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SPECTRAL PROPERTIES OF SKEW-PRODUCT DIFFEOMORPHISMS OF TORI

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Introduction. Let $\alpha = (\alpha_1, \ldots, \alpha_d)$ be a d-tuple of real numbers and M be a $d' \times d$ matrix with integer entries. For every $\mathbb{Z}^d$-periodic measurable mapping $F : \mathbb{R}^d \to \mathbb{R}^{d'}$ we define a skew-product trasformation of $\mathbb{T}^d \times \mathbb{T}^{d'}$ into itself given by the formula

$$T(x, y) = (x + \alpha, y + Mx + F(x)),$$

where the addition is modulo $\mathbb{Z}^d$ and the variables are treated as column vectors. It is clear that T is an invertible measure preserving transformation of $\mathbb{T}^d \times \mathbb{T}^{d'}$ endowed with Lebesgue measure. Moreover, if F is continuous then T is a homeomorphism which is homotopic to the identity transformation (or equivalently $\phi(x) = Mx + F(x)$ is homotopic to zero) if and only if $M = 0$. It is also obvious that T becomes a C^r diffeomorphism if F is C^r. The mapping $\phi(x) = Mx + F(x)$ will be referred to as a *cocycle.*

If $d = d' = 1$ and α is irrational, T reduces to the well-known Anzai skew-product extension of the irrational rotation [A]. Ergodic properties of such transformations have been studied by many authors and are fairly well understood. The aim of the present note is to extend some of these results to the multidimensional case.

In the case of $M = 0$ it was shown in [I2, I3] that if an irrational number α admits a sufficiently good approximation by rationals then for every $r = 1, 2, \ldots, \infty$ and "most" cocycles F in $C^r(\mathbb{T})$ (and in more general spaces of functions) the Anzai skew product defined on the 2-torus $\mathbb{T} \times \mathbb{T}$ admits a good cyclic approximation by periodic transformations and has partly continuous spectrum. In fact, the cocycle is weakly mixing, which means that the only eigenfunctions are of the form $h(x, y) = C \exp(2\pi i n x)$.

We are going to show that analogous results hold true for any $d, d' \in \mathbb{N}$ (Thm. 1 where the cyclic approximation is replaced by a weaker kind of periodic approximation, Thm. 1$'$, and Thm. 2).

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In the case of $M \neq 0$ and $d = d' = 1, f \in C^2(\mathbb{T})$, it is known that $\phi(x) = mx + f(x)$ is always a weakly mixing cocycle and moreover T has infinite Lebesgue spectrum on the orthocomplement $L^2(dx)^\perp$ of the space of L^2 functions $h(x,y) = h(x)$ depending only on $x \in \mathbb{T}^d$ (see [K, C, ILR, I4]). We will prove a similar result for any d, d', where the smoothness of F is expressed in terms of its Fourier coefficients as in [I4]. This extends some recent results of Frączek [F], where a different technique is used. As a by-product we obtain examples of ergodic skew-product diffeomorphisms with maximal spectral measure of mixed type: partly discrete, partly continuous singular, and partly Lebesgue.

1. Weakly mixing cocycles. If S is an invertible measure preserving ergodic transformation of a probability space (X, μ) and G is a compact metrizable abelian group then a measurable mapping $\phi : X \to G$ is called a *weakly mixing cocycle* if given any nontrivial continuous character γ of G and any $\lambda \in \mathbb{T}$ there is no measurable function $\psi : X \to \mathbb{T}$ with

$$\gamma(\phi(x)) = \lambda\psi(Sx)/\psi(x) \quad \text{a.e.}$$

It is not difficult to see that ϕ is weakly mixing iff the corresponding skew-product transformation of $X \times G$ given by the formula $(x, y) \to (Sx, y+\phi(x))$ has no eigenfunctions in the orthocomplement $L^2(dx)^\perp$ of the functions depending only on $x \in X$.

Let (X, μ) be a standard Lebesgue space. We denote by $\Phi(X, G)$ the space of all measurable cocycles $\phi : X \to G$ endowed with the topology of convergence in measure, where cocycles that are equal a.e. are identified. Extending some earlier results of Jones and Parry [JP], Thm. 8, it was shown in [IS] that the set of weakly mixing cocycles is residual in $\Phi(X, G)$. (It should be noted that the definition of weakly mixing cocycle given in [IS] is correct only for connected groups G; a correct reasoning in the general case is given in the proof below.) Actually, we have the following sharper result, whose proof is based on an idea of Baggett [B] (cf. [IS], Thm. 4).

