 \nu *_0 \lambda$ is absolutely continuous with respect to m_0.*

(iii) *If $\mu \in M(\mathbb{D}_0)$, then $\lim_{\|\mathbf{n}\|\to\infty} \widehat{\mu}(\mathbf{n}) = 0$.*

(iv) *If $\mu \in M(\mathbb{D})$, then there is a unique decomposition $\mu = \mu_0 + \mu_{\mathbb{T}}$ with $\mu_0 \in M(\mathbb{D}_0)$ and $\mu_{\mathbb{T}} \in M(\mathbb{T})$, and $\lim_{n\to\infty} \widehat{\mu}(n + k, n) = \mathcal{F}\mu_{\mathbb{T}}(k)$.*

(v) *If $\mu, \nu \in M(\mathbb{T})$, then $\mu *_\alpha \nu = \mu * \nu$ where $*$ denotes the classical convolution for $M(\mathbb{T})$.*

R e m a r k. If $\alpha = 0$, then (i) actually fails since $\delta_z *_0 \delta_\zeta$ is a unit mass uniformly distributed on the circle with center $z\zeta$ and radius $(1-|z|^2)^{1/2}(1-|\zeta|^2)^{1/2}$. Nevertheless, (ii) is an adequate substitute, for instance, Kanjin's results about the maximal ideal spaces and idempotents [Kan76, Thms. 3 and 4] for $D(\alpha)$ can be extended to $\alpha = 0$.

P r o o f. See [Kan76, Lem. 2] for (i); to obtain (ii), first define

$$F(z,\zeta;\theta) = z\zeta + (1-|z|^2)^{1/2}(1-|\zeta|^2)^{1/2}e^{i\theta},$$
$$G(z,\zeta,w;\theta,\phi) = F(F(z,\zeta;\theta),w;\phi),$$

so that for each $f \in C(\mathbb{D})$,

$$\int_{\mathbb{D}} f\, d(\delta_z *_0 \delta_\zeta *_0 \delta_w) = (2\pi)^{-2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(G(z,\zeta,w;\theta,\phi))\, d\theta\, d\phi.$$

Now suppose, for the moment, that $z,\zeta,w \in [0,1)$ are fixed, then the mapping

$$(\theta,\phi) \mapsto g(\theta,\phi) = (\Re G(z,\zeta,w;\theta,\phi), \Im G(z,\zeta,w;\theta,\phi))$$

from $\mathbb{S} = [-\pi,\pi]\times[-\pi,\pi]$ to $\mathbb{D}$ has Jacobian

$$\begin{aligned} J(\theta,\phi) &= (1-z^2)^{1/2}(1-\zeta^2)^{1/2}(1-w^2)^{1/2}(1-|F(z,\zeta;\theta)|^2)^{1/2} \\ &\quad \times (w\sin(\phi-\theta) - z\zeta(1-w^2)^{1/2}(1-|F(z,\zeta;\theta)|^2)^{-1/2}\sin\theta) \end{aligned}$$

which vanishes only on a finite set of curves in $\mathbb{S}$. Thus if $J(\theta,\phi) \neq 0$ there is a closed disk U, containing (θ,ϕ), on which g is injective. Thus there is $h \in C(\mathbb{D})$ such that $\int_U f(g(\theta,\phi))\, d\theta\, d\phi = \int_{\mathbb{D}} f(z)h(z)\, dm_0(z)$. Now a tedious but elementary argument can be used to show that there is a non-negative function $E_{z,\zeta,w}$ on $\mathbb{D}$ such that

$$\int_{\mathbb{D}} f\, d(\delta_z *_0 \delta_\zeta *_0 \delta_w) = \int_{\mathbb{D}} f E_{z,\zeta,w}\, dm_0, \tag{3.1}$$

whence $\delta_z *_0 \delta_\zeta *_0 \delta_w \in L^1$. Now for general $z,\zeta,w \in \mathbb{D}^0$, choose $\alpha,\beta,\gamma \in \mathbb{T}$ such that $\alpha z, \beta\zeta, \gamma w \in [0,1)$, so that $\delta_z *_0 \delta_\zeta *_0 \delta_w = \delta_{\overline{\alpha\beta\gamma}} *_0 \delta_{\alpha z} *_0 \delta_{\beta\zeta} *_0 \delta_{\gamma w}$ is in L^1, thus (3.1) is valid for all $z,\zeta,w \in \mathbb{D}^0$, so finally if $\mu,\nu,\lambda \in M(\mathbb{D}_0)$, then

