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TOPOLOGICAL ALGEBRAS WITH AN ORTHOGONAL TOTAL SEQUENCE

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The aim of this paper is an investigation of topological algebras with an orthogonal sequence which is total. Closed prime ideals or closed maximal ideals are kernels of multiplicative functionals and the continuous multiplicative functionals are given by the "coefficient functionals". Our main result states that an orthogonal total sequence in a unital Fréchet algebra is already a Schauder basis. Further we consider algebras with a total sequence $(x_n)_{n\in\mathbb{N}}$ satisfying $x_n^2 = x_n$ and $x_n x_{n+1} = x_{n+1}$ for all $n \in \mathbb{N}$.

Introduction. Let A be a topological algebra. A family $(z_i)_{i\in I}$ is called *orthogonal* if $z_i z_j = 0$ for all $i \neq j$ and $z_i^2 = z_i \neq 0$ for all $i \in I$, and it is called *total* if the linear span of the family is dense. A sequence $(x_n)_{n\in\mathbb{N}}$ is called a *basis* if for each $x \in A$ there exists a unique sequence of scalars $(\alpha_n)_{n\in\mathbb{N}}$ such that $x = \sum_{n=1}^{\infty} \alpha_n x_n$. The concept of a topological algebra with an orthogonal basis was introduced in [10] and since then there has been an extensive literature; cf. [1], [6], [11], [12]. The aim of this paper is to show that many results can be carried over to topological algebras with an orthogonal sequence which is only total in the algebra. For example, closed prime ideals or closed maximal ideals are the kernels of multiplicative functionals. The continuous multiplicative functionals are given by the "coefficient functionals" δ_i, $i \in I$. Unital algebras are semisimple and in the general case a description of the radical is given. The investigation of algebras with an orthogonal total sequence was motivated by algebras of holomorphic functions endowed with the Hadamard product: Let G be a domain containing 0 and let $H(G)$ be the set of all holomorphic functions with the compact-open topology. The Hadamard product $f * g$ of $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ is defined locally as $f * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. If G^{c} is a semigroup then $f * g$ has a holomorphic continuation to G and $H(G)$ is a commutative B_0-algebra. If G is in addition simply connected then $H(G)$

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is a B_0-algebra with the orthogonal total sequence z^n, $n \in \mathbb{N}_0$. Examples are $\mathbb{C}_- := \mathbb{C} \setminus [1, \infty)$ or $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ $(r > 1)$; cf. [2], [4], [13].

The first section of this paper is devoted to the study of algebras with an orthogonal sequence. Our main result states that an orthogonal total sequence in a unital Fréchet algebra is already a Schauder basis. Examples show that the assumption of having a unit element or being a Fréchet algebra cannot be omitted. Hence, by a result of T. Husain, the space $\mathbb{C}^{\mathbb{N}}$ is the only unital Fréchet algebra with an orthogonal total (infinite) sequence. In the second section we consider algebras with a total sequence $(x_n)_{n\in\mathbb{N}}$ satisfying $x_n^2 = x_n$ and $x_n x_{n+1} = x_{n+1}$ for all $n \in \mathbb{N}$ and we give some improvements of results in [1].

1. Total orthogonal families. Let A be a topological Hausdorff algebra over the field K of real or complex numbers. A family of distinct points $z_i \in A$, $i \in I$, is called *strongly orthogonal* if $z_i z_i = z_i \neq 0$ for all $i \in I$ and $az_i \in Kz_i$ for all $a \in A$, $i \in I$. Note that a linear functional $\delta_i : A \to K$ is induced via the formula $az_i = \delta_i(a)z_i$. Further, the kernel of a linear functional δ is denoted by $\ker(\delta)$. Lemmas 3.1 and 3.2 in [13] yield the following result:

1.1. PROPOSITION. *Let $(z_i)_{i\in I}$ be a strongly orthogonal family. Then the following statements hold*:

(a) *δ_i is a continuous multiplicative functional.*
(b) *$z_i z_j = 0$ for all $i \neq j$, hence $(z_i)_{i\in I}$ is orthogonal.*
(c) *Let M be a right ideal. Then either $M \subset \ker(\delta_i)$ or $z_i \in M$.*

1.2. PROPOSITION. *A total family $(z_i)_{i\in I}$ is orthogonal if and only if it is strongly orthogonal.*

Proof. Let P be the linear span of $\{z_i : i \in I\}$. If $p \in P$ then $pz_i \in Kz_i$ by orthogonality. Now let $x \in A$ and $(p_j)_j$ be a net in P converging to x. Since $p_j z_i = \lambda_j z_i$ for some λ_j and $p_j z_i$ converges (to xz_i) we infer that $(\lambda_j)_j$ is a Cauchy net. Hence there exists $\lambda \in \mathbb{C}$ such that $p_j z_i \to \lambda z_i$. On the other hand, $p_j z_i \to xz_i$. ∎

Let A be a topological algebra with an orthogonal total family. With the same methods as in 1.2 it is easy to see that $z_i a = az_i = \delta_i(a)z_i$. By a continuity argument one obtains $ab = ba$ for all $a, b \in A$, i.e., A is necessarily commutative.

1.3. PROPOSITION. *Let A be a topological algebra with an orthogonal total family. Then the Jacobson radical* $\mathrm{rad}(A)$ *is given by*

$$\mathrm{rad}(A) = \bigcap_{i\in I} \ker(\delta_i) = \{a \in A : ab = 0 \textit{ for all } b \in A\} = \{a \in A : a^2 = 0\}.$$

If A contains a unit element then A is semisimple.

Proof. The first inclusion is trivial. Suppose now that $\delta_i(a) = 0$ for all $i \in I$. Let $b \in A$ be arbitrary and $(p_j)_j$ be a net in P converging to b. Then $ap_j = 0$ since $p_j \in P$ and $az_i = \delta_i(a)z_i = 0$. By continuity we infer $ab = 0$. Now let $a \in A$ with $ab = 0$ for all $b \in A$. Let M be a maximal modular ideal and choose $b \in A \setminus M$. Then $ab = 0$ and $(a + M)(b + M) = 0$. Since A/M is a field we obtain $a + M = 0$. Hence $a \in \mathrm{rad}(A)$. Finally, it is clear that $\{a \in A : a^2 = 0\}$ is contained in $\mathrm{rad}(A)$. If $x \in \mathrm{rad}(A)$ then $ab = 0$ for all b by the above and therefore $a^2 = 0$. ■

The next two results have been established in [13]:

1.4. Theorem. *Let $(z_i)_{i\in I}$ be an orthogonal total family in the topological algebra A. Let M be an ideal of A. Then the following statements are equivalent*:

(a) *M is a prime ideal which is contained in a closed ideal.*
(b) *M is a closed prime ideal.*
(c) *M is a closed maximal ideal.*
(d) *There exists $i \in I$ with $M = \ker(\delta_i)$.*

If A has a unit element e, then the closed maximal ideals are generated by the elements $e - z_i$, $i \in I$.

1.5. Theorem. *Let A be a unital topological algebra with an orthogonal total family. Let M be a closed ideal and $B := \{i \in I : \delta_i(a) = 0$ for all $a \in M\}$. Then $M = \bigcap_{i\in B} \ker(\delta_i)$.*

The following is now an easy consequence; cf. Theorem 1.1 in [6] and Corollary 1.5 in [11].

