ON A COMBINATORIAL PROBLEM CONNECTED WITH
FACTORIZATIONS

BY

WEIDONG GAO (BEIJING)

0. Let $K$ be an algebraic number field with classgroup $G$ and integer ring $R$. For $k \geq 1$ and a real number $x > 0$, let $a_k = a_k(G)$ be the maximal number of nonprincipal prime ideals which can divide a squarefree element of $R$ with at most $k$ distinct factorizations into irreducible elements, and let $F_k(x)$ be the number of elements $\alpha \in R$ (up to associates) having at most $k$ different factorizations into irreducible elements of $R$. W. Narkiewicz [8] derived the asymptotic expression

$$F_k(x) \sim c_k x (\log x)^{-1 + 1/|G| (\log \log x)^{a_k}},$$

where $c_k$ is positive and depends on $k$ and $K$.

Recently, F. Halter-Koch [6–7] used the characterizations of $a_k(G)$ to study nonunique factorizations. In [8], Narkiewicz showed that $a_k(G)$ depends only on $k$ and $G$, gave a combinatorial definition of it and proposed the problem of determining $a_k(G)$ (Problem 1145).

Let $G$ be a finite abelian group (written additively). The Davenport constant $D(G)$ of $G$ is defined to be the minimal integer $d$ such that for every sequence of $d$ elements in $G$ there is a nonempty subsequence with sum zero. Narkiewicz and ´Sliwa [8–9] derived several properties of $a_1(G)$ involving $D(G)$ and proposed the following conjecture:

**Conjecture 1.** Let $G = C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $1 < n_1 \mid \ldots \mid n_r$. Then $a_1(G) = n_1 + \ldots + n_r$, where $C_n$ denotes the cyclic group of order $n$.

They affirmed Conjecture 1 for $G = C_2^{n_1}, C_2^{n_1} \oplus C_4, C_2^{n_1} \oplus C_4^{n_2}$ or $C_3^{n_1}$.

In this paper we derive several properties of $a_k(G)$, affirm this conjecture for a more general case and determine $a_2(C_2^n)$ and $a_k(C_n)$ provided that $n$ is substantially larger than $k$. The paper is organized in the following way: In Section 1 we repeat the combinatorial definition of $a_k(G)$ due to Narkiewicz [8] and give some preliminaries on $a_1(G)$ and $D(G)$. In Section 2 we derive some new properties of $a_k(G)$ and show the following:

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Theorem 1. Let \( G = C_{n_1} \oplus \ldots \oplus C_{n_r} \) with \( 1 < n_1 | \ldots | n_r \), let \( p \) be a prime with \( 2 \leq p \leq 151 \), and let us adopt the convention \( C^{d}_{0} = C_{1} \). Then \( a_{1}(G) = n_{1} + \ldots + n_{r} \) provided that \( G \) is of one of the following forms \((m \geq 1)\):

1. \( C_{2^{t}}^{3^r} \oplus C_{2^{s}}^{3^m}, 0 \leq t \leq 1 \) or \( 0 \leq s \leq 1 \),
2. \( C_{2^{t}}^{3^r} \oplus C_{2^{t}}^{3^m}, 0 \leq t \leq 1 \) or \( 0 \leq s \leq 1 \),
3. \( C_{4}^{2^{t}} \),
4. \( C_{2^{t}}^{3^r} \oplus C_{2^{t}}^{3^m}, 0 \leq t \leq 1 \),
5. \( C_{2^{t}}^{3^r} \oplus C_{2^{t}}^{3^m}, 0 \leq t \leq 1 \),
6. \( C_{2^{t}}^{3^r} \oplus C_{2^{t}}^{5^r}, 0 \leq t \leq 1 \),
7. \( C_{2^{t}}^{3^r} \oplus C_{2^{t}}^{5^r}, 0 \leq t \leq 1 \),
8. \( C_{2^{t}}^{3^r} \oplus C_{2^{t}}^{3^m}, 0 \leq t \leq 1 \),
9. \( C_{2^{t}}^{3^r} \oplus C_{2^{t}}^{3^m}, 0 \leq t \leq 1, l \geq 4 \) and \( 2^{m} \geq n + 3t + 1 \),
10. \( C_{2^{t}}^{3^r} \oplus C_{2^{t}}^{3^m}, 0 \leq t \leq 1 \),
11. \( C_{2^{t}}^{3^r} \oplus C_{2^{t}}^{3^m}, 0 \leq t \leq 1, l \geq 4 \) and \( 3^{m} \geq 2n + 8t + 1 \),
12. \( C_{2^{t}}^{3^r} \oplus C_{2^{t}}^{5^r}, 0 \leq t \leq 1 \),

In Section 3 we derive some properties of \( a_{k}(G) \) and show the following:

Theorem 2. If \( k \geq 2 \) and if

\[
\log_{2}n + \sqrt{\left(\log_{2}n\right)^{2} + n} + 1,
\]

then \( a_{k}(C_{n}) = n \).

Remark 1. It is proved in [8, Proposition 9] that \( \max\{D(G), \sum_{i=1}^{r} n_{i}\} \leq a_{k}(G) \leq a_{l}(G) \) for \( 1 \leq k \leq l \); therefore if Conjecture 1 is true, then \( D(G) \leq n_{1} + \ldots + n_{r} \) and the best known estimation (see [3])

\[
D(G) \leq n_{r} \left(1 + \frac{\log |G|}{\log n_{r}}\right)
\]

would be improved. So it seems very difficult to settle Conjecture 1 in general.

1. In what follows we always let \( G \) denote a finite abelian group.

For a sequence \( S = (a_{1}, \ldots, a_{m}) \) of elements in \( G \), we use \( \sum S \) to denote the sum \( \sum_{i=1}^{m} a_{i} \). By \( \lambda \) we denote the empty sequence and adopt the convention that \( \sum \lambda = 0 \). We say \( S \) a zero-sum sequence if \( \sum S = 0 \). A subsequence \( T \) of \( S \) is a sequence \( T = (a_{i_{1}}, \ldots, a_{i_{l}}) \) with \( \{i_{1}, \ldots, i_{l}\} \subseteq \{1, \ldots, m\} \); we denote by \( I_{T} \) the index set \( \{i_{1}, \ldots, i_{l}\} \), and identify two subsequences \( S_{1} \) and \( S_{2} \) if \( I_{S_{1}} = I_{S_{2}} \). We say two subsequences \( S_{1} \) and \( S_{2} \) are disjoint if \( I_{S_{1}} \cap I_{S_{2}} = \emptyset \) (the empty set) and define multiplication of two disjoint subsequences by juxtaposition.
A nonempty sequence $B$ of nonzero elements in $G$ is called a block in $G$ provided that $\sum B = 0$; we call a block irreducible if it cannot be written as a product of two blocks.

By a factorization of a block $B = (b_1, \ldots, b_k)$ we shall understand any surjective map

$$\varphi: \{1, \ldots, k\} \to \{1, \ldots, t\}$$

with a certain positive integer $t = t(\varphi)$ such that, for $j = 1, \ldots, t$, the sequences $B_j = (b_i : \varphi(i) = j)$ are blocks. If they are all irreducible, we speak about an irreducible factorization of $B$. Obviously, we have $B = \prod B_j$. Two such factorizations $\varphi$ and $\psi$ are called strongly equivalent if $t(\varphi) = t(\psi)$ ($= t$ say) and for a suitable permutation $\delta$ the sets $\{i : \varphi(i) = j\}$ and $\{\psi(i) = \delta(j)\}$ coincide for $j = 1, \ldots, t$. For $k \geq 1$, we define $B_k(G)$ to be the set consisting of all blocks which have at most $k$ strongly inequivalent irreducible factorizations, and let $a_k(G) = \max\{|B| : B \in B_k(G)\}$.