$$\int_{\mathbb{D}} f\, d(\mu *_0 \nu *_0 \lambda) = \int_{\mathbb{D}}\int_{\mathbb{D}}\int_{\mathbb{D}} f(\xi)E_{z,\zeta,w}(\xi)\, dm_0(\xi)\, d\mu(z)\, d\nu(\zeta)\, d\lambda(w),$$

hence $\mu *_0 \nu *_0 \lambda \in L^1$, which establishes (ii).

Now if $\mu \in M(\mathbb{D}^0)$ and $\alpha \ge 0$, then $\mu *_\alpha \mu *_\alpha \mu \in L^1$ so by (2.3), $\lim_{\|\mathbf{n}\|\to\infty}(\widehat{\mu}(\mathbf{n}))^3 = 0$, and (iii) is proved. Parts (iv) and (v) are obvious. ∎

4. $D(\alpha)$-Sidon sets. If E is a subset of $\mathbb{N}_0^2$, we shall say that $f \in L^1$ is an *E-function* if $\widehat{f}(\mathbf{n}) = 0$ for every $\mathbf{n} \notin E$. A polynomial which is an

E-function is called an E-*polynomial*. We denote by $\ell^p(E)$ (resp. $C_0(E)$) the functions in ℓ^p (resp. $C_0(\mathbb{N}_0^2)$) which are supported on E. Thus E is a $D(\alpha)$-*Sidon* set if there is a constant B_E such that for every E-polynomial f,

$$\|\widehat{f}\|_1 \le B_E \|f\|_\infty. \tag{4.1}$$

LEMMA 4.1. *Let $E \subset \mathbb{N}_0^2$. Then the following are equivalent*:

(a) *E is a $D(\alpha)$-Sidon set.*

(b) *If f is a bounded E-function*, *then $\|\widehat{f}\|_1 < \infty$.*

(c) *If f is a continuous E-function*, *then $\|\widehat{f}\|_1 < \infty$.*

(d) *To each bounded function ϕ on E there corresponds a measure $\mu \in M(\mathbb{D})$ such that $\widehat{\mu}(\mathbf{n}) = \phi(\mathbf{n})$ for every $\mathbf{n} \in E$.*

(e) *To every $\phi \in C_0(E)$ corresponds a function $f \in L^1$ such that $\widehat{f}(\mathbf{n}) = \phi(\mathbf{n})$ for every $\mathbf{n} \in E$.*

Proof. The proof is an adaptation of [Edw67, §15.1.4]. We illustrate this with two of the arguments.

(a)⇒(b). Assume E is a $D(\alpha)$-Sidon set with constant B_E, and let f be a bounded E-function. Then with F_N as in Lemma 3.1, $F_N * f$ is an E-polynomial and we have $\sum_{\|\mathbf{n}\|\le N} |\widehat{f}(\mathbf{n})| h_{\mathbf{n}}^{(\alpha)} \le \|\widehat{F}_N \widehat{f}\|_1 = \|(F_N * f)\widehat{\ }\|_1 \le B_E \|F_N * f\|_\infty \le B_E \|F_N\|_1 \|f\|_\infty \le 2B_E\|f\|_\infty$. Thus, since N is arbitrary, $\|\widehat{f}\|_1 \le 2B_E\|f\|_\infty$.