1.6. Corollary. *An orthogonal basis in a topological algebra is a Schauder basis. A topological algebra with an orthogonal basis A is semisimple.*

Proof. Let $z = \sum_{n=1}^{\infty} \alpha_n z_n$. Then $\delta_m(z)z_m = zz_m = \alpha_m z_m$, i.e., $\delta_m(z) = \alpha_m$ is continuous for each m. Let us show that a topological algebra with an orthogonal basis is semisimple: if z is in the radical then $\alpha_n = \delta_n(z) = 0$ for all n and hence $z = \sum_{n=1}^{\infty} \alpha_n z_n = 0$. ■

In [10] it is shown that the only unital Fréchet algebra with an orthogonal *basis* is the space $\mathbb{C}^{\mathbb{N}}$. This result can be generalized to the case of a unital Fréchet algebra with an orthogonal *total sequence*. On the other hand, the unital B_0-algebra $H(\mathbb{C}_-)$ (endowed with the Hadamard product) does not have an orthogonal basis (cf. [13]) although z^n, $n \in \mathbb{N}_0$, is an orthogonal total sequence. Hence the assumption of being a Fréchet algebra is essential. Moreover, one cannot omit the assumption of having a unit element, as Remark 1.8 shows.

1.7. THEOREM. *Let A be a unital complex Fréchet algebra. Then an orthogonal total sequence $(z_n)_{n\in\mathbb{N}}$ is a Schauder basis.*

Proof. By Theorem 1.4 we know that the set Δ_A of all continuous multiplicative functionals is equal to $\{\delta_n : n \in \mathbb{N}\}$. Note that $\{h \in \Delta_A : h(z_n) = 0 \text{ for all } n \in \mathbb{N}\} = \emptyset$. By a theorem in [8, p. 136] there exists a sequence $(b_n)_n$ in A such that $e = \sum_{n=1}^{\infty} z_n b_n$. But $z_n b_n = \delta_n(b_n) z_n$ and $z_n e = z_n^2 b_n = z_n b_n = \delta_n(b_n) z_n$. It follows that $\delta_n(b_n) = 1$ for all $n \in \mathbb{N}$. Hence we have proved that $e = \sum_{n=1}^{\infty} z_n$. This implies $x = xe = \sum_{n=1}^{\infty} x z_n = \sum_{n=1}^{\infty} \delta_n(x) z_n$ for $x \in A$. ∎

1.8. Remark. We give an example showing that the assumption of having a unit in Theorem 1.7 is essential. Let $G := \{z \in \mathbb{C} : |z| < 3\} \setminus [2, 3)$. Then $H(G)$ is a non-unital Fréchet algebra (cf. Theorem 2.8 in [13]) with respect to Hadamard multiplication. Clearly the monomials z^n $(n \in \mathbb{N}_0)$ are an orthogonal and total family (since G is simply connected). Suppose that it is a basis. Then $1/(2-z) = \sum_{n=0}^{\infty} a_n z^n$ (compact convergence in G). Hence this power series converges compactly on $\{z \in \mathbb{C} : |z| < 3\}$. But a_n are the Taylor coefficients of $1/(2-z)$, and therefore $a_n = 1/2^{n+1}$ and the convergence radius is only 2, a contradiction.

Let A be a topological algebra with an orthogonal total sequence. By Theorem 1.4 the continuous multiplicative functionals are given by δ_n, $n \in \mathbb{N}$. In [17] Żelazko proved that, if a Fréchet algebra has at most countably many continuous multiplicative functionals, then each multiplicative functional is continuous. An important ingredient of the proof is a result of R. Arens concerning the *joint* spectrum in a Fréchet algebra. We give here a modification of the proof which uses only the description of the spectrum of a single element via multiplicative functionals.