For a sequence $S$ of elements in $G$, we use $\sum(S)$ to denote the set consisting of all elements in $G$ which can be expressed as a sum over a nonempty subsequence of $S$, i.e.,

$$\sum(S) = \{ \sum T : \lambda \neq T, T \subseteq S \},$$

where $T \subseteq S$ means that $T$ is a subsequence of $S$.

**Lemma 1** ([9, Proposition 2]). Let $B = B_1 \ldots B_r \in B(G)$ and let $B_1, \ldots, B_r$ be irreducible blocks. Then $B \in B_1(G)$ if and only if for all disjoint nonempty subsets $X, Y$ of $\{1, \ldots, r\}$ we have

$$\sum(\prod_{i \in X} B_i) \cap \sum(\prod_{i \in Y} B_i) = \{0\}.$$

**Lemma 2** ([9, Proposition 6]). If $B = B_1 \ldots B_r \in B_1(G)$ and if $B_1, \ldots, B_r$ are irreducible blocks, then $|B_1| \cdots |B_r| \leq |G|$.

**Lemma 3** ([9, Proposition 3]). Let $B = B_1 \ldots B_r \in B_1(G)$ and let $B_1, \ldots, B_r$ be irreducible blocks. Then $|B| \leq D(G) + r - 1$.

For a sequence $S$ of elements in $G$, let $f_E(S)$ (resp. $f_O(S)$) denote the number of zero-sum subsequences $T$ of $S$ with $2 | |T|$ (resp. $2 \nmid |T|$), where we count $f_E(S)$ including the empty sequence; hence, we have $f_E(S) \geq 1$.

**Lemma 4.** Let $p$ be a prime. Then the following hold.

(i) $D(C_{n_1} \oplus C_{n_2}) = n_1 + n_2 - 1$ ($n_1 | n_2$) ([11]).

(ii) $D(C_{2p^k}) = 6p^k - 2$ ([2]).

(iii) $D(C_{3 \times 2^r}) = 9 \times 2^r - 2$ ([3]).

(iv) $D(\bigoplus_{i=1}^k C_{p^e_i}) = 1 + \sum_{i=1}^k (p^{e_i} - 1)$ ([10]).
(v) If $S$ is a sequence of elements in $\bigoplus_{i=1}^{k} C_{p^e}$, with $|S| \geq 1 + \sum_{i=1}^{k} (p^{e_i} - 1)$, then $f_E(S) \equiv f_O(S)$ (mod $p$) ([2], [10]).

**Lemma 5.** Let $H = C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $1 < n_1 | \ldots | n_r$, $n_1 | n$, and $D(H \oplus C_n^2) = 2(n-1) + D(H)$. Then $D(H \oplus C_n) = n - 1 + D(H)$.

**Proof.** By the definition of Davenport’s constant one can choose a sequence $T = (a_1, \ldots, a_{D(H \oplus C_n)-1})$ of $D(H \oplus C_n) - 1$ elements in $H \oplus C_n$ such that $0 \notin \sum(T)$. Put $b_i = (a_i, 0)$ with $0 \in C_n$ for $i = 1, \ldots, D(H \oplus C_n) - 1$, and put $b_1 = (0, 1)$ with $0 \in H \oplus C_n$ and $1 \in C_n$ for $i = D(H \oplus C_n), \ldots, D(H \oplus C_n) + n - 2$. Clearly, $b_i \in H \oplus C_n^2$ for $i = 1, \ldots, D(H \oplus C_n) + n - 2$ and the sequence $b_1, \ldots, b_{D(H \oplus C_n)} + n - 2$ contains no nonempty zero-sum subsequence. This implies that

$$D(H \oplus C_n) + n - 1 \leq D(H \oplus C_n^2).$$

Similarly, one can prove that

$$D(H) + n - 1 \leq D(H \oplus C_n),$$

so we have

$$D(H) + 2(n-1) \leq D(H \oplus C_n) + n - 1 \leq D(H \oplus C_n^2) = D(H) + 2(n-1).$$

This forces that $D(H \oplus C_n) = D(H) + n - 1$ as desired.

**Lemma 6.** Let $H = C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $1 < n_1 | \ldots | n_r$, and $n_1 | n$. Suppose that $n \geq D(H)$ and $D(H \oplus C_n^2) = 2(n-1) + D(H)$. Then any sequence $S$ of $2(n-1) + D(H)$ elements in $H \oplus C_n$ contains a nonempty zero-sum subsequence $T$ with $|T| \leq n$.

**Proof.** Suppose $S = (a_1, \ldots, a_{2(n-1)+D(H)})$. For $i = 1, \ldots, 2(n-1) + D(H)$ we define $b_i = (a_i, 1)$ with $1 \in C_n$. Clearly, $b_i \in H \oplus C_n^2$. Since $D(H \oplus C_n^2) = 2(n-1) + D(H)$, the sequence $b_1, \ldots, b_{2(n-1)+D(H)}$ contains a nonempty zero-sum subsequence $T$. By the definition of $b_i$, we must have $n | |T|$. But $n \geq D(H) - 1$, so $|T| \leq 2(n-1) + D(H) \leq 3n-1$, and this forces that

$$|T| = n \quad \text{or} \quad |T| = 2n.$$

If $|T| = n$ we are done. Otherwise, $|T| = 2n$. By Lemma 5, $D(H \oplus C_n) = n - 1 + D(H) \leq 2n - 1$, so one can find a nonempty zero-sum subsequence $M$ of $T$ with $|M| < |T|$. Setting $W$ equal to the shorter of $M$ and $T - M$ (the subsequence with index set $I_T - I_M$) completes the proof.

**Lemma 7.** Let $H = C_{n_1} \oplus \ldots \oplus C_{n_r}$ with $1 < n_1 | \ldots | n_r$, and $n_1 | n$. Suppose that $n \geq D(H)$ and $D(H \oplus C_n^2) = 2(n-1) + D(H)$. Then any zero-sum sequence $S$ of elements in $H \oplus C_n$ with $|S| \geq n + D(H)$ contains a zero-sum subsequence $T$ with $|S| - n \leq |T| < |S|$.
Proof. We distinguish three cases.

Case 1: $|S| \geq 2(n - 1) + D(H)$. Then the lemma follows from Lemma 6.

Case 2: $n + D(G) \leq |S| \leq 2n$. By Lemma 5, we have $D(H \oplus C_n) = n - 1 + D(G)$, thus there exists a zero-sum subsequence $W$ of $S$ with $1 \leq |W| < |S|$. Setting $T$ equal to the longer of $W$ and $S - W$ proves the lemma in this case.

Case 3: $2n + 1 \leq |S| \leq 2n - 3 + D(H)$. We define

$$b_i = \begin{cases} (a_i, 1) & \text{with } i \in C_n \text{ if } i = 1, \ldots, |S|, \\ (0, 1) & \text{with } 0 \in H \oplus C_n \text{ and } 1 \in C_n \text{ if } i = |S| + 1, \ldots, 2(n - 1) + D(H), \end{cases}$$

and similarly to the proof of Lemma 6 we find a zero-sum subsequence $W$ of $b_1, \ldots, b_{2(n - 1) + D(H)}$ with $|W| = n$ or $2n$. Put

$$J = \begin{cases} \{1, \ldots, |S|\} - I_W & \text{if } |W| = n \text{ (not necessarily } I_W \subseteq \{1, \ldots, |S|\}), \\ I_W - \{\{S\} + 1, \ldots, 2(n - 1) + D(H)\} & \text{if } |W| = 2n, \end{cases}$$

and let $T$ be the subsequence of $S$ with $I_T = J$. Clearly, $\sum T = 0$ and $|S| - n \leq |T| < |S|$. This completes the proof.

