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ON WEYL PSEUDOSYMMETRIC HYPERSURFACES

BY

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1. Introduction. It is known that every pseudosymmetric manifold is Weyl pseudosymmetric but the converse of the statement is not true (see [8]). However, under some additional assumptions, a Weyl pseudosymmetric manifold is pseudosymmetric. For instance, in [4] (see Corollary) it was shown that every Weyl pseudosymmetric hypersurface of dimension ≥ 4, isometrically immersed in a Euclidean space, is pseudosymmetric.

In the present paper we consider a more general case. Namely, we examine the Weyl pseudosymmetric hypersurfaces in semi-Riemannian spaces of constant curvature. Our main result (see Theorem 3.1) states that every Weyl pseudosymmetric hypersurface of dimension ≥ 4, isometrically immersed in a semi-Riemannian space of constant curvature, is pseudosymmetric. Recently, pseudosymmetric as well as Ricci-pseudosymmetric hypersurfaces, isometrically immersed in spaces of constant curvature, were investigated in [2], [3] and [22]–[24] (see also [14]).

A semi-Riemannian manifold (M, g), $n = \dim M \geq 3$, is said to be *pseudosymmetric* [16] if at every point of M its Riemann–Christoffel curvature tensor R satisfies the following condition:

$(*)_1$ the tensors $R \cdot R$ and $Q(g, R)$ are linearly dependent.

More precisely, the manifold (M, g) is pseudosymmetric if and only if

$$R \cdot R = L_R Q(g, R) \tag{1}$$

holds on the set $U_R = \{x \in M : Z(R) \neq 0 \text{ at } x\}$, for some function L_R on U_R. For precise definitions of the symbols used, we refer to Section 2. One of the important subclasses of pseudosymmetric manifolds is the class of semisymmetric manifolds. A semi-Riemannian manifold (M, g), $n \geq 3$, is

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called *semisymmetric* if

$$R \cdot R = 0, \tag{2}$$

holds on M. The notion of pseudosymmetry is the proper generalization of semisymmetry. E.g. certain warped product manifolds (see [5], [12], [16]) provide the examples of non-semisymmetric pseudosymmetric manifolds. It is easy to verify that if $(*)_1$ holds at a point of a semi-Riemannian manifold (M, g) then its Weyl conformal curvature tensor C satisfies at this point the following condition:

$(*)_2$ the tensors $R \cdot C$ and $Q(g, C)$ are linearly dependent.

The manifold (M, g), $n \geq 4$, is called *Weyl pseudosymmetric* ([9]) if $(*)_2$ holds on M. The manifold (M, g) is Weyl pseudosymmetric if and only if

$$R \cdot C = L_C Q(g, C) \tag{3}$$

is fulfilled on the set $U_C = \{x \in M : C \neq 0 \text{ at } x\}$, for some function L_C on U_C. The class of Weyl pseudosymmetric manifolds forms an extension of the class of *Weyl-semisymmetric* manifolds. A semi-Riemannian manifold (M, g), $n \geq 4$, is called *Weyl-semisymmetric* if

$$R \cdot C = 0, \tag{4}$$

holds on M. It is clear that any Weyl-semisymmetric manifold is Weyl pseudosymmetric. The converse of this statement is not true (see [8]). In [26, Proposition] it was shown that (2) and (4) are equivalent on the subset U_C of any semi-Riemannian manifold of dimension ≥ 5. In [15, Lemma 2] it was shown that any totally umbilical submanifold isometrically immersed in a semi-Riemannian Weyl pseudosymmetric manifold is also Weyl pseudosymmetric. In [17, Theorem 1] it was shown that (3) and

$$R \cdot R = L_C Q(g, R), \tag{5}$$

are equivalent on the subset U_C of every semi-Riemannian manifold of dimension $n \geq 5$. Further, (3) and (5) are equivalent on the subset U_C of every 4-dimensional warped product manifold [9, Theorem 3]. Of course, every conformally flat Riemannian manifold of dimension $n \geq 4$ is Weyl pseudosymmetric. However, there exist conformally flat manifolds, which are not pseudosymmetric (see, e.g. [5, Example 4.1]). An example of a 4-dimensional, non-semisymmetric and non-conformally flat Riemannian manifold satisfying (4) was presented in [6, Lemme 1.1]. A certain conformal deformation of that manifold gives a non-pseudosymmetric Weyl pseudosymmetric manifold with non-zero tensor $R \cdot C$ (see [8]).