1.9. THEOREM. *Let A be a unital topological algebra with an orthogonal total sequence. If A is Baire and $\sigma(a) = \{h(a) : h \in \Delta_A\}$ then each multiplicative functional is continuous.*

Proof. Let $A_{n,m} := \ker(\delta_n - \delta_m)$ for $n \neq m$. Then $A_{n,m}$ is closed and nowhere dense (since it is a hyperplane). By Baire's category theorem there exists $a \in A$ with $a \notin A_{n,m}$, i.e., $\delta_n(a) \neq \delta_m(a)$ for all $n \neq m$. Now let ψ be a multiplicative functional. Since $a - \psi(a)$ is not invertible there exists $n \in \mathbb{N}$ with $\delta_n(a - \psi(a)) = 0$, i.e., $\psi(a) = \delta_n(a)$. It follows that $\delta_m(a - \psi(a) + z_n) \neq 0$ for all $m \in \mathbb{N}_0$. Hence there exists $b \in A$ with $(a - \psi(a) + z_n)b = 1$. It follows that $1 - \delta_n(b) z_n = 1 - b z_n = (a - \psi(a))b \in \ker(\psi)$. Therefore $\psi = \delta_n$. ∎

2. Total sequences with $x_n^2 = x_n$ and $x_n x_{n+1} = x_{n+1}$.

Let A be a commutative topological Hausdorff algebra over the field K of real or complex numbers. We assume that there exists a sequence of distinct points x_n with $x_n^2 = x_n$ and $x_n x_{n+1} = x_{n+1}$ for all $n \in \mathbb{N}$. Such algebras have been discussed in [1] and we give here some improvements of the results therein. Note that $z_n := x_n - x_{n+1} \neq 0$ is a strongly orthogonal sequence; see Theorem 2.1(a) below. Hence we can define multiplicative functionals δ_n via the formula $zz_n = \delta_n(z) z_n$ for $n \in \mathbb{N}$. It is easy to see that $1 = \delta_n(x_1) = \ldots = \delta_n(x_n)$ and $\delta_n(x_k) = 0$ for all $k > n$. Statement (c) of Theorem 2.1 was proved in [1] only for complete LMC algebras. The following is a consequence of the theorem of Żelazko and Theorem 2.1(c):

If A is a Fréchet algebra with a total sequence $(x_n)_{n\in\mathbb{N}}$ of distinct points satisfying $x_n^2 = x_n$ and $x_n x_{n+1} = x_{n+1}$ for all $n \in \mathbb{N}$ then each multiplicative functional is continuous.

2.1. Theorem. *Let A be a topological algebra with a total sequence $(x_n)_{n\in\mathbb{N}}$ of distinct points satisfying $x_n^2 = x_n$ and $x_n x_{n+1} = x_{n+1}$ for all $n \in \mathbb{N}$. Then the following statements hold*:

(a) $z_n := x_n - x_{n+1}$ *induces a strongly orthogonal sequence and x_1 is the unit element.*

(b) *If f is a non-trivial multiplicative functional then either $f(x_n) = 1$ for all $n \in \mathbb{N}$ or there exists $n \in \mathbb{N}$ with $f = \delta_n$.*

(c) *There are at most countably many continuous multiplicative functionals.*

(d) *If $(x_n)_n$ is a basis then A is semisimple.*

Proof. (a) It is easy to see that $z_n^2 = z_n$. Let Q be the linear span of $\{x_n : n \in \mathbb{N}\}$. For $x \in A$ there exists a net q_j converging to $x \in A$. It is easy to see that $qz_n \in Kz_n$ for all $n \in \mathbb{N}$ and $q \in Q$. Hence $q_j z_n = \lambda_j z_n$ and λ_j is a Cauchy sequence since $q_j z_n$ converges to xz_n. It follows that $xz_n = \lambda z_n$, where λ is the limit of λ_j. In order to show that x_1 is the unit element note that $x_1 q = q$ for all $q \in Q$. Since each $x \in A$ is a limit of some q_j a continuity argument shows that $x_1 x = x$.