We say two nonempty sequences $S = (a_1, \ldots, a_m)$ and $T = (b_1, \ldots, b_m)$ of elements in $C_n$ with the same size $m$ are similar (written $S \sim T$) if there exist an integer $c$ coprime to $n$ and a permutation $\sigma$ of $1, \ldots, m$ such that $a_i = cb_{\sigma(i)}$ for $i = 1, \ldots, m$. Clearly, $\sim$ is an equivalence relation. For any $x \in C_n$, we denote by $[x]_n$ the minimal nonnegative inverse image of $x$ under the natural homomorphism from the additive group of integers onto $C_n$.

Lemma 8 ([1], [4]). Let $S = (a_1, \ldots, a_{n-k})$ be a sequence of $n-k$ elements in $C_n$ with $n \geq 2$. Suppose that $0 \notin \sum(S)$ and suppose that $k \leq n/4 + 1$. Then

$$S \sim (1, \ldots, 1, x_1, \ldots, x_k-1),$$

with all $x_i \neq 0$.

2. In this section we derive some properties of $a_1(G)$ and prove Theorem 1.

Proposition 1. Let $G = \bigoplus_{i=1}^k C_{p^i}$, with $p$ an odd prime, let $B = B_1 \ldots B_t \in B_1(G)$ and let $B_1, B_2, \ldots, B_r$ be irreducible blocks. Suppose that exactly $t$ of $|B_1|, \ldots, |B_r|$ are odd. Then $|B| \leq D(G) + t - 1$.

Proof. Without loss of generality, we assume that $|B_1|, \ldots, |B_t|$ are odd and that $|B_{t+1}|, \ldots, |B_r|$ are even. Let $D_i \subseteq B_i$ with $|D_i| = |B_i| - 1$ for $i = 1, \ldots, t$, and put $S = D_1 \ldots D_t B_{t+1} \ldots B_r$. By the choice of $D_1, \ldots, D_t$ and the hypothesis of the proposition, all zero-sum subsequences of $S$ consist
of all products of the form $B_{i_1} \cdots B_{i_t}$ with $l \geq 0$ and $t+1 \leq i_1 < \cdots < i_t \leq r$. This gives

$$f_B(S) = \binom{r-t}{0} + \binom{r-t}{1} + \binom{r-t}{2} + \cdots + \binom{r-t}{r-t} = 2^{r-t}$$

and $f_B(S) = 0$. But $p \nmid 2$, therefore $f_B(S) \not\equiv f_B(S) \pmod{p}$. Now it follows from Lemma 4(v) that $|B| - t = |S| \leq \sum_{i=1}^{k} (p^i - 1) = D(G) - 1$, that is, $|B| \leq D(G) + t - 1$.

**Proposition 2.** Let $H = C_{n_1} \oplus \cdots \oplus C_{n_l}$ be a finite abelian group with $1 < n_1 | \cdots | n_l$, and let $G = H \oplus C_{nm}$ with $n_l | n$. Suppose that (i) $m \geq 4$ and $n \geq D(H)$, and (ii) $D(H \oplus C^2_2) = 2(n-1) + D(H)$. Then $a_1(G) \leq a_1(H \oplus C_n) + nm - n$; moreover, if $a_1(H \oplus C_n) = n + n + \cdots + n$ then $a_1(G) = nm + n + \cdots + n$.

**Remark 2.** From Lemma 4(ii)–(iv) we see that there exists a large class of pairs of $(H,n)$ satisfying conditions (i) and (ii) of Proposition 2.

**Lemma 9.** Let $s$, $r$, $a$, $b$ be positive integers such that $a \geq 2$, $2a < b$ and $(r-1)b > s \geq ar$. Let $l, x_1, \ldots, x_l$ be positive integers satisfying

(i) $l \geq r$,
(ii) $x_1 + \cdots + x_l = s$,
(iii) $a \leq x_1, \ldots, x_l \leq b$.

Suppose $x_1 = n_1, \ldots, x_l = n_l$ are such that the product $x_1 \cdots x_l$ attains its minimal possible value. Then (a) there is at most one $i$ such that $a \neq n_i \neq b$; and we may assume (b) $l = r$.

**Proof.** (a) If there are $i, j$ with $1 \leq i \neq j \leq l$ such that $a < n_i, n_j < b$, without loss of generality, we assume that $a < n_i \leq n_j < b$. Then $(n_i - 1)(n_j + 1) < n_inj$, therefore if we take $x_i = n_i - 1, x_j = n_j + 1$ and $x_k = n_k$ for $k \neq i, j$, then $x_1 \cdots x_l < n_1 \cdots n_l$, a contradiction. This proves (a).

(b) Let $l$ be the smallest integer satisfying $l \geq r$ and the hypothesis of the lemma. If $l \geq r + 1$, then since $s \leq (r-1)b$, there are at most $r-2$ distinct indices $i$ such that $n_i = b$, so by (a), there are at least two indices $i$ and $j$ such that $n_i = n_j = a$; without loss of generality, we assume $n_{l-1} = n_l = a$. Now let $x_i = n_i$ for $i = 1, \ldots, l-2$ and set $x_{l-1} = n_{l-1} + n_l = 2a < b$. Then $x_1 \cdots x_{l-1} \leq n_1 \cdots n_l$, a contradiction. This proves (b) and completes the proof.

**Proof of Proposition 2.** Let $t = a_1(G) - nm - n_1 - \cdots - n_l \geq 0$. It is sufficient to prove that there exists a block in $B_1(H \oplus C_n)$ of length not less than $n_1 + \cdots + n_l + n + t$. To do this we consider a block $A = A_1 \cdots A_r \in B_1(G)$ with $|A| = a_1(G) = nm + n_1 + \cdots + n_l + t$, where $A_1, \ldots, A_r$ are irreducible blocks.
By rearranging the indices we may assume that

\[ A = (a_1, \ldots, a_{mn+n_1+\ldots+n_t+t-r}, b_1, \ldots, b_r) \]

with \( b_i \in A_i \) for \( i = 1, \ldots, r \).

We assert that

\( r \leq n_1 + \ldots + n_t \).

Assume \( r > n_1 + \ldots + n_t \). Since it is well known that \( D(H) \geq n_1 + \ldots + n_t - l + 1 \) (see for example [2]), we have \( n \geq D(H) \geq n_1 + \ldots + n_t - l + 1 \). Now by Lemma 9,

\[ |A_1| \ldots |A_r| \geq (nm + n_1 + \ldots + n_t + t - 2r)2^r \]
\[ > (nm + t - n_1 - \ldots - n_t)2^{n_1+\ldots+n_t} \]
\[ \geq ((m-1)n - l + 1)2^{n_1} \ldots 2^{n_t} \]
\[ \geq (m-1)n - l + 1)(2n_1)\ldots(2n_t) \]
\[ \geq mnm_1 \ldots n_t = |G| \]

this contradicts Lemma 2 and proves (1).

It is well known that there exists a homomorphism \( \varphi \) from \( H \oplus C_m \) onto \( H \oplus C_n \) with \( \ker \varphi = C_m \) (up to isomorphism).