(e)⇒(a). By the open mapping theorem, to each $\phi \in C_0(E)$ corresponds an $f \in L^1$ such that $\widehat{f} = \phi$ on E and $\|f\|_1 \le B\|\phi\|_\infty$. Let g be an E-polynomial, and define $\phi(\mathbf{n}) = |\widehat{g}(\mathbf{n})|/\widehat{g}(\mathbf{n})$ if $\widehat{g}(\mathbf{n}) \ne 0$ and $\phi(\mathbf{n}) = 0$ otherwise. Then $\phi \in C_0(E)$ and $\|\phi\|_\infty \le 1$, so there is $f \in L^1$ such that $\widehat{f} = \phi$ on E and $\|f\|_1 \le B$, so $\|\widehat{g}\|_1 = \sum_{\mathbf{n}\in E} |\widehat{g}(\mathbf{n})| h_{\mathbf{n}}^{(\alpha)} = \sum_{\mathbf{n}\in E} \widehat{f}(\mathbf{n})\widehat{g}(\mathbf{n}) h_{\mathbf{n}}^{(\alpha)} = (f *_\alpha g)\widehat{\ }(1) \le \|f\|_1\|g\|_\infty \le B\|g\|_\infty$. ∎

We also need the following criterion for $\mathbb{T}$-Sidon sets (that is, Sidon sets for classical Fourier analysis on $\mathbb{T}$); see [Edw67, §15.1.5].

LEMMA 4.2. *Let $E \subset \mathbb{Z}$. Then E is a $\mathbb{T}$-Sidon set if and only if there is a number $\eta > 0$ such that for each $\phi : E \to \{-1, 1\}$ there is $\mu \in M(\mathbb{T})$ such that* $\sup_{n\in E} |\phi(n) - \mathcal{F}\mu(n)| \le 1 - \eta$.

Let $\#(E)$ denote the cardinality of the set $E \subset \mathbb{N}_0^2$, let $d_k = \{(n+k, n) : n \in \mathbb{N}_0\}$, and let $E_\infty = \{k : E \cap d_k \ne \emptyset\}$.

THEOREM 1. *Let $\alpha \ge 0$. Then $E \subset \mathbb{N}_0^2$ is a $\mathbb{D}(\alpha)$-Sidon set if and only if* (i) *for all $|k|$ sufficiently large $\#(E \cap d_k) \le 1$, and* (ii) *E_∞ is a $\mathbb{T}$-Sidon set.*

Proof. The following observation is the key: by Lemma 3.2(iv) and the remark preceding the lemma, any $\mu \in M(\mathbb{D})$ may be uniquely decomposed into $\mu = \mu_0 + \mu_{\mathbb{T}}$ where $\mu_0 \in M(\mathbb{D}^0)$ and $\mu_{\mathbb{T}} \in M(\mathbb{T})$. Moreover, $\widehat{\mu}_{\mathbb{T}}(m,n)$ depends only on $m - n$ as $m + n \to \infty$.

Assume E is a $\mathbb{D}(\alpha)$-Sidon set. Therefore, arguing by way of contradiction if (i) failed for E, it would be possible to exhibit $\mu_{\mathbb{T}} \in M(\mathbb{T})$ such that for some strictly increasing sequence $\{k_j\}_{j=0}^{\infty}$, $\mathcal{F}\mu_{\mathbb{T}}(k_j)$ would take values close to $+1$ and -1 for each sufficiently large j. This is absurd.

When finitely many points are added to or deleted from a Sidon set, the result is still a Sidon set, so assume $E = \{(n_k + k, n_k) : k \in E_\infty\}$. We must show E_∞ is a $\mathbb{T}$-Sidon set. Choose $\phi_\infty : E_\infty \to \{-1, 1\}$ and let ϕ be any function on $\mathbb{N}_0^2$ with values in $\{-1, 1\}$ and which satisfies $\phi(n_k + k, n_k) = \phi_\infty(k)$ for every $k \in E_\infty$. By Lemma 4.1 there is $\mu \in M(\mathbb{D})$ such that $\widehat{\mu} = \phi$ on E. For k large enough, we will get $|\mathcal{F}\mu_{\mathbb{T}}(k) - \phi_\infty(k)| \le 1/2$. It is trivial to extend this to finitely more values of k, so E_∞ is a $\mathbb{T}$-Sidon set by Lemma 4.2 with $\eta = 1/2$.

For the converse assume $E = \{(n_k + k, n_k) : k \in E_\infty\}$ and let ϕ be a bounded function on E and $\phi_\infty(k) = \phi(n_k + k, n_k)$; E_∞ is a $\mathbb{T}$-Sidon set, so that there is $\mu \in M(\mathbb{T})$ with $\mathcal{F}\mu = \phi_\infty$ on E_∞, so $\widehat{\mu} = \phi$ on E. ∎

As a consequence of Theorem 1, knowledge about $\mathbb{T}$-Sidon sets yields knowledge about $\mathbb{D}(\alpha)$-Sidon sets. See [Rud62, §5.75 and §5.76] with $\Gamma = \mathbb{Z}$, and [Edw67, §15.2].