For (b) note that $f(x_{n+1}) = f(x_n x_{n+1}) = f(x_n) f(x_{n+1})$ and $f(x_1) = 1$ since x_1 is the unit element. Suppose that $f(x_{n_0+1}) = 0$ for some $n_0 \in \mathbb{N}$ and that n_0 is minimal with this property. Then $f(y_{n_0}) \neq 0$ for $y_{n_0} = x_{n_0} - x_{n_0+1}$. Since $ay_{n_0} = \delta_{n_0}(a) y_{n_0}$ we infer

$$f(a)f(y_{n_0}) = f(ay_{n_0}) = f(\delta_{n_0}(a)y_{n_0}) = \delta_{n_0}(a) f(y_{n_0}).$$

Hence $f(a) = \delta_{n_0}(a)$. Finally, suppose that $f(x_n) \neq 0$ for all $n \in \mathbb{N}$. Then $f(x_n) = 1$ by the above recursion formula.

For (c) let f, g be continuous multiplicative functionals different from δ_n ($n \in \mathbb{N}$). Then $f(x_n) = 1 = g(x_n)$ for all $n \in \mathbb{N}$. Since the linear span of $\{x_n : n \in \mathbb{N}\}$ is dense one obtains $f = g$ by continuity.

For (d) we refer to the proof of Proposition 4.3 in [1]. ∎

2.2. Proposition. *Let A be a topological algebra with a total sequence $(x_n)_{n\in\mathbb{N}}$ satisfying $x_n^2 = x_n$ and $x_n x_{n+1} = x_{n+1}$ for all $n \in \mathbb{N}$. Then $z_n := x_n - x_{n+1}$, $n \in \mathbb{N}$ is total provided that x_n converges to zero.*

Proof. Note that $z_1+\ldots+z_n = (x_1-x_2)+\ldots+(x_n-x_{n+1}) = x_1-x_{n+1}$. Hence $\sum_{n=1}^{\infty} z_n = x_1$ is contained in the closure of the linear span of $\{z_n : n \in \mathbb{N}\}$, which will be denoted by M. As $x_1 \in M$ and $z_1 + \ldots + z_{n-1} = x_1 - x_n \in M$ we obtain $x_n \in M$. Hence $M = A$. ∎

The following result was proved in [1] for complete LMC algebras (hence (i) is automatically satisfied) and for $c_n = 1$. Our theorem can be applied to the B_0-algebra $H(\mathbb{D})$ which is not a Fréchet algebra but satisfies the assumptions; cf. [3].

2.3. Theorem. *Let A be a topological algebra with a total sequence $(x_n)_{n\in\mathbb{N}}$ satisfying $x_n^2 = x_n$ and $x_n x_{n+1} = x_{n+1}$ for all $n \in \mathbb{N}$. Suppose that A satisfies the following two conditions*:

(i) *$\sigma(x)$ is contained in the closure of $\{h(x) : h \in \Delta_A\}$ for all $x \in A$, and*

(ii) *there exist $c_n \geq 0$ such that $\sum_{n=1}^{\infty} c_n x_n \in A$ and $\sum_{n=1}^{\infty} c_n = \infty$.*

Then each multiplicative functional is continuous.

Proof. If f is a non-trivial multiplicative functional different from all δ_n then $f(x_n) = 1$ for all $n \in \mathbb{N}$ by Theorem 2.1(b). Put $y_n := \sum_{k=n}^{\infty} c_k x_k$. Then $\sigma(y_n) \subset [0,\infty)$: for each multiplicative continuous functional we have $h(y_n) = \sum_{k=n}^{\infty} c_k h(x_k)$ and $h(x_k) \in \{0,1\}$ (since $x_k^2 = x_k$) and $c_k \geq 0$. By (i) the result follows. Since $f(y_n) \in \sigma(y_n)$ by multiplicativity we infer $f(y_n) \geq 0$. Hence $f(\sum_{n=0}^{\infty} c_n x_n) = c_1 + \ldots + c_n + f(y_{n+1}) \geq c_1 + \ldots + c_n$. Since $\sum_{n=0}^{\infty} c_n = \infty$ we obtain a contradiction. ∎

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