For a sequence \( S = (s_1, \ldots, s_u) \) of elements of \( H \oplus C_m \), let \( \varphi(S) \) denote the sequence \( (\varphi(s_1), \ldots, \varphi(s_u)) \) of elements of \( H \oplus C_n \). Since \( nm + n_1 + \ldots + n_t + t - r \geq nm = (m-2)n + 2n \) and \( n \geq D(H) \), by Lemmas 6 and 7 one can find \( m - 1 \) disjoint nonempty subsequences \( B_1, \ldots, B_{m-1} \) of \( (a_1, \ldots, a_{mn+n_1+\ldots+n_t+t-r}) \) with \( \sum \varphi(B_i) = 0 \) for \( i = 1, \ldots, m - 1 \), and \( |B_i| \leq n \) for \( i = 1, \ldots, m - 2 \). Therefore

\[ \sum B_i \in \ker \varphi = C_m \]

for \( i = 1, \ldots, m - 1 \).

Since \( A = A_1 \ldots A_r \) is the unique irreducible factorization of \( A \) and \( b_i \in A_i \) for \( i = 1, \ldots, r \), the sequence \( \sum B_1, \ldots, \sum B_{m-1} \) contains no nonempty zero-sum subsequence, and it follows from Lemma 8 that \( \sum B_1 = \ldots = \sum B_{m-1} = a \) (say) and \( a \) generates \( C_m \).

Let \( A_{i_1}, \ldots, A_{i_v} \) (\( v \geq 0 \)) be all irreducible blocks contained in \( A - B_1 - \ldots - B_{m-2} \). Since \( A \in B_1(G) \), it follows that \( A_{i_1}, \ldots, A_{i_v} \) are disjoint, so one can write

\[ A - B_1 - \ldots - B_{m-2} = A_{i_1} \ldots A_{i_v} B' \]

Then \( B' \) contains no nonempty zero-sum subsequence and

\[ \sum B' = \sum A - \sum B_1 - \ldots - \sum B_{m-2} - \sum A_{i_1} - \ldots - \sum A_{i_v} = 2a. \]

Now we split the proof into steps.
Step 1: $\varphi(B_1), \ldots, \varphi(B_{m-2})$ and $\varphi(A_{i_1}), \ldots, \varphi(A_{i_r})$ are irreducible blocks in $H \oplus C_n$. If for some $i$ with $1 \leq i \leq m-2$, $\varphi(B_i)$ is not an irreducible block in $H \oplus C_n$, then there exist two disjoint nonempty subsequences $B_i', B_i''$ of $B_i$ such that $\sum \varphi(B_i') = \sum \varphi(B_i'') = 0$ (in $H \oplus C_n$) and $B_i = B_i'B_i''$. Then $\sum B_i' \in C_m, \sum B_i'' \in C_m$, and the sequence $\sum B_1, \ldots, \sum B_{i-1}, \sum B_i', \sum B_i'', \sum B_{i+1}, \ldots, \sum B_{m-1}$ contains a nonempty zero-sum subsequence. This contradicts $b_i \in A_i$ for $i = 1, \ldots, r$ and proves $\varphi(B_1), \ldots, \varphi(B_{m-2})$ are irreducible blocks.

If for some $j$, $\varphi(A_{i_j})$ is not an irreducible block in $H \oplus C_n$, then there exist two disjoint nonempty subsequences $A_{i_j}', A_{i_j}''$ of $A_{i_j}$ such that $\sum \varphi(A_{i_j}') = \sum \varphi(A_{i_j}'') = 0$ (in $H \oplus C_n$) and $A_{i_j} = A_{i_j}'A_{i_j}''$. It follows from $A \in B_1(G)$ that $\sum B_1, \ldots, \sum B_{m-2}, \sum A_{i_j}'$ contains no nonempty zero-sum subsequence, so by Lemma 8, $\sum A_{i_j}' = a$, and therefore, $\sum B'A_{i_j}'B_1 \ldots B_{m-3} = 0$. This clearly contradicts $A = A_1 \ldots A_r \in B_1(G)$ and completes the proof of this step.

Step 2: $\varphi(B_1)\varphi(A_{i_1}) \ldots \varphi(A_{i_r}) \in B_1(H \oplus C_n)$. Assume otherwise. Then there exist $B_1' \subseteq B_1, A_{i_j}' \subseteq A_{i_j}, \ldots, A_{i_r}' \subseteq A_{i_r}$ such that $\sum \varphi(B_1') = \sum \varphi(A_{i_j}') = \lambda$ for at least one $j$ with $1 \leq j \leq v$. Therefore, $\sum B_1' = \sum A_{i_1}' \ldots A_{i_r}' \in C_m$, so $\sum (B_1 - B_1')A_{i_1}' \ldots A_{i_r}' \in C_m$. Noting that $m \geq 4, \sum B_2 = a$ and $\sum B' = 2a$, it follows from Lemma 8 that the sequence $\sum (B_1 - B_1')A_{i_1}' \ldots A_{i_r}', \sum B_2, \ldots, \sum B_{m-2}, \sum B'$ contains a nonempty zero-sum subsequence. Clearly, such a subsequence must contain the term $\sum (B_1 - B_1')A_{i_1}' \ldots A_{i_r}'$, contrary to $A \in B_1(G)$.

Step 3: We distinguish two cases.

Case 1: $|B'| \leq 2n$. Then

$$|\varphi(B_1)\varphi(A_{i_1}) \ldots \varphi(A_{i_r})| = |B_1A_{i_1} \ldots A_{i_r}|$$

$$= |A| - |B'| - |B_2| - \ldots - |B_{m-2}|$$

$$\geq |A| - 2n - (m - 3)n \geq n + n_1 + \ldots + n_t,$$

as desired.

Case 2: $|B'| > 2n$. Then $|B'| > n + D(H)$. By Lemma 7, there exists a subsequence $T$ of $B'$ such that $\sum \varphi(T) = 0$ and $|B'| - n \leq |T| < |B'|$. Put $W = B' - T$. Then

$$1 \leq |W| \leq n.$$

Since $a$ generates $C_m$ and $B'$ contains no nonempty zero-sum subsequence,

$$\sum T = fa \text{ with } 1 \leq f \leq m - 1.$$

If $3 \leq f \leq m - 1$, let $A_{u_1}, \ldots, A_{u_m}$ be all irreducible blocks which meet $T$ (i.e. $I_{A_{u_i}} \cap I_T \neq \emptyset$ for $i = 1, \ldots, h$). Since $\sum TB_1 \ldots B_{m-f} = \sum TB_2 \ldots B_{m-f+1} = 0$, it follows from $A = A_1 \ldots A_r \in$
$B_1(G)$ that $B_1 \ldots B_{m-f} = A_{u_1} \ldots A_{u_h} - T = B_2 \ldots B_{m-f+1}$. This contradicts the disjointness of $B_1, \ldots, B_{m-2}$. Hence

$$\sum T = a \text{ or } 2a.$$

But $\sum T + \sum W = 2a$ and $\sum W \neq 0$, so we must have $\sum T = \sum W = a$. Let $T'$ be a nonempty subsequence of $T$ with $\sum \varphi(T') = 0$. Then by using the same method one can prove that $\sum T' = a$. This forces that $T' = T$ and implies that

$$\varphi(T)$$

is an irreducible block in $H \oplus C_n$.