PROPOSITION. *Let S be an ergodic invertible measure preserving tranformation of a standard Lebesgue space (X, μ) and G be a compact metrizable abelian group. Then the weakly mixing cocycles form a dense G_δ subset of $\Phi(X, G)$.*

Proof. We first prove the G_δ-ness. In the proof it will be covenient to identify $\mathbb{T}$ with the circle group in the complex plane. Let D be a countable linearly dense subset of the complex space $L^2(X, \mu)$. It suffices to show that for every $\gamma \in \widehat{G} \setminus \{1\}, h \in D$, and $k \in \mathbb{N}$ the set $\Phi^\gamma_{h,k}$ of all cocycles

$\phi \in \Phi(X,G)$ such that

$$\exists\lambda \in \mathbb{T}\ \exists\psi \in \Phi(X,\mathbb{T}) \quad \gamma\circ\phi = \lambda\psi\circ S/\psi, \quad \Big|\int \psi h\, d\mu\Big| \ge 1/k$$

is closed in $\Phi(X,G)$ (indeed, the union of all such sets is exactly the family of cocycles that are not weakly mixing). Suppose $\phi_n \in \Phi^{\gamma}_{h,k}$ and $\phi_n \to \phi$ in measure. Choose appropriate λ_n and ψ_n for each ϕ_n. By the weak compactness of the unit ball in $L^2(X,\mu)$ there exists a subsequence $\psi_{n'} \to \widetilde{\psi}$ weakly in L^2. By passing to a further subsequence we may assume $\lambda_{n'} \to \lambda$ in $\mathbb{T}$. Observe that

$$(\gamma\circ\phi_{n'})\psi_{n'} - (\gamma\circ\phi)\widetilde{\psi} = (\gamma\circ\phi_{n'} - \gamma\circ\phi)\psi_{n'} + (\psi_{n'} - \widetilde{\psi})\gamma\circ\phi \to 0$$

weakly in L^2. We also have $\lambda_{n'}\psi_{n'}\circ S \to \lambda\widetilde{\psi}\circ S$ hence

$$(\gamma\circ\phi)\widetilde{\psi} = \lambda\widetilde{\psi}\circ S.$$

Clearly $|\int \widetilde{\psi} h\, d\mu| \ge 1/k$ so $\widetilde{\psi} \ne 0$. Moreover, $|\widetilde{\psi}| = |\widetilde{\psi}\circ S|$, so $|\widetilde{\psi}|$ is constant by ergodicity. Letting $\psi = \widetilde{\psi}/|\widetilde{\psi}|$ we get $|\int \psi h\, d\mu| \ge 1/k$ and $\gamma\circ\phi = \lambda\psi\circ S/\psi$ so $\phi \in \Phi^{\gamma}_{h,k}$ as required.

In order to show that the set of weakly mixing cocycles is dense, we will use the fact that for every nontrivial character γ of G the set

$$\Phi_\gamma = \{\phi \in \Phi(X,G) : \exists\lambda \in \gamma(G)\ \exists\psi \in \Phi(X,\gamma(G))\ \gamma\circ\phi = \lambda\psi\circ S/\psi\}$$

is of the first category ([IS], p. 72). What we need to prove is that

$$\widetilde{\Phi}_\gamma = \{\phi \in \Phi(X,G) : \exists\widetilde{\lambda} \in \mathbb{T}\ \exists\widetilde{\psi} \in \Phi(X,\mathbb{T})\ \gamma\circ\phi = \widetilde{\lambda}\widetilde{\psi}\circ S/\widetilde{\psi}\}$$

is of the first category. If G is connected then $\gamma(G) = \mathbb{T}$, the two sets coincide and there is nothing to prove. Without loss of generality we may now assume that $\gamma(G)$ is a cyclic subgroup of order N in $\mathbb{T}$. Denote by $h_1, h_2, \ldots$ a maximal orthonormal system of eigenfunctions for S. If $\phi, \widetilde{\psi}$, and $\widetilde{\lambda}$ are as in the definition of $\widetilde{\Phi}_\gamma$ then

$$1 = (\gamma\circ\phi)^N = \widetilde{\lambda}^N(\widetilde{\psi}\circ S)^N/\widetilde{\psi}^N,$$

which implies that $\widetilde{\psi}^N$ is an eigenfunction for S. For some $k \in \mathbb{N}$ and $c \in \mathbb{R}$ we have

$$\widetilde{\psi}^N(x) = e^{2\pi i c} h_k(x) = e^{2\pi i(g_k(x)+c)},$$

where $g_k(x)$ is a real-valued function with

$$g_k(Sx) = g_k(x) + \beta_k + m_k(x)$$

for some $\beta_k \in \mathbb{R}$ and $m_k(x)$ a measurable integer-valued function on X. One gets

$$\widetilde{\psi}(x) = e^{2\pi i(g_k(x)+n(x)+c)/N}$$

for another integer-valued function $n(x)$. Consequently,

$$\begin{aligned}\gamma(\phi(x)) &= \widetilde{\lambda}\widetilde{\psi}(Sx)/\widetilde{\psi}(x)\\ &= \widetilde{\lambda}e^{2\pi i\beta_k/N}\frac{e^{2\pi in(Sx)/N}}{e^{2\pi in(x)/N}}e^{2\pi im_k(x)/N}\\ &= e^{2\pi im_k(x)/N}\lambda\psi(Sx)/\psi(x),\end{aligned}$$

where ψ and λ are as in Φ_γ. For every eigenfunction h_k and every γ as above we may choose a measurable function $\eta_{\gamma,k}: X \to G$ such that

$$\gamma(\eta_{\gamma,k}(x)) = e^{2\pi im_k(x)/N}.$$

It is now clear that $\phi\eta^{-1}_{\gamma,k} \in \Phi_\gamma$ or $\phi \in \eta_{\gamma,k}\Phi_\gamma$. Since ϕ was arbitrary in $\widetilde{\Phi}_\gamma$, we conclude that $\widetilde{\Phi}_\gamma$ is covered by a countable union of translates of first category sets, so it is first category itself, which ends the proof of the proposition.

