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CONVERGENCE WITH RESPECT TO F_σ-SUPPORTED IDEALS

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Let $(X, \mathcal{S})$ be a measurable space and $\mathcal{I} \subset \mathcal{S}$ a proper σ-ideal in a σ-field $\mathcal{S}$. We shall denote by $\mathcal{M}(\mathcal{S})$ the family of all $\mathcal{S}$-measurable real functions defined on X.

Definition 1 (cf. [7]). We shall say that a sequence $\{f_n\}_{n\in\mathbb{N}}$ of $\mathcal{S}$-measurable functions defined on X *converges with respect to* $\mathcal{I}$ to an $\mathcal{S}$-measurable function f defined on X if and only if every subsequence $\{f_{m_n}\}_{n\in\mathbb{N}}$ contains a subsequence $\{f_{m_{p_n}}\}_{n\in\mathbb{N}}$ converging to f $\mathcal{I}$-a.e., which means that the set $\{x \in X : f_{m_{p_n}}(x) \nrightarrow f(x)\}$ is a member of $\mathcal{I}$. We then write $f_n \xrightarrow[n\to\infty]{\mathcal{I}} f$.

It is easy to observe that the space $\mathcal{M}(\mathcal{S})$ equipped with convergence with respect to a σ-ideal $\mathcal{I}$ is an $\mathcal{L}^*$ space (cf. [3, Problem Q, p. 90]). Hence it is possible to define the closure operator on $\mathcal{M}(\mathcal{S})$ by letting $f \in \overline{A}$ if and only if A contains a sequence converging with respect to $\mathcal{I}$ to a function f (cf. [7]). This closure operator has the properties: $\overline{\emptyset} = \emptyset$, $A \subset \overline{A}$, $\overline{A \cap B} = \overline{A} \cap \overline{B}$, for any sets $A, B \in \mathcal{M}(\mathcal{S})$. Moreover, $\overline{\overline{A}} = \overline{A}$ for every $A \in \mathcal{M}(\mathcal{S})$ if and only if the following condition, usually labelled by (L4), is satisfied:

(L4) If $f_{j,n} \xrightarrow[n\to\infty]{\mathcal{I}} f_j$ for each $j \in \mathbb{N}$ and $f_j \xrightarrow[j\to\infty]{\mathcal{I}} f$, then there exist sequences $\{j_p\}_{p\in\mathbb{N}}$ and $\{n_p\}_{p\in\mathbb{N}}$ of positive integers such that $f_{j_p,n_p} \xrightarrow[n\to\infty]{\mathcal{I}} f$.

If condition (L4) is satisfied, then the topology determined by the closure operator described above is often called the *Fréchet topology*, and the space $\mathcal{M}(\mathcal{S})$ equipped with the Fréchet topology is a *Fréchet space*.

It is well known that the space of Lebesgue measurable functions over $\mathbb{R}$ is a Fréchet space, whereas the space of all functions with the Baire property is not (cf. [7] and [4]).

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DEFINITION 2. We shall say that a double sequence $\{B_{j,n}\}_{j,n\in\mathbb{N}}$ of $\mathcal{S}$-measurable sets *exhausts* X if

(i) $B_{j,n} \subset B_{j,n+1}$ for each $j \in \mathbb{N}$,
(ii) $\bigcup_{n=1}^{\infty} B_{j,n} = X$ for each $j \in \mathbb{N}$.