We assert that

$$\varphi(T)\varphi(A_{i_1}) \ldots \varphi(A_{i_v}) \in B_1(H \oplus C_n).$$

Assume to the contrary that there exist $T' \subseteq T, A'_{i_1} \subseteq A_{i_1}, \ldots, A'_{i_v} \subseteq A_{i_v}$ such that $\sum \varphi(T'A'_{i_1} \ldots A'_{i_v}) = 0$ and $A'_{i_j} \neq \lambda$ for some $1 \leq j \leq v$. Then $\sum T'A'_{i_1} \ldots A'_{i_v} \in C_m$. Notice that the sequence $\sum B_1, \ldots, \sum B_{m-2}, \sum W, \sum T'A'_{i_1} \ldots A'_{i_v}$ must contain a nonempty zero-sum subsequence and such a subsequence must contain the term $\sum T'A'_{i_1} \ldots A'_{i_v}$. This clearly contradicts $A = A_1 \ldots A_r \in B_1(G)$ and proves the assertion. Now the theorem follows from $|\varphi(T)\varphi(A_{i_1}) \ldots \varphi(A_{i_v})| = nm + n_1 + \ldots + n_t + t - |B_1| - \ldots - |B_{m-2}| - |W| \geq n + n_1 + \ldots + n_t + t$. This completes the proof.

**Proposition 3.** If $D(C_n^t) = 3n - 2$, then

(i) $a_1(C_n \oplus C_{2n}) \leq a_1(C_n^2) + n$;
(ii) $a_1(C_n \oplus C_{3n}) \leq a_1(C_n^2) + 2n$;
(iii) $a_1(C_{2n}^2) \leq a_1(C_n^3) + 2n$, and
(iv) $a_1(C_{3n}^2) \leq a_1(C_n^3) + 4n$.

**Proof.** Put $H = C_k \oplus C_n$ and $G = C_k \oplus C_{nm}$. It is well known that there exists a homomorphism $\varphi$ from $G$ onto $H$ such that $\ker \varphi = C_l \oplus C_{m'}$ (up to isomorphism). We use the same notation $A = A_1 \ldots A_r \in B_1(G)$, $\varphi$, $\varphi(S)$ as in the proof of Proposition 2.

(i) $k = 1, l = n, m = 2$. Let $t = a_1(C_n \oplus C_{2n}) - 3n$. Clearly, it is sufficient to prove that there exists a block in $B_1(C_n^t)$ of length not less than $2n + t$. If $t = 0$, then the proposition follows from Remark 1, so we may assume that $t \geq 1$, and $r \geq 3$ follows from Lemma 3. We assert that

$$\max\{|A_1|, \ldots, |A_r|\} \geq 2n + t.$$

Otherwise by Lemma 9 we get $|A_1| \ldots |A_r| > (2n + t)n > 2n^2 = |C_n \oplus C_{2n}|$; this contradicts Lemma 2 and proves the assertion. So we may assume that

$$|A_r| \geq 2n + t.$$

By using Lemmas 7 and 4(i) one can find a subsequence $B_1$ of $A_r$ such that $\sum \varphi(B_1) = 0$ and $|A_r| - n \leq |B_1| < |A_r|$. Put $B_2 = A_r - B_1$. Then
\[\sum \varphi(B_2) = 0.\] So \(B_1 \in C_2, B_2 \in C_2,\) and clearly \(B_1 = B_2 = 1.\) It is easy to prove that \(\varphi(B_1), \varphi(B_2), \varphi(A_1), \ldots, \varphi(A_{r-1})\) are all irreducible blocks in \(C_n^r,\) and similarly to the proof of Proposition 2 one can get \(\varphi(B_1), \varphi(A_1), \ldots, \varphi(A_{r-1}) \in B_1(C_n^r).\) Now (i) follows from \(\varphi(B_1), \varphi(A_1), \ldots, \varphi(A_{r-1}) \in B_1(C_n^r).\) Noticing that \(D\) may assume that subsequence \(B_n\) we may assume that \(t = a_1(C_n \oplus C_{3n}) - 4n.\) Similarly to (i) we may assume that \(t \geq 1\) and by Lemma 3 we have \(r \geq 3,\) and similarly to (i) we get max\(|A_1|, \ldots, |A_r|\) \(\geq 3n + t,\) so we may assume that \(|A_r| \geq 3n + t.\) By using Lemmas 4(i), 5, and 7 we get three disjoint subsequences \(B_1, B_2, B_3\) of \(A_r\) such that \(\sum \varphi(B_1) = \sum \varphi(B_2) = \sum \varphi(B_3) = 0\) and \(|B_1| < |A_r - B_1| - n \leq |B_2| < |A_r - B_1|,\) and \(B_3 = A_r - B_1 - B_2.\) Clearly, \(\sum B_1 = \sum B_2 = \sum B_3 = a\) (say) and \(a = 1\) or \(2.\) Now (ii) follows in a similar way to (i).

(iii) \(k = n, l = m = 2.\) Let \(t = a_1(C_{2n}^2) - 4n.\) If \(t = 0,\) then (iii) follows from Remark 1, so we may assume that \(t \geq 1.\) Clearly, it is sufficient to prove that there exists a block in \(B_1(C_{2n}^2)\) of length not less than \(2n + t.\)

Since \(a_1(C_{2n}^2) \geq 4n + 1,\) by Lemmas 3 and 4(i) we have \(r \geq 3.\) If max\(|A_1|, \ldots, |A_r|\) \(< 3n,\) then by Lemma 9 we have \(|A_1| \ldots |A_r| \geq 2(n + 2 - 2)(3n - 1) > 4n^2 = |C_{2n}^2|.\) This contradicts Lemma 2, so we may assume that \(|A_r| \geq 3n,\) and by using Lemmas 6 and 7 we find three disjoint subsequences \(B_1, B_2, B_3\) of \(A_r\) such that \(\sum \varphi(B_1) = \sum \varphi(B_2) = \sum \varphi(B_3) = 0\) and \(|B_1| \leq |A_r - B_1| - n \leq |B_2| < |A_r - B_1|,\) and \(B_3 = A_r - B_1 - B_2.\) Noticing that \(D(C_{2n}^2) = 3\) we can prove (iii) similarly to (i).

(iv) \(k = n, l = m = 3.\) Let \(t = a_1(C_{3n}) - 6n.\) Similarly to (iii) we may assume that \(t \geq 1,\) and \(r \geq 3\) follows from Lemmas 3 and 4(i). Furthermore, we may assume \(n \geq 3\) for otherwise (iv) reduces to (iii). If max\(|A_1|, \ldots, |A_r| < 5n,\) then by Lemma 9 we have \(|A_1| \ldots |A_r| \geq 2(n + 2 - 2)(5n - 1) > 9n^2 = |C_{3n}^3|.\) This contradicts Lemma 2 and proves that max\(|A_1|, \ldots, |A_r| \geq 5n.\) Now (iv) follows in a similar way to (iii) upon noting that \(D(C_{2n}^2) = 5.\) This completes the proof.

**Corollary 1.** If \(a_1(C_n^2) = 2n\) and \(D(C_{2n}^3) = 3n - 2,\) then

(i) \(a_1(C_n \oplus C_{2n}) = 3n;\)

(ii) \(a_1(C_n \oplus C_{3n}) = 4n;\)

(iii) \(a_1(C_{2n}^2) = 4n,\) and

(iv) \(a_1(C_{3n}^3) = 6n.\)

**Proof.** This follows from Remark 1 and Proposition 3.

**Lemma 10 ([2, Theorem (2.8)]).** Let \(p\) be a prime, \(H\) a finite abelian \(p\)-group, and let \(S\) be a sequence of \(D(H) - 2\) elements in \(H.\) Suppose that \(f_k(S) - f_0(S) \neq 0\) (mod \(p\)). Then all elements not in \(\sum(S)\) are contained in a fixed proper coset of a subgroup of \(H.\)
P. van Emde Boas ([2, Theorem (2.8)]) stated the conclusion of Lemma 10 for the case \( f_E(S) = 1 \) and \( f_O(S) = 0 \), but his method does work for the general case. For convenience, we repeat the proof here.