In order to prove that the weakly mixing cocycles form a G_δ set also in certain subspaces $E \subset \Phi(X,G)$, where E is endowed with a topology stonger than that of convergence in measure, it will now suffice to remark that the intersection of any G_δ set with E is a G_δ in E.

In proofs of the existence (and denseness) of weakly mixing cocycles in various subspaces the following simple principle may be useful.

Let $1, \alpha_1, \ldots, \alpha_d$ be rationally independent and suppose $\beta = \sum l_k\alpha_k$, where $(l_1,\ldots,l_d) \in \mathbb{Z}^d \setminus \{0\}$. If $\phi \in \Phi(\mathbb{T},\mathbb{T})$ is weakly mixing over the β rotation then $\widetilde{\phi}(x_1,\ldots,x_d) = \phi(\sum l_kx_k)$ is weakly mixing over α.

This statement can be checked directly. It is also a special case of the following lemma suggested to the author by M. Lemańczyk.

LEMMA 1. *Let X be a compact metrizable monothetic group with ergodic rotation α and h be a continuous homomorphism of X onto another compact metrizable group Y. If $\phi \in \Phi(Y,\mathbb{T})$ is weakly mixing over $\beta = h(\alpha)$ then $\widetilde{\phi} = \phi \circ h \in \Phi(X,\mathbb{T})$ is weakly mixing over α.*

Proof. Denote by $\mu_y, y \in Y$, the canonical system of conditional measures concentrated on the fibers $h^{-1}(y)$. For every bounded Borel complex-valued function f on X one has

$$\int_X f(x)\,dx = \int_Y\int f(x)\,d\mu_y(x)\,dy$$

and, for almost every y,

$$\int f(x+\alpha)\,d\mu_y(x) = \int f(x)\,d\mu_{y+\beta}(x).$$

The latter formula is a consequence of the invariance of the Haar measure on Y and can be checked directly by integrating against an arbitrary bounded Borel function $g(y)$.

In order to prove that $\widetilde{\phi}$ is weakly mixing suppose, to the contrary, that for some $m \neq 0$ and $|\lambda| = 1$,

$$\widetilde{\phi}(x)^m = \lambda\psi(x+\alpha)/\psi(x),$$

where $\psi : X \to \mathbb{T}$ is a measurable mapping. We then have

$$\phi(h(x))^m\psi(x) = \lambda\psi(x+\alpha).$$

On the other hand, there exists a character $\chi \in \widehat{X}$ such that

$$\int \psi(x)\chi(x)\,dx \neq 0.$$

By multiplying the last equality by $\chi(x)$ and integrating with respect to μ_y we get

$$\begin{aligned}\phi(y)^m \int \chi(x)\psi(x)\,d\mu_y(x) &= \lambda\overline{\chi}(\alpha)\int \chi(x+\alpha)\psi(x+\alpha)\,d\mu_y(x)\\ &= \lambda' \int \chi(x)\psi(x)\,d\mu_{y+\beta}(x),\end{aligned}$$

where $|\lambda'| = 1$ and $\eta(y) = \int \chi(x)\psi(x)d\mu_y(x)$ does not vanish a.e. on Y. By ergodicity, $|\eta(y)|$ is constant a.e. on Y so without loss of generality we may assume $\eta \in \Phi(Y,\mathbb{T})$. Consequently,

$$\phi(y)^m = \lambda'\eta(x+\beta)/\eta(x),$$

which contradicts the weak mixing of ϕ.

Let α be an ergodic rotation of $\mathbb{T}^d$. Then it follows directly from the definition of weak mixing that for every $d' \in \mathbb{N}$ a cocycle $\phi : \mathbb{T}^d \to \mathbb{T}^{d'}$ is weakly mixing over α iff $\chi \circ \phi$ is weakly mixing over α in $\Phi(\mathbb{T}^d,\mathbb{T})$ for every nontrivial character χ of $\mathbb{T}^{d'}$. The mapping $\phi \to \chi\circ\phi$ is a continuous open homomorphism of $\Phi(\mathbb{T}^d,\mathbb{T}^{d'})$ onto $\Phi(\mathbb{T}^d,\mathbb{T})$ so the inverse image of a dense G_δ (residual) set is a dense G_δ (residual) set in $\Phi(\mathbb{T}^d,\mathbb{T}^{d'})$. We may use this remark along with Lemma 1, the Proposition, and results from [I2, I3] to prove that weakly mixing cocycles are generic in some subspaces of $\Phi(\mathbb{T}^d,\mathbb{T}^{d'})$. As a sample result of this type we prove that "most" cocycles among those defining skew-product C^∞ diffeomorphisms which are homotopic to the identity are weakly mixing cocycles. To this end denote by $\Phi_0^\infty(\mathbb{T}^d,\mathbb{T}^{d'})$ the set of cocycles of the form $F(x) = (f_1(x),\ldots,f_{d'}(x))$ modulo $\mathbb{Z}^{d'}$, where each of the functions $f_k(x_1,\ldots,x_d)$ is C^∞ and 1-periodic in all its variables. The space $\Phi_0^\infty(\mathbb{T}^d,\mathbb{T}^{d'})$ will be endowed with its usual C^∞ topology.