5. $D(\alpha)$**-Riesz sets.** These are sets which generalize the classical F. and M. Riesz Theorem; see Rudin [Rud62, §8.2.1] where a more detailed discussion and additional references are found. The following two results list a necessary condition and a sufficient condition for $D(\alpha)$-Riesz sets. Theorem 3 is stronger than the earlier one of Kanjin [Kan76, Thm. 7], and it is based on his proof.

THEOREM 2. *If E is a $D(\alpha)$-Riesz set then $E^{c} \cap d_k$ is an infinite set for every k.*

Proof. If $E^{c} \cap d_k$ is finite for some k, let

$$p(z) = \sum_{\mathbf{n} \in E^{c} \cap d_k} h_{\mathbf{n}}^{(\alpha)} R_{\mathbf{n}}^{(\alpha)}(z)$$

and let

$$d\mu(z) = \frac{1}{2\pi} e^{ik\theta} d\theta - p(z)\, dm_\alpha(z) \qquad (z = re^{i\theta}).$$

Then $\operatorname{supp}(\widehat{\mu}) \subset E \cap d_k \subset E$, but μ is not absolutely continuous since $|\mu|(\mathbb{T}) = 1$. ∎

THEOREM 3. *Suppose $\{\phi(k)\}_{k=0}^{\infty}$ is a non-negative sequence such that*

$$\limsup_{k \to \infty} ((\log\, k)/k)\phi(k) = 0. \tag{5.1}$$

Then the following are $D(\alpha)$-Riesz sets:

$$R_1(\phi) = \{(n+k, n) : n \le \phi(k)\} \cup \{(n, n+k) : n \le \phi(k)\},$$
$$R_2(\phi) = \{(m, n) : m \le \phi(n) \text{ or } n \le \phi(m)\}.$$

Proof. Assume ϕ satisfies (5.1) and that $\mu \in M(\mathbb{D})$ with $\operatorname{supp}(\widehat{\mu}) \subset R_1(\phi)$. Stirling's formula yields the bound

$$h^{(\alpha)}_{n+k,n}|R^{(\alpha)}_{n+k,n}(z)| = h^{(\alpha)}_{n,n+k}|R^{(\alpha)}_{n,n+k}(z)| < Ck^{n+\alpha+1}r^k \qquad (n \le k).$$

The same bound holds for $n \le \phi(k)$ since $\phi(k) \le k$ if k is sufficiently large. Thus

$$\sum_{k=2}^{\infty}\sum_{n=0}^{\phi(k)} h^{(\alpha)}_{n+k,n}|R^{(\alpha)}_{n+k,n}(z)| < C\sum_{k=2}^{\infty}\sum_{n=0}^{\phi(k)} k^{n+\alpha+1}r^k = C\sum_{k=2}^{\infty} k^{\alpha+1}r^k\frac{k^{\phi(k)+1}-1}{k-1}$$
$$\le 2C\sum_{k=2}^{\infty} k^{\phi(k)+\alpha+1}r^k.$$

This converges for each $r < 1$ by the root test provided

$$\limsup_{k\to\infty} k^{\phi(k)/k} \le 1,$$

which will be the case if (5.1) holds. It follows that $\sum_{\mathbf{n}\in\mathbb{N}_0^2}\widehat{\mu}(\mathbf{n})R^{(\alpha)}_{\mathbf{n}}$ converges to a continuous function f on compact subsets of $\mathbb{D}^0$. Finally, with F_N as in Lemma 3.1, Fatou's lemma yields

$$\int_{\mathbb{D}} |f|\,dm_\alpha = \int_{\mathbb{D}} \lim_{N\to\infty}|F_N *_\alpha \mu|\,dm_\alpha \le \liminf_{N\to\infty}\int_{\mathbb{D}} |F_N *_\alpha \mu|\,dm_\alpha$$
$$= \liminf_{N\to\infty}\|F_N *_\alpha \mu\|_1 \le 2\|\mu\|,$$

so $f \in L^1$ and $d\mu = f\,dm_\alpha$ and thus $R_1(\phi)$ is a Riesz set.