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2. Preliminary results. Let (M, g), $n = \dim M \geq 3$, be a connected semi-Riemannian manifold of class C^∞. We denote by ∇, S and κ the Levi-Civita connection, the Ricci tensor and the scalar curvature of (M, g), respectively. We define on M the endomorphisms $\widetilde{R}(X, Y)$, $X \wedge Y$ and $\widetilde{\mathcal{C}}(X, Y)$ by

$$\widetilde{R}(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z,$$
$$(X \wedge Y)Z = g(Y,Z)X - g(X,Z)Y,$$
$$\widetilde{\mathcal{C}}(X,Y) = \widetilde{\mathcal{R}}(X,Y) + \frac{1}{n-2}\left(\frac{\kappa}{n-1}X \wedge Y - (X \wedge \widetilde{\mathcal{S}}Y + \widetilde{\mathcal{S}}X \wedge Y)\right),$$

where $X, Y, Z \in \Xi(M)$, the Lie algebra of vector fields on M. The Ricci operator $\widetilde{\mathcal{S}}$ is defined by $g(\widetilde{\mathcal{S}}X, Y) = S(X, Y)$. The Riemann–Christoffel curvature tensor R, the Weyl conformal curvature tensor C, the $(0, 4)$-tensor G and the tensor $Z(R)$ are defined by

$$R(X_1, \ldots, X_4) = g(\widetilde{R}(X_1, X_2)X_3, X_4),$$
$$C(X_1, \ldots, X_4) = g(\widetilde{\mathcal{C}}(X_1, X_2)X_3, X_4),$$
$$G(X_1, \ldots, X_4) = g((X_1 \wedge X_2)X_3, X_4),$$
$$Z(R) = R - \frac{\kappa}{n(n-1)}G,$$

where $X_1, \ldots, X_4 \in \Xi(M)$. Further, for a symmetric $(0, 2)$-tensor field A on M we define the endomorphism $X \wedge_A Y$ of $\Xi(M)$ by

$$(X \wedge_A Y)Z = A(Z,Y)X - A(Z,X)Y,$$

where $X, Y, Z \in \Xi(M)$. Evidently, we have $X \wedge_g Y = X \wedge Y$. For a $(0, k)$-tensor field T on M, $k \geq 1$, and a symmetric $(0, 2)$-tensor field A on M, we define the $(0, k+2)$-tensor fields $R \cdot T$ and $Q(A, T)$ by

$$\begin{aligned}(R \cdot T)(X_1, \ldots, X_k; X, Y) = &- T(\widetilde{\mathcal{R}}(X,Y)X_1, X_2, \ldots, X_k)\\ &- \ldots - T(X_1, \ldots, X_{k-1}, \widetilde{\mathcal{R}}(X,Y)X_k),\\ Q(A,T)(X_1, \ldots, X_k; X, Y) = &- T((X \wedge_A Y)X_1, X_2, \ldots, X_k)\\ &- \ldots - T(X_1, \ldots, X_{k-1}, (X \wedge_A Y)X_k).\end{aligned}$$

Lemma 2.1. *Let (M, g), $n \geq 4$, be a semi-Riemannian manifold satisfying at a point $x \in U_C \subset M$ the equation* (3).

(i) *The following equality is fulfilled at x*:

$$\begin{aligned}(6)\quad &(n-2)(R \cdot S)_{hklm} + g_{kl}V_{hm} - g_{km}V_{lh} + g_{lh}V_{km} - g_{hm}V_{kl}\\ &\quad + (R \cdot S)_{lkhm} + (R \cdot S)_{hmlk} - (R \cdot S)_{mkhl} - (R \cdot S)_{hlmk} = 0,\end{aligned}$$

where V is the $(0, 2)$-tensor field on M with the local components V_{ij} defined *by*

$$V_{ij} = g^{hk}(R \cdot S)_{hijk}.$$

(ii) *If for a symmetric* $(0,2)$*-tensor* W *the equality* $R \cdot S = Q(g, W)$ *is satisfied at* x *then* (5) *is fulfilled at* x.

Proof. (i) The equality (6) was obtained in [17].

(ii) By our assumption, from [20, Lemma 2] it follows that

$$R \cdot S = L_C Q(g, S) \tag{7}$$

holds at x. But (7) reduces (3) to (5), which completes the proof.