COROLLARY 1. *Let $d, d' \in \mathbb{N}$. There exists a residual subset A of $\mathbb{R}^d$ such that for every $\alpha \in A$ the set of weakly mixing cocycles in $\Phi_0^\infty(\mathbb{T}^d,\mathbb{T}^{d'})$ is a dense G_δ in $\Phi_0^\infty(\mathbb{T}^d,\mathbb{T}^{d'})$.*

Proof. Let A be the set of vectors $\alpha = (\alpha_1,\ldots,\alpha_d)$ such that the numbers $1,\alpha_1,\ldots,\alpha_d$ are rationally independent and α_1 is a Liouville number.

By [I2], Thm. 2, the weakly mixing cocycles over α_1 form a dense G_δ set in $\Phi_0^\infty(\mathbb{T},\mathbb{T})$. It therefore follows from Lemma 1 that there exist weakly mixing cocycles over α in $\Phi_0^\infty(\mathbb{T}^d,\mathbb{T})$. On the other hand, by the Proposition, the set of weakly mixing cocycles is a G_δ, so to prove that it is a dense G_δ in $\Phi_0^\infty(\mathbb{T}^d,\mathbb{T})$, it remains to show the denseness. This follows directly from the following observations:

(a) the trigonometric polynomials in d variables with zero constant term are coboundaries over any ergodic rotation of $\mathbb{T}^d$, in fact if

$$p(x)=\sum a_j\exp(2\pi ijx),$$

where $jx=\sum j_kx_k$, then $p(x)=g(x+\alpha)-g(x)$, where

$$g(x)=\sum a_j(\exp(2\pi ij\alpha)-1)^{-1}\exp(2\pi ijx),$$

(b) the constant cocycles with values $\beta=\sum l_k\alpha_k$, where l is any vector in $\mathbb{Z}^d$, are coboundaries since if $g(x)=\sum l_kx_k$ then $g(x+\alpha)-g(x)=\beta$,

(c) if $1,\alpha_1,\ldots,\alpha_d$ are rationally independent then modulo 1 the set of all β as above is dense in $\mathbb{T}$; consequently, the set of coboundary polynomials is dense in $\Phi_0^\infty(\mathbb{T}^d,\mathbb{T})$,

(d) if ϕ is weakly mixing and ψ is a coboundary then $\phi+\psi$ is weakly mixing.

The final step in the proof consists in passing from 1 to d'. Here we simply use the open homomorphism $\phi\mapsto\chi\circ\phi$ from $\Phi_0^\infty(\mathbb{T}^d,\mathbb{T}^{d'})$ onto $\Phi_0^\infty(\mathbb{T}^d,\mathbb{T})$ as indicated in the preceding discussion.

In the same manner, using Corollary 3 of [I2], we obtain the following result for C^1 cocycles. Here $\Phi_0^1(\mathbb{T}^d,\mathbb{T}^{d'})$ denotes the corresponding space of C^1 cocycles endowed with its C^1 topology.

COROLLARY 2. *Let $d,d'\in\mathbb{N}$. There exists a residual set A of full measure in $\mathbb{R}^d$ such that for every $\alpha\in A$ the set of weakly mixing cocycles in $\Phi_0^1(\mathbb{T}^d,\mathbb{T}^{d'})$ is a dense G_δ in $\Phi_0^1(\mathbb{T}^d,\mathbb{T}^{d'})$.*

It is clear that results of this type can also be obtained for other spaces such as C^r, analytic, and entire cocycles (homotopic to constant functions), thus extending the 1-dimensional case discussed in [I2, I3, KLR, Z].

2. Periodic approximation. We know from [R, IS, I2, I3] that if α admits a sufficiently good diophantine approximation then for "most" cocycles in spaces such as $\Phi(\mathbb{T},\mathbb{T})$, $\Phi_0^r(\mathbb{T},\mathbb{T})$, etc., the skew product T admits a good approximation by cyclic transformations in the sense of Katok and Stepin. This implies that T is rigid and rank-one, so in particular it has simple singular spectrum. Moreover, the spectral measure must be concentrated on a small set if the rate of approximation is high (see [I1] for more details).

In the present section we indicate how to obtain analogous results for skew products on multidimensional tori. The essential difference between $d = 1$ and $d > 1$ is that now we will use an approximation of the second type by periodic transformations (briefly, a.p.t.II; the definition, due to Katok and Stepin [KS], is recalled below). This kind of approximation, however, is sufficient to imply singular spectral type (see [I1], Thm. 4). The proofs are only sketched to indicate differences with the one-dimensional arguments. In the whole section we assume $M = 0$.