THEOREM 1. *The space $\mathcal{M}(\mathcal{S})$ is a Fréchet space if and only if, for each double sequence $\{B_{j,n}\}_{j,n\in\mathbb{N}}$ of $\mathcal{S}$-measurable sets exhausting X, there exist an increasing sequence $\{j_p\}_{p\in\mathbb{N}}$ of positive integers and a sequence $\{n_p\}_{p\in\mathbb{N}}$ of positive integers such that $X - \liminf_{p\in\mathbb{N}} B_{j_p,n_p} \in \mathcal{I}$.*

Proof. *Sufficiency.* We prove that condition (L4) is fulfilled. We may suppose, choosing subsequences if necessary, that $\{f_{j,n}\}_{j,n\in\mathbb{N}}$ is a sequence of $\mathcal{S}$-measurable functions such that $f_{j,n} \underset{n\to\infty}{\longrightarrow} f_j$ everywhere except on a set $A_j \in \mathcal{I}$ and $f_j \underset{j\to\infty}{\longrightarrow} f$ everywhere except on $A_0 \in \mathcal{I}$. Let $A = \bigcup_{j=0}^{\infty} A_j$.

Putting $B_{j,n} = \{x \in X - A : |f_{j,k}(x) - f_j(x)| \le 1/j \text{ for } k \ge n\}$ and $B^*_{j,n} = B_{j,n} \cup A$, we see that the double sequence $\{B^*_{j,n}\}_{j,n\in\mathbb{N}}$ exhausts X. Thus there exist an increasing sequence $\{j_p\}_{p\in\mathbb{N}}$ of positive integers and a sequence $\{n_p\}_{p\in\mathbb{N}}$ of positive integers such that $B = X - \liminf_{p\in\mathbb{N}} B_{j_p,n_p} \in \mathcal{I}$. If $x \notin B$, we conclude that there exists a positive integer s such that $x \in B_{j_p,n_p}$ for each $p \ge s$. Hence

$$|f_{j_p,k}(x) - f_{j_p}(x)| \le 1/j_p \quad \text{for } k \ge n_p.$$

Thus

$$|f_{j_p,n_p}(x) - f_{j_p}(x)| \le 1/j_p \quad \text{for } p \ge s.$$

Since $\{j_p\}_{p\in\mathbb{N}}$ is increasing, it follows that the sequence $\{f_{j_p,n_p}\}_{p\in\mathbb{N}}$ is convergent to f everywhere except on the set $A \cup B$ which is a member of $\mathcal{I}$.

Necessity. Suppose to the contrary that a double sequence $\{B_{j,n}\}_{j,n\in\mathbb{N}}$ of $\mathcal{S}$-measurable sets is exhausting and, for every increasing sequence $\{j_p\}_{p\in\mathbb{N}}$ of positive integers and every sequence $\{n_p\}_{p\in\mathbb{N}}$ of positive integers, $X - \liminf_{p\in\mathbb{N}} B_{j_p,n_p} \notin \mathcal{I}$.

Putting

$$f_{j,n} = \chi_{X - B_{j,n}} + 1/j, \quad f_j = 1/j, \quad f = 0,$$

we see that $f_{j,n} \xrightarrow[n\to\infty]{\mathcal{I}} f$ and $f_j \xrightarrow[j\to\infty]{\mathcal{I}} f$. Thus, by condition (L4), there exist an increasing sequence $\{j_p\}_{p\in\mathbb{N}}$ of positive integers and a sequence $\{n_p\}_{p\in\mathbb{N}}$ of positive integers such that $f_{j_p,n_p} \underset{p\to\infty}{\longrightarrow} 0$. Observe that $\lim_{p\to\infty} f_{j_p,n_p}(x) = 0$ if and only if $x \notin X - \liminf_{p\in\mathbb{N}} B_{j_p,n_p}$. In that way, we have a contradiction with the fact that no subsequence of $\{f_{j_p,n_p}\}_{p\in\mathbb{N}}$ is $\mathcal{I}$-a.e. convergent to 0. ■

In the case when the pair $(\mathcal{S}, \mathcal{I})$ satisfies c.c.c., the paper of Wagner [7] contains a necessary and sufficient condition for convergence with respect to $\mathcal{I}$ to yield the Fréchet topology in the space $\mathcal{M}(\mathcal{S})$. Also, a condition without the property c.c.c. is given in [8].