**Proof of Lemma 10.** In the proof we shall use multiplicative notation for \( H \), and in all other cases in this paper, additive notation will be used.

Let \( H = C_{p^e_1} \oplus \ldots \oplus C_{p^e_r} \) with \( 1 \leq e_1 \leq \ldots \leq e_r \), and suppose \( S = (g_1, \ldots, g_k) \), where \( k = D(H) - 2 = -k - 1 + \sum_{i=1}^k p^{e_i} \). Put \( N(S,g) := N_{\text{even}} - N_{\text{odd}} \) where \( N_{\text{even(odd)}} \) is the number of solutions of the equation \( g_1^{m_1} g_2^{m_2} \ldots g_k^{m_k} = g \), \( m_i = 0, 1 \), with \( \sum_{i=1}^k m_i \) even (odd).

We denote by \( F_p \) the \( p \)-element field. We multiply out the product 
\[
(1 - g_1)(1 - g_2)\ldots(1 - g_k)
\]
in the group ring \( F_p[H] \). Then
\[
\prod_{i=1}^k (1 - g_i) = \sum_{g \in H} N(S,g) g.
\]

If \( g^{p^n} = 1 \) (\( g \in H \)), then it is well known that the following equalities hold in \( F_p[H] \):
\[
(1 - g)^{p^n} = 0,
\]
\[
(1 - g)^{p^n-1} = \sum_{v=0}^{p^n-1} g^v,
\]
\[
(1 - g)^{p^n-2} = \sum_{v=1}^{p^n-1} vg^{v-1}.
\]

Let \( x_1, \ldots, x_r \) be a basis for \( H \) where \( x_i \) has order \( p^{e_i} \). Then \( g_i = x_1^{f_{i_1}} \ldots x_r^{f_{i_r}} \), \( 0 \leq f_{ij} \leq p^{e_j} - 1 \), \( i = 1, \ldots, k \), \( j = 1, \ldots, r \). Now, we have 
\[
\prod_{i=1}^k (1 - g_i) = \prod_{i=1}^k (1 - x_1^{f_{i_1} \ldots x_r^{f_{i_r}}})
\]
\[
= \prod_{i=1}^k (1 - (1 - x_1)^{f_{i_1}} \ldots (1 - x_r)^{f_{i_r}})
\]
\[
= \prod_{i=1}^k \sum_{j=1}^r (f_{ij}(1 - x_j) + h_{ij}(1 - x_j)^2 + \alpha_{ij}(1 - x_j)^3),
\]
where \( h_{ij} = \frac{1}{2}(f_{ij} - 1)f_{ij} \) and \( \alpha_{ij} \in F_p[H] \). Now from (3) and \( k = -1 + \sum_{i=1}^{r} (p^{\sigma_i} - 1) \) we derive that
\[
\prod_{i=1}^{k} (1 - g_i) = \prod_{i=1}^{k} \sum_{j=1}^{r} (f_{ij}(1 - x_j) + h_{ij}(1 - x_j)^2),
\]
and it follows from (3)–(5) that
\[
(6) \quad \prod_{i=1}^{k} (1 - g_i) = c_0 \prod_{i=1}^{r} \sum_{j=0}^{x_i - 1} x_j^i + \sum_{i=1}^{r} c_i \left( \sum_{v=1}^{p^{\sigma_i} - 1} x_v^i \right) \prod_{j=1}^{r} \sum_{j \neq i}^{x_j} x_j^i
\]
where \( c_i \in F_p \).

For every \( g \in H \), write \( g = x_1^{\tau_1(g)} \ldots x_r^{\tau_r(g)} \). Then from (6) we derive that
\[
\prod_{i=1}^{k} (1 - g_i) = \sum_{g \in H} (c_0 + c_1 (\tau_1(g) + 1) + \ldots + c_r (\tau_r(g) + 1))g.
\]
This together with (2) implies
\[
N(S, g) = \sum_{i=1}^{r} c_i \tau_i(g) + \sum_{i=0}^{r} c_i.
\]
Now by the hypothesis of the lemma we have
\[
\sum_{i=0}^{r} c_i = N(S, 1) = f_E(S) - f_O(S) \neq 0 \quad \text{in } F_p.
\]
It follows that all elements \( g \) not in \( \sum(S) \) satisfy the equation
\[
\sum_{i=1}^{r} c_i \tau_i(g) = - \sum_{i=0}^{r} c_i \neq 0,
\]
and this equation defines a proper coset. This completes the proof.

**Lemma 11.** Let \( p \) be an odd prime, and let \( A = A_1 \ldots A_r \in B_1(C_p^2) \) with \( A_1, \ldots, A_r \) irreducible blocks. Suppose that \( |A| = 2p + t \) and \( t \geq 1 \). Then at least \( 4 + t \) of \( |A_1|, \ldots, |A_r| \) are odd.

**Proof.** Suppose that exactly \( l \) of \( |A_1|, \ldots, |A_r| \) are odd. Then \( l \geq 2 + t \) follows from Proposition 1 and Lemma 4(iv).

Assume the conclusion of the lemma is false. Then \( l = 2 + t \) follows from the obvious fact \( l \equiv 2p + t \equiv t \pmod{2} \). Without loss of generality, we may assume that \( |A_1|, \ldots, |A_{2+t}| \) are odd and that \( |A_{3+t}|, \ldots, |A_r| \) are even. We next show that
\[
p \mid |A_1|.
\]
We fix $a_i \in A_i$ for $i = 1, \ldots, 2 + t$, take any $x \in A_1 - (a_1)$, and set

$$S = (A_1 - (a_1)) x (A_2 - (a_2)) \cdots (A_{2+t} - (a_{2+t})) A_{3+t} \cdots A_r.$$ 

Clearly, $f_E(S) = 2r - 2t$, $f_O(S) = 0$, $|S| = 2p - 3 = D(C_p^2) - 2$, and

$$\{-a_1, -a_1 - a_2, \ldots, -a_1 - a_2 + t, -x, -x - a_2, \ldots, -x - a_2 + t\} \cap \sum(S) = \emptyset.$$ 

Now it follows from Lemma 10 that there exist a subgroup $H$ of $C_p^2$ and an element $g \in C_p^2 - H$ such that

$$\{-a_1, -a_1 - a_2, \ldots, -a_1 - a_2 + t, -x, -x - a_2, \ldots, -x - a_2 + t\} \subseteq g + H.$$ 

This implies that $x - a_1 = (-a_1) - (-x) \in H$, $a_2 = (-a_1) - (-a_1 - a_2) \in H$, so we have $H = \langle a_2 \rangle$. Since $x$ was arbitrary, any element of $A_1$ is in $a_1 + H = g + H$. Now $|A_1|(g + H) = 0$ (in $C_p^2/H$) follows from $\sum A_1 = 0$; but $g + H \neq 0$ (in $C_p^2/H$), hence, $p | A_1$. Similarly, one can prove that $p | |A_2|, \ldots, p | |A_2| + t$. This yields $|A| \geq |A_1| + \ldots + |A_{2+t}| \geq (2 + t)p > 2p + t$, a contradiction. This completes the proof.

**Lemma 12.** Let $p$ be a prime with $2 \leq p \leq 151$. Then $a_1(C_p^2) = 2p$.

**Proof.** We may assume that $p \geq 5$; for $p \leq 3$ see [9].

Assume to the contrary that $a_1(C_p^2) \neq 2p$. Then one can find a block $A = A_1 \cdots A_r \in B_1(C_p^2)$ with $|A| = 2p + t$ and $t \geq 1$, where $A_1, \ldots, A_r$ are irreducible blocks. Suppose exactly $l$ of $|A_1|, \ldots, |A_r|$ are odd. Then $l \geq 4 + t$ follows from Lemma 11.