One important ingredient is the following lemma, which for $d = 1$ can be found in [CSF], 16.3. In the present generality the proof is essentially the same with the gradient ∇F replacing the derivative F' and the curvilinear integral

$$\int_{x^1}^{x^2} \sum_{j=0}^{r-1} \nabla F(\xi + j\alpha)\, dl$$

in place of the corresponding definite integral.

LEMMA 2. *Let $1, \alpha_1, \ldots, \alpha_d$ be rationally independent real numbers and $F : \mathbb{R}^d \to \mathbb{R}$ be C^1 and 1-periodic in each variable. Then*

$$\max_{1\le r\le q,\, \|x^1 - x^2\|\le 1/q} \Big| \sum_{j=0}^{r-1} F(x^1 + j\alpha) - F(x^2 + j\alpha) \Big| \to 0$$

as $q \to \infty$.

It is clear that the lemma carries over to functions $F : \mathbb{R}^d \to \mathbb{R}^{d'}$ by considering each coordinate separately. Thanks to Lemma 2 we will be able to prove that approximating partitions converge to the point partition of the space $\mathbb{T}^d \times \mathbb{T}^{d'}$.

If α is as in Lemma 2 then $l\alpha \notin \mathbb{Z}$ for every nonzero integer vector $l = (l_1, \ldots, l_d)$. By the continuity of the inner product, for every $r \in \mathbb{N}$ there exists a neighborhood U of α such that $l\beta \notin \mathbb{Z}$ whenever $\beta \in U$, $l \neq 0$, and $|l_j| \le r$, $j = 1, \ldots, d$. Now let $P(x)$ be a real-valued trigonometric polynomial in d variables,

$$P(x) = \sum_{l_1=-r}^{r} \cdots \sum_{l_d=-r}^{r} a_{l_1 \ldots l_d} e^{2\pi i l x}.$$

If $\alpha^n = (p_1^{(n)}/q_n, \ldots, p_d^{(n)}/q_n) \to \alpha$, where α^n is written in its reduced form, then for all sufficiently large n we have $\alpha_n \in U$ whence $l\alpha^n \notin \mathbb{Z}$ for all nonzero l occurring in the representation of $P(x)$. Consequently, for all such l, q_n does not divide $\sum l_j p_j^{(n)}$. It follows that

$$\sum_{j=0}^{q_n-1} P(x+j\alpha^n) = q_n a_{0\ldots 0}$$

as in the 1-dimensional case (cf. [I3]).

According to [KS], T is said to *admit a.p.t.II with speed* $u(n)$ if there exist a sequence of partitions

$$\{C_0, \ldots, C_{Q_n-1}\} \to \varepsilon$$

and a sequence of p_n-periodic transformations T_n permuting the sets C_j and such that $T_n \to T$ and

$$\sum_{j=0}^{Q_n-1} |TC_j \bigtriangleup T_n C_j| < u(p_n).$$

If $u(n) = o(v(n))$, we will say that T *admits a.p.t.II with speed* $o(v(n))$. If $Q_n = p_n$, the approximation is called *cyclic*.

As in [I3] we will consider rather general spaces of smooth cocycles contained in $\Phi_0^1(\mathbb{T}^d, \mathbb{T}^{d'})$. Let E be an additive subgroup of $\Phi_0^1(\mathbb{T}^d, \mathbb{T}^{d'})$ endowed with its own topology, stronger than the C^1-convergence, and such that

(1) E is a complete metric group,
(2) E contains the constant mappings with natural topology,
(3) E has a dense subset of trigonometric polynomial mappings.

The next theorem is proved in a similar way to Theorem 1 of [I3].

THEOREM 1. *Let α, α^n, E be as above and $v(n)$ be a sequence of positive numbers converging to zero. Suppose there exist integers $s_n \to \infty$ such that*

$$\|\alpha - \alpha^n\| = o(v(s_n q_n)/q_n).$$

Then the set of cocycles $F \in E$ such that the corresponding skew-product diffeomorphism admits a.p.t.II with speed $o(v(n))$ is residual in E.

Proof. We first show how to approximate a skew product whose cocycle is a special polynomial function. We begin by choosing a sequence of integers $s_n \to \infty$ as in the statement of the theorem. Without loss of generality we may assume that $s_n = s_1^{(n)} \ldots s_{d'}^{(n)}$, where the greatest common divisor of $s_1^{(n)}, \ldots, s_{d'}^{(n)}$ is equal to 1 and $\min(s_1^{(n)}, \ldots, s_{d'}^{(n)}) \to \infty$. We define

$$C_0 = [0, 1/q_n)^d \times [0, 1/s_n)^{d'}$$

to be the first cell of an approximating partition. If $\widetilde{P}$ is a trigonometric polynomial mapping then according to the preceding discussion there exists a vector c_n of size not exceeding $1/q_n$ such that $\widetilde{Q}_n = \widetilde{P} + c_n$ satisfies the equation

$$\sum_{j=0}^{q_n-1} \widetilde{Q}_n(x+j\alpha^n) = (1/s_1^{(n)},\dots,1/s_{d'}^{(n)}) \quad \text{modulo } \mathbb{Z}^{d'}.$$