Now, we consider a topological space X. We shall denote by $\mathcal{B}(X)$ the family of Borel sets in X. For any proper σ-ideal $\mathcal{I}$ of subsets of X, we set $\mathcal{S} = \mathcal{B}(X) \vartriangle \mathcal{I}$, the smallest σ-field containing both $\mathcal{B}(X)$ and $\mathcal{I}$. One can easily check that $\mathcal{S}$ is the collection of all sets of the form $(A-B)\cup C$ where $A \in \mathcal{S}$ and $B, C \in \mathcal{I}$. Moreover, without any difficulties we have

THEOREM 2. *Convergence with respect to $\mathcal{I}$ yields the Fréchet topology in the space $\mathcal{M}(\mathcal{S})$ if and only if, for every sequence $\{B_{j,n}\}_{j,n\in\mathbb{N}}$ of Borel sets in X exhausting X, there exists an increasing sequence $\{j_p\}_{p\in\mathbb{N}}$ of positive integers and a sequence $\{n_p\}_{p\in\mathbb{N}}$ of positive integers such that $X - \liminf_{p\in\mathbb{N}} B_{j_p,n_p} \in \mathcal{I}$.*

We say that a σ-ideal $\mathcal{I}$ is *uniform* if it contains all singletons $\{x\}$ for each $x \in X$, and it is *F_σ-supported* if, for any $A \in \mathcal{I}$, there exists an F_σ-set B belonging to $\mathcal{I}$ such that $A \subset B$.

Now, our goal is to prove the following

THEOREM 3. *If X is a Polish space, then convergence with respect to a uniform and proper F_σ-supported σ-ideal $\mathcal{I}$ does not yield the Fréchet topology in the space $\mathcal{M}(\mathcal{B}(X) \vartriangle \mathcal{I})$.*

Before we carry out the proof we shall present some theorems which are evidently necessary in our considerations.

THEOREM 4 (cf. [4]). *Convergence with respect to the σ-ideal $\mathcal{K}(X)$ of all meager sets in a second countable topological space X yields the Fréchet topology in the space of all real functions on X with the Baire property if and only if $X = A \cup B$ where A is an open set of the first category and B is the countable set of all isolated points of X.*

THEOREM 5. *If $(X, \mathcal{S})$ is a measurable space and there exists an $\mathcal{S}$-measurable real function f such that $f^{-1}(\{z\}) \notin \mathcal{I}$ for each $z \in \mathbb{R}$, then $\mathcal{M}(\mathcal{S})$ is not a Fréchet space.*

Proof. It is known (see, for example, Th. 2 in [5]) that convergence with respect to the σ-ideal $\mathcal{I} = \{\emptyset\}$ does not generate the Fréchet topology in the space of all real Borel functions on $\mathbb{R}$. Hence, by Theorem 1, there exists a sequence $\{B_{j,n}\}_{j,n\in\mathbb{N}}$ of Borel sets exhausting $\mathbb{R}$ such that, for each increasing sequence $\{j_p\}_{p\in\mathbb{N}}$ of positive integers and each sequence $\{n_p\}_{p\in\mathbb{N}}$ of positive integers, $\mathbb{R} - \liminf_{p\in\mathbb{N}} B_{j_p,n_p} \neq \emptyset$. Now, putting $B^*_{j,n} = f^{-1}(B_{j,n})$, we have an exhausting sequence of $\mathcal{S}$-measurable sets such that, for each increasing sequence $\{n_p\}_{p\in\mathbb{N}}$ of positive integers, $X - \liminf_{p\in\mathbb{N}} B^*_{j_p,n_p} \notin \mathcal{I}$, which ends the proof of the theorem. ∎

THEOREM 6 (Kechris and Solecki [6]). *If $\mathcal{I}$ is an F_σ-supported, proper and uniform σ-ideal of subsets of a Polish space X, then exactly one of the following possibilities holds*:

(i) $\mathcal{I} = \mathcal{MGR}(\mathcal{F})$ *for a countable family* $\mathcal{F}$ *of closed subsets of* X (*which can be assumed to be well-ordered by reverse inclusion*), *where* $\mathcal{MGR}(\mathcal{F}) = \{A \subset X : \forall_{F\in\mathcal{F}},\ A \cap F$ *is meager in* $F\}$;

(ii) *there is a homeomorphic embedding* $\Phi : 2^\omega \times \omega^\omega \to X$ *such that* $\Phi(\{\alpha\} \times \omega^\omega) \notin \mathcal{I}$ *for any* $\alpha \in 2^\omega$.