If $p = 5$, then $2 \times 5 + t = |A| \geq 3l \geq 3(4 + t) = 10 + t$, a contradiction. Hence, $7 \leq p \leq 151$ and it follows from $l \geq 4 + t \geq 5$ that $|A_1| \cdots |A_r| \geq 3^l(2p + 1 - 12) = 162(p - 5.5) > p^2$, a contradiction to Lemma 2. This completes the proof.

**Lemma 13.** $a_1(C_p^2) = 2 \times 5^s$.

**Proof.** We proceed by induction on $s$. If $s = 1$, then the assertion follows from Lemma 12.

Taking $s \geq 2$ we assume that the lemma is true for $s - 1$. Assume to the contrary that $a_1(C_p^2) \neq 2 \times 5^s$. Then one can find a block $A = A_1 \cdots A_r \in B_1(C_p^2)$ with $|A| = 2 \times 5^s + t$ and $t \geq 1$, where $A_1, \ldots, A_r$ are irreducible blocks. By Proposition 1, at least three of $|A_1|, \ldots, |A_r|$ are odd. If $\max\{|A_1|, \ldots, |A_r|\} < 9 \times 5^{s-1}$, then by Lemma 9 we have $|A_1| \cdots |A_r| \geq 3 \times (5^{s-1} - 1)(9 \times 5^{s-1} - 1) > (5^s)^2 = |C_p^2|$. This contradicts Lemma 2 and shows that $\max\{|A_1|, \ldots, |A_r|\} \geq 9 \times 5^{s-1}$. Note $D(C_p^2) = 9$ and similarly to the proof of Proposition 3 one can derive a contradiction. So we complete the proof.

**Proof of Theorem 1.** Obviously, (1)–(7) follow from Corollary 1, Lemma 12, Lemma 13, Lemma 4 and Proposition 2. So to prove the theorem we only need to consider (8)–(12).
We assert that \( B_D \) of \( B \) and \( B = 1 \) for \( j \geq 1 \), where \( A_1, \ldots, A_r \) are irreducible blocks. It follows from Lemma 3 that \( r \geq n + 3 \) and this implies that \( |A_1| \cdots |A_r| \geq 2^{n+2}(2^m + 1) > |C_3^0 \oplus C_4 \oplus C_2^m| \), a contradiction to Lemma 2.

(9) follows from Proposition 2, Lemma 4 and the conclusion of (8).

(10) As in (8) we only consider the case of \( t = 1 \). Assume to the contrary that \( a_1(C_3^0 \oplus C_6 \oplus C_2^m) \neq 3n + 9 + 3^m \). Then one can find a block \( A = A_1 \cdots A_r \in B_1(C_3^0 \oplus C_6 \oplus C_2^m) \) with \( |A| = 3n + 9 + 3^m + t \) and \( t \geq 1 \), where \( A_1, \ldots, A_r \) are irreducible blocks. It follows from Proposition 1 that at least \( n + 3 \) of \( |A_1|, \ldots, |A_r| \) are odd. This implies that \( |A_1| \cdots |A_r| \geq 3^{n+3}(3^m + 1) > |C_3^0 \oplus C_6 \oplus C_2^m| \), a contradiction to Lemma 2.

(11) follows from Proposition 2, Lemma 4 and the conclusion of (10).

(12) The proof is similar to that of (10) and we omit it here. Now the proof is complete.

3. In this section we consider \( a_k(G) \) with \( k \geq 2 \).

Proposition 4. Let \( B \in B_2(G) - B_1(G) \), and let \( B = \prod_{i=1}^{t} B_i \), \( i = 1, 2 \), be the two strongly inequivalent irreducible factorizations of \( B \), where \( B_j, 1 \leq i \leq 2, 1 \leq j \leq r_i \), are all irreducible blocks. Then

\[
|B| \leq \max\{r_1, r_2\} + D(G) - 1.
\]

Proof. Suppose \( r_1 \geq r_2 \) and \( B = (b_1, \ldots, b_k) \). Put \( E_j = I_{B_j} \) for \( j = 1, \ldots, r_1 \) and \( F_j = I_{B_{r_1+j}} \) for \( j = 1, \ldots, r_2 \). We have \( B_{r_1+j} = (b_i : i \in E_j) \) and \( B_{r_1+j} = (b_i : i \in F_j) \).

For \( j = 1, \ldots, r_2 \), we define \( D_j \) to be the set \( \{i : E_i \cap F_j \neq \emptyset, 1 \leq i \leq r_1\} \).

We assert that \( D_1, \ldots, D_{r_2} \) has a system of distinct representatives.

Deny the assertion; by Hall’s Theorem ([5], p. 45) there exists a nonempty subset \( \{i_1, \ldots, i_t\} \) of \( \{1, \ldots, r_2\} \) such that

\[
|D_{i_1} \cup \cdots \cup D_{i_t}| < t.
\]

Suppose \( D_{i_1} \cup \cdots \cup D_{i_t} = \{f_1, \ldots, f_m\} \). Then \( m < t \). By the definition of \( D_j, 1 \leq j \leq r_2 \), we have

\[
F_1 \cup \ldots \cup F_t \subseteq E_{f_1} \cup \ldots \cup E_{f_m}.
\]

Set \( E = (E_{f_1} \cup \ldots \cup E_{f_m}) - (F_1 \cup \ldots \cup F_t) \) and \( B_0 = (b_i : i \in E) \). Clearly, \( B_0 \) is a block or the empty sequence, and we have

\[
B = B_0 B_{i_1} \cdots B_{i_t} \prod_{l \neq f_1, \ldots, f_m} B_{i_l}.
\]
This implies that $B$ can be factored into a product of at least $r_1 - m + t > r_1$ irreducible blocks. Obviously, such an irreducible factorization is not strongly equivalent to $B = \prod_{i=1}^{r_1} B_{i_1}$ or $B = \prod_{i=2}^{r_2} B_{i_2}$, a contradiction to $B \in B_2(G)$. This proves the assertion.

Let $\{s_1, \ldots, s_{r_1}\}$ be a system of distinct representatives of $D_1, \ldots, D_{r_2}$. Then $F_j \cap E_{s_j} \neq \emptyset$, $j = 1, \ldots, r_2$. Take $u_i \in E_i$ for $i = 1, \ldots, r_1$ so that $u_{s_{ij}} \in F_j \cap E_{s_{ij}}$ for $j = 1, \ldots, r_2$. Put $M = \{1, \ldots, k\} - \{u_1, \ldots, u_{r_1}\}$. Clearly, no nonempty subset of $M$ can be expressed as a union of some $E_j$, or as a union of some $F_j$. This implies that for any nonempty subset $W$ of $M$, the sequence $(b_i : i \in W)$ is not a block, so $|M| \leq D(G) - 1$ and $|B| = |M| + r_1 \leq r_1 + D(G) - 1$. This completes the proof.

**Corollary 2.** $a_2(C^n_2) = 2n$.

**Proof.** Since it is proved in [9] that $a_1(C^n_2) = 2n$, we have $a_2(C^n_2) \geq a_1(C^n_2) = 2n$.

To prove the upper bound we consider any $B \in B_2(C^n_2)$ and show that $|B| \leq 2n$.

If $B \in B_1(C^n_2)$, the estimate is trivial.