It is clear that the skew product

$$T_{\widetilde{Q}_n}(x,y) = (x+\alpha, y+\widetilde{Q}_n(x))$$

is approximated by the $q_n s_n$-periodic transformation

$$T_n(x,y) = (x+\alpha^n, y+\widetilde{Q}_n(x)).$$

The first $q_n s_n$ cells of the partition C_j are defined to be the images of C_0 by T_n^j. This gives the first tower—a cycle of length $q_n s_n$. To define $C_{q_n s_n}$ we pick any unoccupied $1/q_n$-cube C in $[0,1)^d$ and let $C_{q_n s_n} = C \times [0,1/s_n)^{d'}$. Now put

$$C_{q_n s_n + j} = T_n^j C_{q_n s_n}$$

for $j = 0,\dots,q_n s_n - 1$. By continuing in this manner we will produce q_n^{d-1} disjoint towers of height $q_n s_n$, each being a T_n-invariant set on which T_n acts periodically with period $q_n s_n$. We obtain a partition consisting of $Q_n = q_n^d s_n$ elements C_j. The approximation error for a single $1/q_n$-cube is majorized by

$$2d\|\alpha-\alpha^n\|/q_n^{d-1}$$

so the total error, which is at most q_n^d times larger, is bounded by

$$q_n o(v(q_n s_n)/q_n) = o(v(q_n s_n)).$$

To pass from polynomial mappings to generic cocycles in E we use small neighborhoods of the $\widetilde{Q}_n$'s from a dense set so that the approximation error remains of the same magnitude; to obtain the convergence of the partitions to the point partition ε we use Lemma 2 (see [I2] or [I3] for the details).

If we only care about a residual set of α's, then the approximation can be made cyclic.

THEOREM 1′. *Let E and $v(n)$ be as above. There exists a residual subset A of $\mathbb{R}^d$ such that for every $\alpha \in A$ the set of cocycles $F \in E$ such that the corresponding skew-product diffeomorphism admits a cyclic approximation with speed $o(v(n))$ is residual in E.*

Proof. To construct A we fix a sequence of integer vectors $(q_1^{(n)},\dots,q_d^{(n)})$ such that the numbers $q_j^{(n)}$ are distinct primes for any fixed n and

$$\min_j q_j^{(n)} \to \infty.$$

It is clear that for every N the set of vectors of the form

$$\beta = (p_1^{(n)}/q_1^{(n)},\dots,p_d^{(n)}/q_d^{(n)}),$$

where $n \geq N$ and $1 \leq p_j^{(n)} < q_j^{(n)}$, is dense in the d-torus. Now let s_n be as in the previous proof, $q_n = q_1^{(n)} \ldots q_d^{(n)}$, and $\eta_n = o(v(s_n q_n)/s_n q_n)$. Denote by $U(n)$ the union of the η_n-neighborhoods of all the vectors β for fixed n. We define

$$A = \bigcap_N \bigcup_{n \geq N} U(n).$$

It is clear that A is a dense G_δ set, hence residual. For every $\alpha \in A$ we choose a sequence of approximations β_{n_k} and repeat the procedure from the proof of Theorem 1, but with different partitions C_j. For simplicity we write $n_k = n$ and $\beta_{n_k} = \alpha^n$. Now let

$$C_0 = [0, 1/q_1^{(n)}) \times \ldots \times [0, 1/q_d^{(n)}) \times [0, 1/s_n)^{d'}$$

and $C_j = T_n^j C_0$ for $j = 1, \ldots, s_n q_n$, where T_n is defined as before. We obtain a single cycle of length $s_n q_n$ for T_n and the proof is completed without difficulty.

By combining the above theorem with the results of the previous section we may obtain cocycles which are both weakly mixing and well approximable. In particular, by intersecting two residual sets we may produce "large" sets of exponentially approximable weakly mixing analytic or even entire cocycles in $\Phi_0^1(\mathbb{T}^d, \mathbb{T}^{d'})$ (cf. [I3] and the spaces E^λ therein). The resulting skew products are analytic diffeomorphisms which are rigid and have partly continuous spectrum with spectral measure concentrated on a set of Hausdorff dimension zero (for the latter statement see [I1], Cor. 6).

It is also clear that in the special case of $d = 1$ we may replace a.p.t.II by cyclic approximation in Theorem 1, with a proof as in [I3].

3. Lebesgue spectrum. In the present section we study cocycles of the form

$$\phi(x) = mx + f(x) \quad \text{modulo } \mathbb{Z}^{d'},$$

where m is a nonzero vector in $\mathbb{Z}^d$ and $f(x)$ is $\mathbb{Z}^d$-periodic. The following theorem generalizes a 1-dimensional result from [I4].

THEOREM 2. *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a C^1 function 1-periodic in each variable and suppose*

$$\sum_{j \in \mathbb{Z}^d} \|j\|^3 |\widehat{f}(j)|^2 < \infty.$$

Then for every $\alpha \in \mathbb{R}^d$ such that $1, \alpha_1, \ldots, \alpha_d$ are rationally independent and for every $m \neq 0$ in $\mathbb{Z}^d$ the skew product

$$T(x, y) = (x + \alpha, y + mx + f(x))$$

acting on $\mathbb{T}^{d+1}$ has countable Lebesgue spectrum on the space $L^2(dx)^\perp$.