Let ϕ satisfy (5.1). In the light of the definition of lim sup and the fact that any subset of a Riesz set is obviously a Riesz set, there is no loss of generality in assuming that ϕ is an unbounded non-decreasing function. We show that $R_2(\phi) \subset R_1(\psi)$ where

$$\limsup_{k\to\infty}((\log\, k)/k)\psi(k) = 0.$$

Since ϕ satisfies (5.1) there is n_0 such that $\phi(2n) \le n$ for $n \ge n_0$. Hence if $n \ge n_0$, then $n \le \frac{1}{2}\phi^{-1}(n)$, so if $n_0 \le n \le \phi(m)$, we have $\frac{1}{2}\phi^{-1}(n) + n \le \phi^{-1}(n) \le m$, whence

$$n \le \phi(2(m-n)) \qquad \text{for } n \ge n_0.$$

Thus $n \le \psi(m-n)$ where $\psi(k) = n_0 + \phi(2k)$, so $R_2(\phi) \subset R_1(\psi)$ as required. ■

REFERENCES

[AT74] H. Annabi et K. Trimèche, *Convolution généralisée sur le disque unité*, C. R. Acad. Sci. Paris 278 (1974), 21–24.

[BG91] M. Bouhaik and L. Gallardo, *A Mehler–Heine formula for disk polynomials*, Indag. Math. 1 (1991), 9–18.

[BG92] —, —, *Un théorème limite central dans un hypergroupe bidimensionnel*, Ann. Inst. H. Poincaré 28 (1992), 47–61.

[BH95] W. R. Bloom and H. Heyer, *Harmonic Analysis of Probability Measures on Hypergroups*, de Gruyter Stud. Math. 20, de Gruyter, Berlin, New York, 1995.

[CS92] W. C. Connett and A. L. Schwartz, *Fourier analysis off groups*, in: The Madison Symposium on Complex Analysis (Providence, R.I.), A. Nagel and L. Stout (eds.), Contemp. Math. 137, Amer. Math. Soc. 1992, 169–176.

[CS95] —, —, *Continuous* 2*-variable polynomial hypergroups*, in: Applications of Hypergroups and Related Measure Algebras (Providence, R.I.), O. Gebuhrer, W. C. Connett and A. L. Schwartz (eds.), Contemp. Math. 183, Amer. Math. Soc., 1995, 89–109.

[Edw67] R. E. Edwards, *Fourier Series*, Vols. I, II, Holt, Rinehart and Winston, New York, 1967.

[HK93] H. Heyer and S. Koshi, *Harmonic Analysis on the Disk Hypergroup*, Mathematical Seminar Notes, Tokyo Metropolitan University, 1993.

[Kan76] Y. Kanjin, *A convolution measure algebra on the unit disc*, Tôhoku Math. J. (2) 28 (1976), 105–115.

[Kan85] —, *Banach algebra related to disk polynomials*, ibid. 37 (1985), 395–404.

[Koo72] T. H. Koornwinder, *The addition formula for Jacobi polynomials*, *II*, *the Laplace type integral representation and the product formula*, Tech. Report TW 133/72, Mathematisch Centrum, Amsterdam, 1972.

[Koo78] —, *Positivity proofs for linearization and connection coefficients of orthogonal polynomials satisfying an addition formula*, J. London Math. Soc. (2) 18 (1978), 101–114.

[Rud62] W. Rudin, *Fourier Analysis on Groups*, Interscience Publishers, 1962.

[Sze67] G. Szegő, *Orthogonal Polynomials*, 2nd ed., Colloq. Publ. 23, Amer. Math. Soc., Providence, R.I., 1967.

Département de Mathématique
Institut de Recherche Mathématique Avancée
Université Louis Pasteur
67100 Strasbourg, France
E-mail: gebuhrer@math.u-strasbg.fr

Department of Mathematics
and Computer Science
University of Missouri–St. Louis
St. Louis, Missouri 63121, U.S.A.
E-mail: schwartz@arch.umsl.edu

Received 30 August 1995;
revised 22 April 1996