Proof of Theorem 3. We see that only the case when X is dense in itself is essential because the countable set of all isolated points is a member of $\mathcal{I}$. Suppose that the σ-ideal $\mathcal{I}$ satisfies condition (i) of Theorem 5. Let $\mathcal{F} = \{F_\alpha\}_{\alpha<\beta}$ where $\beta < \omega_1$. It is demonstrated in the proof of the Kechris–Solecki theorem that, for any $F_{\alpha_1}, F_{\alpha_2} \in \mathcal{F}$, if $\alpha_1 < \alpha_2$, then F_{α_2} is meager in F_{α_1}. We consider two cases:

Case 1: $\operatorname{card}\mathcal{F} = 1$. If $X - F_0 = \emptyset$, then the σ-ideal $\mathcal{I}$ is identical with the σ-ideal of all meager sets in X and, by Theorem 3, $\mathcal{M}(\mathcal{B}(X) \vartriangle \mathcal{I})$ is not a Fréchet space. Let $X - F_0 \neq \emptyset$. We see that $X - F_0 \in \mathcal{I}$, $F_0 \notin \mathcal{I}$ and $\mathcal{I} \cap 2^{F_0} = \mathcal{K}(F_0)$ where $\mathcal{K}(F_0)$ is the σ-ideal of all meager sets in F_0. By Theorem 4, convergence with respect to $\mathcal{K}(F_0)$ does not yield the Fréchet topology in the space of all real Baire functions in F_0 and, in consequence, by Theorem 1, $\mathcal{M}(\mathcal{B}(X) \vartriangle \mathcal{I})$ is not a Fréchet space.

Case 2: $\operatorname{card}\mathcal{F} > 1$. If $X - F_0 \neq \emptyset$, we argue as in the previous case. Let $X - F_0 = \emptyset$. Then, by the property of the family $\mathcal{F}$, $X - F_1$ is a nonempty open set of the second category. By Theorem 4, convergence with respect to the σ-ideal $\mathcal{K}(X - F_1)$ of meager sets in $X - F_1$ does not yield the Fréchet topology in the family of all $\mathcal{B}(X - F_1) \vartriangle \mathcal{K}(X - F_1)$-measurable functions. Thus, by Theorem 2, there exists a sequence $\{B_{j,n}\}_{j,n\in\mathbb{N}}$ of Borel sets in $X - F_1$ exhausting $X - F_1$ such that, for each increasing sequence $\{j_p\}_{p\in\mathbb{N}}$ of positive integers and each sequence $\{n_p\}_{p\in\mathbb{N}}$ of positive integers, $(X - F_1) - \liminf_{p\in\mathbb{N}} B_{j_p,n_p} \in \mathcal{K}(X - F_1)$. Putting $B^*_{j,n} = B_{j,n} \cup F_1$ for any $j, n \in \mathbb{N}$, we have a sequence of Borel sets exhausting X such that $X - \liminf_{p\in\mathbb{N}} B^*_{j_p,n_p} = (X - F_1) - \liminf_{p\in\mathbb{N}} B_{j_p,n_p} \in \mathcal{K}(X - F_1)$. For every $A \in \mathcal{K}(X - F_1)$ and for each set $F_\alpha \in \mathcal{F}$, we have $F_\alpha \cap A = \emptyset$ for $\alpha \neq 0$ and $F_0 \cap A = A$; thus for every $A \in \mathcal{K}(X - F_1)$, $F_\alpha \cap A$ is meager in F_α for each $F_\alpha \in \mathcal{F}$. This implies that $X - \liminf_{p\in\mathbb{N}} B_{j_p,n_p} \in \mathcal{I}$ and the proof in the case when condition (i) is satisfied is complete.