Proof (sketch). The proof differs from the 1-dimensional case [I4] in some details; we outline the necessary changes.

For every $N \neq 0$ the operator

$$(V_N g)(x) = e^{2\pi i N(f(x)+mx)} g(x+\alpha)$$

acts on $L^2(dx_1 \dots dx_d) = L^2(dx)$. The partial sums S_n are defined as

$$S_n = \sum_{\|j\| \le n} a_{j_1 \dots j_d} e^{2\pi i (j_1 x_1 + \dots + j_d x_d)},$$

where $a_{j_1 \dots j_d} = \widehat{f}(j_1, \dots, j_d)$ and $\| \ \|$ is the sup norm. We have

$$\begin{aligned} |({V_N}^n 1, 1)| &\le 2\pi |N| \int |f^{(n)}(x) - S_n^{(n)}(x)| \, dx \\ &\quad + \left| \int e^{2\pi i N(S_n^{(n)}(x) + nmx)} \, dx \right| \\ &= I_n + II_n, \end{aligned}$$

where $g^{(n)}(x) = \sum_{k=0}^{n} g(x+k\alpha)$ for $n \ge 0$. It suffices to show as in [I4] that the sequences I_n and II_n are square summable. The proof for I_n does not require any change. For II_n^2 we may assume $m_1 > 0$ and write f' for the partial derivative with respect to x_1. The sets

$$A_n = \{x_1 \in [0,1] : S_n^{(n)\prime}(x)/n < 1/2 - m_1\}$$

depend on $x_2, \dots, x_d$, but still ${f'}^{(n)}/n \to 0$ uniformly and

$$\|S_n' - f'\|_2^2 = 4\pi^2 \sum_{\|j\| > n} j_1^2 |\widehat{f}(j)|^2 \le 4\pi^2 \sum_{\|j\| > n} \|j\|^2 |\widehat{f}(j)|^2,$$

which allows us to prove $\sum |A_n| < \infty$ uniformly in $x_2, \dots, x_d$. The sets B_n are defined without any change by

$$B_n = [0,1] \setminus A_n = [a_1, b_1] \cup [a_2, b_2] \cup \dots \cup [a_r, b_r],$$

We obtain

$$\begin{aligned} II_n &\le \int_{\mathbb{T}^{d-1}} \left| \int_{A_n} e^{2\pi i N(S_n^{(n)}(x) + nmx)} \, dx_1 \right| dx_2 \dots dx_d \\ &\quad + \int_{\mathbb{T}^{d-1}} \left| \int_{B_n} e^{2\pi i N(S_n^{(n)}(x) + nmx)} \, dx_1 \right| dx_2 \dots dx_d \\ &\le \sup_{x_2, \dots, x_d} |A_n| + \int_{\mathbb{T}^{d-1}} \left| \sum_{k=1}^{r} \int_{a_k}^{b_k} e^{2\pi i n(S_n^{(n)}(x) + nmx)} \, dx_1 \right| dx_2 \dots dx_d. \end{aligned}$$

We estimate the inner integral of the second term of II_n by integration by parts and the van der Corput lemma as in [I4] (cf. also [ILR]). The total variation in x_1 of the integrand on the set A_n is evaluated as the L^1-norm of its partial derivative and the same calculation as in [I4] carries over due to the uniformity of the estimates in the remaining variables (m will have to be replaced by m_1). The second derivative of S_n, which occurs in the integration by parts, is also majorized as in dimension one:

$$\|S_n''\|_2^2 \le \sum_{\|j\|\le n} (4\pi^2\|j\|^2|\widehat{f}(j)|)^2.$$

As a result we will get $\sum II_n^2 < \infty$, completing the proof of the theorem.

It should be noted that if $f \in C^2(\mathbb{T}^d)$ then $\sum_j \|j\|^4|\widehat{f}(j)|^2 < \infty$, so f satisfies the assumption of Theorem 2.

We may also consider the more general case of $d' \ge 1, M \ne 0$, and $F : \mathbb{R}^d \to \mathbb{R}^{d'}$. Assume for simplicity that $F \in \Phi_0^2(\mathbb{T}^d, \mathbb{T}^{d'})$. For every $N \in \mathbb{Z}^{d'} \setminus \{0\}$ the character $\chi_N(y) = \exp(2\pi i N y)$ of $\mathbb{T}^{d'}$ defines the T-invariant subpace $H_N \subset L^2(\mathbb{T}^d \times \mathbb{T}^{d'})$ consisting of the functions $g(x)\chi_N(y)$. The action of T on H_N is unitarily conjugate to the operator