If $B \in B_2(C^n_2) - B_1(C^n_2)$, suppose $B = \prod_{i=1}^{r_1} B_{i_1}, i = 1, 2$, are the two strongly inequivalent irreducible factorizations of $B$, where $B_{i_1}, 1 \leq i \leq 2, 1 \leq j \leq r_1$, are irreducible blocks. We assume without loss of generality that $r_1 \leq r_2$. It follows from Proposition 4 that $D(C^n_2) + r_1 - 1 \geq |B| = \sum_{i=1}^{r_1} |B_{i_1}| \geq 2r_1$, thus, $r_1 \leq D(C^n_2) - 1$, and $|B| \leq 2(D(C^n_2) - 1) = 2n$ by Lemma 4(iv). This completes the proof.

**Lemma 14.** Let $B \in B_k(G) - B_{k-1}(G)$ with $k \geq 2$, and let $B = \prod_{j=1}^{r_1} B_{j_1}, i = 1, \ldots, k$, be the $k$ strongly inequivalent irreducible factorizations of $B$, where $B_{j_1}, 1 \leq i \leq k, 1 \leq j \leq r_1$, are irreducible blocks. Suppose that $r_1 = \max\{r_1, \ldots, r_k\} \geq k$. Then there exists a subset $X$ of $\{1, \ldots, r_1\}$ such that $\prod_{j \in X} B_{j_1} \in B_1(G)$ and $|X| \geq r_1 - k + 1$.

**Proof.** Clearly, for any $i = 2, \ldots, k$ there exists an $f = f(i)$ such that $I_{B_{j_1}} \neq I_{B_{i_1}}$ for any $t = 1, \ldots, r_1$. Put $Y = \bigcup_{2 \leq i \leq k} \{f(i)\}$. Then $|Y| \leq k - 1$. Set $X = \{1, \ldots, r_1\} - Y$. Clearly, $\prod_{j \in X} B_{j_1} \in B_1(G)$ and $|X| \geq r_1 - k + 1$. This completes the proof.

**Lemma 15.** Let $G$ be a finite abelian group of order $n$, let $B \in B_k(G) - B_{k-1}(G)$ with $k \geq 2$, and let $B = \prod_{j=1}^{r_1} B_{j_1}, i = 1, \ldots, k$, be the $k$ strongly inequivalent irreducible factorizations of $B$, where $B_{j_1}, 1 \leq i \leq k, 1 \leq j \leq r_1$, are irreducible blocks. Then

$$\max\{r_1, \ldots, r_k\} \leq k - 1 + \log_2 n.$$

**Proof.** Without loss of generality, assume that $r_1 = \max\{r_1, \ldots, r_k\} \geq k$. By using Lemma 14 one can find a subset $X$ of $\{1, \ldots, r_1\}$ such
that \( \prod_{j \in X} B_{1_j} \in B_1(G) \) and \( |X| \geq r_1 - k + 1 \). Now \( \prod_{j \in X} |B_{1_j}| \leq n \) follows from Lemma 2. Note that all \( |B_{1_j}| \geq 2 \), we have \( |X| \leq \log_2 n \), and \( r_1 \leq k - 1 + \log_2 n \) follows. This completes the proof.

**Proof of Theorem 2.** Assume to the contrary that \( a_k(C_n) \neq n \). Since \( a_k(C_n) \geq a_{k-1}(C_n) \geq \ldots \geq a_1(C_n) = n \), we have \( a_k(C_n) = n + 1 + t \) for some \( t \geq 0 \). Let \( B \in B_k(C_n) \) with \( |B| = n + 1 + t \). Since \( a_1(C_n) = n \), we must have \( B \in B_m(C_n) - B_{m-1}(C_n) \) for some \( 2 \leq m \leq k \). Let \( B = \prod_{j=1}^{n} B_{1_j}, 1 \leq i \leq m \), be the \( m \) strongly inequivalent irreducible factorizations of \( B \), where \( B_{1_j}, 1 \leq i \leq m, 1 \leq j \leq r_i \), are irreducible blocks.

Suppose \( B = (b_1, \ldots, b_r) \). Put \( E_{1_j} = I_{B_{1_j}} \) for \( i = 1, \ldots, m \) and \( j = 1, \ldots, r_i \). For \( j = 1, \ldots, r_2 \), we define \( D_j \) to be the set \( \{ t : E_{1_i} \cup E_{2_j} \neq \emptyset, 1 \leq t \leq r_1 \} \). Similarly to the proof of Proposition 4 one can show that \( D_1, \ldots, D_{r_2} \) has a system of distinct representatives. Therefore one can find an \( r_1 \)-subset of \( \{1, \ldots, s\} \) which meets all \( E_{1_j} \) and all \( E_{2_j} \). Hence, one can find an \( (r_1 + r_2 + \ldots + r_k) \)-subset \( I \) of \( \{1, \ldots, s\} \) such that \( I \cap E_{i_j} \neq \emptyset \) for \( i = 1, \ldots, m \) and \( j = 1, \ldots, r_i \). Put \( J = \{1, \ldots, s\} - I \) and let \( T \) be the subsequence of \( B \) with \( I_T = J \). Clearly, \( T \) contains no nonempty zero-sum subsequence. Put \( l = n - |T| \). Notice that

\[
\begin{align*}
l &= n - |T| = n - |J| = n - (n + 1 + t - |I|) \leq |I| - 1 \\
&= r_1 + r_3 + \ldots + r_m - 1 \leq (m - 1)r_1 - 1 \\
&\leq (m - 1)(m - 1 + \log_2 n) - 1 \quad \text{(by Lemma 15)} \\
&\leq (k - 1)(k - 1 + \log_2 n) \leq n/4 \quad \text{(by the hypothesis of the theorem)},
\end{align*}
\]

so by using Lemma 8 we see that, \( T \) contains an \( (n - 2l + 1) \)-subsequence which is similar to the sequence \((1, \ldots, 1)\). Therefore, \( B \) contains an \( (n - 2l + 1) \)-subsequence which is similar to the sequence \((1, \ldots, 1)\): without loss of generality, we may assume that

\[
B = (1, \ldots, 1, x_1, \ldots, x_{t+2l}).
\]

If \( |x_i| \geq 2l \), since \((1, \ldots, 1, x_i)\) is an irreducible block and

\[
\left( \frac{n - 2l + 1}{n - |x_i|} \right) \geq n - 2l + 1 \geq n/2 + 1 > k
\]

(from the hypothesis of the theorem), we must have \( B \notin B_k(C_n) \), a contradiction. Hence,
1 ≤ |x_i|_n ≤ 2l - 1
for i = 1, ..., t + 2l, and so 2 ≤ |x_1|_n + |x_2|_n ≤ 4l - 2 ≤ n - 2, hence,
2 ≤ |x_1 + x_2|_n = |x_1|_n + |x_2|_n ≤ n - 2.
If |x_1 + x_2|_n ≥ 2l, since \( (1, \ldots, 1, x_1, x_2) \) is an irreducible block and
\[
\binom{n - 2l + 1}{n - |x_1 + x_2|_n} \geq n - 2l + 1 > k,
\]
we have \( B \notin B_k(G) \), a contradiction. Hence, \(|x_1|_n + |x_2|_n = |x_1 + x_2|_n \leq 2l - 1 \). Continuing the same process we finally get
\[
\sum_{i=1}^{2l+t} |x_i|_n = \sum_{i=1}^{2l+t} x_i |_n \leq 2l - 1;
\]
but
\[
\sum_{i=1}^{2l+t} |x_i|_n \geq 2l + t \geq 2l,
\]
a contradiction. This completes the proof.

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Department of Information Engineering
Beijing University of Posts and Telecommunications
Beijing 100088, China
E-mail: zmhu@bupt.edu.cn

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