Suppose that $\mathcal{I}$ satisfies condition (ii) of Theorem 6. By Proposition 2 of [2], there exists a Borel function $f : X \to \mathbb{R}$ such that $f^{-1}(\{z\}) \notin \mathcal{I}$ for any $z \in \mathbb{R}$. An application of Theorem 5 finishes the proof. ■

COROLLARY 1. *If $\mathcal{I}$ is a proper and uniform σ-ideal in a Polish space X such that convergence with respect to $\mathcal{I}$ yields the Fréchet topology in the space of $\mathcal{M}(\mathcal{B} \triangle \mathcal{I})$-measurable functions, and $\mathcal{I}^* = \{A \in I : \exists_{B \in F_\sigma}\ A \subset B \wedge B \in I\}$, then $\mathcal{I} - \mathcal{I}^* \neq \emptyset$.*

Proof. If $\mathcal{I} - \mathcal{I}^* = \emptyset$, then $\mathcal{I} = \mathcal{I}^*$, which contradicts Theorem 3 stating that convergence with respect to $\mathcal{I}^*$ does not generate the Fréchet topology in the space of $\mathcal{M}(\mathcal{B} \triangle \mathcal{I}^*)$-measurable functions. ■

Now, we will be concerned with product ideals.

Let $\mathcal{I}$ and $\mathcal{J}$ be proper σ-ideals of X and Y, respectively. We consider a σ-ideal of $X \times Y$ which is the product of $\mathcal{I}$ and $\mathcal{J}$ according to the following definition (cf. [1]):

$$\mathcal{I} \times \mathcal{J} = \{A \subset X \times Y : \{x : A_x \notin \mathcal{J}\} \in \mathcal{I}\}$$

where A_x denotes the section of A, i.e. $A_x = \{y \in Y : (x, y) \in A\}$.

If $\mathcal{I}$ and $\mathcal{J}$ are proper σ-ideals, then $\mathcal{I} \times \mathcal{J}$ is also a proper σ-ideal of subsets of $X \times Y$ (see [1]).

Let $\mathcal{I}$ and $\mathcal{J}$ be proper σ-ideals of topological spaces X and Y, respectively.

THEOREM 7. *If convergence with respect to $\mathcal{I} \times \mathcal{J}$ yields the Fréchet topology in the space of $\mathcal{B}(X \times Y) \triangle (\mathcal{I} \times \mathcal{J})$-measurable functions, then convergence with respect to $\mathcal{I}$ and $\mathcal{J}$ yields the Fréchet topology in the space of $\mathcal{B}(X) \triangle \mathcal{I}$- and $\mathcal{B}(Y) \triangle \mathcal{J}$-measurable functions, respectively.*

Proof. Suppose that convergence with respect to $\mathcal{I} \times \mathcal{J}$ yields the Fréchet topology in the space of $\mathcal{B}(X \times Y) \triangle (\mathcal{I} \times \mathcal{J})$-measurable functions. Assume that the space of $\mathcal{B}(X) \triangle \mathcal{I}$-measurable functions is not a Fréchet space. By Theorem 1, there exists a double sequence $\{B_{j,n}\} \subset \mathcal{B}(X)$ exhausting X such that, for each increasing sequence $\{j_p\}_{p\in\mathbb{N}}$ of positive integers and for each sequence $\{n_p\}_{p\in\mathbb{N}}$ of positive integers, $X - \liminf_{p\in\mathbb{N}} B_{j_p,n_p} \notin \mathcal{I}$. Putting $A_{j,n} = B_{j,n} \times Y$ for any $j, n \in \mathbb{N}$, we have a double sequence $\{A_{j,n}\} \subset \mathcal{B}(X \times Y)$ of sets exhausting $X \times Y$ such that, for each increasing sequence $\{j_p\}_{p\in\mathbb{N}}$ and each sequence $\{n_p\}_{p\in\mathbb{N}}$, we have $\{x : (X \times Y - \liminf_{p\in\mathbb{N}} A_{n_p,j_p})_x \notin \mathcal{J}\} = X - \liminf_{p\in\mathbb{N}} B_{j_p,n_p} \notin \mathcal{I}$. This means that $X \times Y - \liminf_{p\in\mathbb{N}} A_{j_p,n_p} \notin \mathcal{I} \times \mathcal{J}$, which contradicts the fact that the last set should be a member of $\mathcal{I} \times \mathcal{J}$.