$$V_N g(x) = e^{2\pi i N(F(x)+Mx)} g(x+\alpha)$$

acting on $L^2(\mathbb{T}^d)$, so formally V_N is the same as for $d' = 1$. The argument reduces to $d' = 1$ with $f(x) = NF(x)$ and $m = NM$. If $m \ne 0$ we are in the position of Theorem 2 and obtain Lebesgue spectrum. If, on the other hand, $m = 0$, then, at least for certain F's as in Sections 1 and 2, we obtain singular continuous spectrum on H_N. Moreover, the singular part of T will have multiplicity one. To see the latter property we assume $0 < \operatorname{rank} M = d_1' < d'$, write $\mathbb{T}_1 = M\mathbb{T}^d, \mathbb{T}_2 = \mathbb{T}^{d'}/\mathbb{T}_1$, and let $\pi : \mathbb{T}^{d'} \to \mathbb{T}_2$ be the quotient homomorphism. Note that $\mathbb{T}_2$ can be identified as a $d' - d_1'$-dimensional torus and its dual group is just the annihilator $\mathbb{T}_1^\perp$ of $\mathbb{T}_1$ in the dual of $\mathbb{T}^{d'}$ (use the formula $\chi\pi = \widetilde{\chi}$ for $\chi \in \mathbb{T}_1^\perp$). The homomorphism π is now represented by an integer matrix. By the preceding discussion, the spectrum is Lebesgue on the direct sum $\bigoplus_{\chi_N \in \widehat{\mathbb{T}}^{d'} \setminus \mathbb{T}_1^\perp} H_N$. On the other hand, the spectral behavior of T on $\bigoplus_{\chi_N \in \mathbb{T}_1^\perp} H_N$ is the same as that of the skew product $T_2(x, y) = (x + \alpha, y + \pi F(x))$ on the space $L^2(\mathbb{T}^d \times \mathbb{T}_2)$. By Theorem 1$'$, for most α and most cocycles πF (hence for most F's) this skew product admits a good cyclic approximation so it has simple singular continuous spectrum on the orthocomplement of $L^2(dx)$ in $L^2(\mathbb{T}^d \times \mathbb{T}_2)$.

Consequently, we have the following corollary.

COROLLARY 3. *Let α and M be as above. If* $\operatorname{rank} M = d'$ *then for every $F \in \Phi_0^2(\mathbb{T}^d, \mathbb{T}^{d'})$ the skew product diffeomorphism T has countable Lebesgue spectrum on $L^2(dx)^\perp$. If* $0 < \operatorname{rank} M < d'$ *then for a residual set of vectors α and a residual set of functions $F \in \Phi_0^r(\mathbb{T}^d, \mathbb{T}^{d'})(r = 2, 3, \ldots, \infty)$ the skew product T is ergodic and its spectrum on $L^2(dx)^\perp$ splits into a Lebesgue part with infinite multiplicity and a singular continuous part with multiplicity one.*

It is clear that the functions F in the second part of the corollary can also be chosen analytic or entire at the expense of diminishing the set of α's.

REFERENCES

[A] H. Anzai, *Ergodic skew product transformations on the torus*, Osaka Math. J. 3 (1951), 83–99.

[B] L. Baggett, *On functions that are trivial cocycles for a set of irrationals*, Proc. Amer. Math. Soc. 104 (1988), 1212–1217.

[C] G. H. Choe, *Spectral properties of cocycles*, Ph.D. Thesis, University of California, Berkeley, 1987.

[CFS] I. P. Cornfeld, S. V. Fomin and Ya. G. Sinai, *Ergodic Theory*, Springer, 1982.

[F] K. Frączek, *Spectral properties of cocycles over rotations*, preprint.

[I1] A. Iwanik, *Approximation by periodic transformations and diophantine approximation of the spectrum*, in: Ergodic Theory of Z^d-Actions, Proc. Warwick Symposium 1993–4, Cambridge Univ. Press, 1996, 387–401.

[I2] —, *Generic smooth cocycles of degree zero over irrational rotations*, Studia Math. 115 (1995), 241–250.

[I3] —, *Cyclic approximation of analytic cocycles over irrational rotations*, Colloq. Math. 70 (1996), 73–78.

[I4] —, *Anzai skew products with Lebesgue component of infinite multiplicity*, Bull. London Math. Soc., to appear.

[ILR] A. Iwanik, M. Lemańczyk and D. Rudolph, *Absolutely continuous cocycles over irrational rotations*, Israel J. Math. 83 (1993), 73–95.

[IS] A. Iwanik and J. Serafin, *Most monothetic extensions are rank-1*, Colloq. Math. 66 (1993), 63–76.

[JP] R. Jones and W. Parry, *Compact abelian group extensions of dynamical systems II*, Compositio Math. 25 (1972), 135–147.

[KS] A. B. Katok and A. M. Stepin, *Approximations in ergodic theory*, Uspekhi Mat. Nauk 22 (5) (1967), 81–106 (in Russian).

[K] A. G. Kushnirenko, *Spectral properties of some dynamical systems with polynomial divergence of orbits*, Vestnik Moskov. Univ. 1974 (1), 101–108 (in Russian).

[R] A. Robinson, *Non-abelian extensions have nonsimple spectrum*, Compositio Math. 65 (1988), 155–170.

[Z] Q. Zhang, *Rigidity of smooth cocycles over irrational rotations*, preprint.

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