Now, suppose that convergence with respect to $\mathcal{J}$ does not yield the Fréchet topology in the space of $\mathcal{B}(Y) \triangle \mathcal{J}$-measurable functions. By Theorem 1, there exists a double sequence $\{B_{j,n}\}_{j,n\in\mathbb{N}} \subset \mathcal{B}(X)$ exhausting X such that, for each increasing sequence $\{j_p\}_{p\in\mathbb{N}}$ and each sequence $\{n_p\}_{p\in\mathbb{N}}$, we have $Y - \liminf_{p\in\mathbb{N}} B_{j_p,n_p}, \notin \mathcal{J}$. Putting $A_{j,n} = X \times B_{j,n}$ for any $j, n \in \mathbb{N}$, we have a double sequence of Borel sets exhausting $X \times Y$ such that for each increasing sequence $\{j_p\}_{p\in\mathbb{N}}$ and each sequence $\{n_p\}_{p\in\mathbb{N}}$, we have $\{x : (X \times Y - \liminf_{p\in\mathbb{N}} A_{j_p,n_p})_x \notin \mathcal{J}\} = X \notin \mathcal{I}$. This means that $X \times Y - \liminf_{p\in\mathbb{N}} A_{j_p,n_p} \notin \mathcal{I} \times \mathcal{J}$, which contradicts the fact that the last set should be a member of $\mathcal{I} \times \mathcal{J}$. ∎

COROLLARY 2. *If $\mathcal{L}$ is the σ-ideal of all Lebesgue null sets over $\mathbb{R}$ and $\mathcal{K}$ is the σ-ideal of all meager sets, then convergence with respect to the σ-ideal $\mathcal{K} \times \mathcal{L}$ (respectively $\mathcal{L} \times \mathcal{K}$) does not yield the Fréchet topology in the space of $\mathcal{B}(\mathbb{R}^2) \triangle (\mathcal{K} \times \mathcal{L})$-measurable functions (respectively $\mathcal{B}(\mathbb{R}^2) \triangle (\mathcal{L} \times \mathcal{K})$-measurable functions).*

REFERENCES

[1] M. Balcerzak, *Some Properties of Ideals of Sets in Polish Spaces*, Łódź University Press, 1991.

[2] M. Balcerzak and D. Rogowska, *Solution of the Baire order problem of Mauldin*, Proc. Amer. Math. Soc. 123 (1995), 3413–3416.

[3] R. Engelking, *General Topology*, PWN, Warszawa, 1977.

[4] J. Hejduk, *Convergence with respect to σ-ideal of meager sets in separable category bases*, Demonstratio Math. 28 (1995), 619–623,

[5] —, *Convergence with respect to some σ-ideals*, Rev. Res. Faculty of Science of Novi Sad, Math. Ser. 21 (1991).

[6] A. S. Kechris and S. Solecki, *Approximation of analytic by Borel sets and definable countable chain conditions*, Israel J. Math. 89 (1995), 343–356.

[7] E. Wagner, *Sequences of measurable functions*, Fund. Math. 112 (1981), 89–102.

[8] E. Wagner and W. Wilczyński, *Spaces of measurable functions*, Rend. Circ. Mat. Palermo (2) 30 (1981), 97–110.

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