Let M be a hypersurface isometrically immersed in a semi-Riemannian manifold $(N, \widetilde{g})$, $n = \dim M \geq 4$. We denote by g the induced metric tensor from $\widetilde{g}$. Let the equations $x^r = x^r(y^h)$ be the local parametric expression of M in $(N, \widetilde{g})$, where y^h and x^r are the local coordinates of M in N, respectively, and $h, i, j, k, l, m \in \{1, \ldots, n\}$ and $r, s, t, u \in \{1, \ldots, n+1\}$. The Gauss equation of M in $(N, \widetilde{g})$ can be written in the form

$$R_{hijk} = \widetilde{R}_{rstu} B^r_h B^s_i B^t_j B^u_k + \varepsilon(H_{hk} H_{ij} - H_{hj} H_{ik}), \tag{8}$$

where $\widetilde{R}_{rstu}$, R_{hijk} and H_{hk} denote the local components of the Riemann–Christoffel curvature tensor $\widetilde{R}$ of $(N, \widetilde{g})$, the Riemann–Christoffel curvature tensor R of M and the second fundamental tensor H of M in $(N, \widetilde{g})$, respectively. Furthermore, $B^r_k = \partial x^r / \partial y^k$, $\varepsilon = \widetilde{g}(\xi, \xi) = \pm 1$ and ξ is the local unit normal vector field. If M is a hypersurface isometrically immersed in a semi-Riemannian space of constant curvature $N^{n+1}(c)$ then (8) becomes

$$R_{hijk} = \varepsilon(H_{hk} H_{ij} - H_{hj} H_{ik}) + \frac{\widetilde{\kappa}}{n(n+1)} G_{hijk}, \tag{9}$$

where $G_{hijk} = g_{hk} g_{ij} - g_{hj} g_{ik}$ are the local components of the tensor G. We denote by $H^p_{ij} = g^{hk} H^{p-1}_{hi} H_{kj}$, $p = 2, 3, \ldots$, the local components of the tensor H^p. We have also $\operatorname{tr}(H) = g^{hk} H_{hk}$. Using (9) we can obtain the following result.

Lemma 2.2 ([2], Lemma 2.1). *The following identity is satisfied on every hypersurface* M *isometrically immersed in a semi-Riemannian space of constant curvature* $N^{n+1}(c)$, $n \geq 3$:

$$R \cdot S = Q(H, \operatorname{tr}(H) H^2 - H^3) + \frac{\varepsilon \widetilde{\kappa}}{n(n+1)} Q(g, \operatorname{tr}(H) H - H^2). \tag{10}$$

Lemma 2.3. *Let* M *be a hypersurface isometrically immersed in a semi-Riemannian space of constant curvature* $N^{n+1}(c)$, $n \geq 4$, *and let* (3) *be satisfied at a point* $x \in U_C \subset M$. *Then* (5) *holds at* x.

Proof. From Lemma 2.1(i) it follows that (6) is fulfilled at x. Substituting the identity (10) in (6), after straightforward calculations, we obtain

$$n\,\mathrm{tr}(H)Q(H,H^2) - nQ(H,H^3) + Q(g,V) + \frac{\varepsilon\widetilde{\kappa}}{n+1}(\mathrm{tr}(H)Q(g,H) - Q(g,H^2)) = 0.$$

We can write the above equality in the form

$$Q(H, \mathrm{tr}(H)H^2 - H^3) = Q(g, \overline{W}),$$

$\overline{W}$ being a symmetric $(0,2)$-tensor. By (10) we get $R \cdot S = Q(g,W)$, where W is a symmetric $(0,2)$-tensor. Now Lemma 2.1(ii) completes the proof.

LEMMA 2.4. *Let M be a hypersurface isometrically immersed in a semi-Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$. Then at every point of the set $M - U_C$ the condition $(*)_1$ is fulfilled.*

Proof. Our assertion is trivial at all points of $M - U_C$ at which the tensor $Z(R)$ of M vanishes. Let x be a point of $M - U_C$ at which $Z(R) \neq 0$. From [21, Theorem 4.1] it follows that M is quasi-umbilical at x, i.e. the relation $H = \widetilde{\alpha}g + \widetilde{\beta}u \otimes u$ holds at x, where $u \in T^*_x(M)$ and $\widetilde{\alpha}, \widetilde{\beta} \in \mathbb{R}$. From the last relation we get easily

$$H^2 = \alpha H + \beta g, \qquad \alpha, \beta \in \mathbb{R}. \tag{11}$$

Now, [22, Lemma 1] implies $R \cdot R = (\widetilde{\kappa}/(n(n+1)) - \varepsilon\beta)Q(g,R)$, which completes the proof.

Remark 2.1. As we have stated in the proof of Lemma 2.4, every quasi-umbilical hypersurface M in a semi-Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, fulfils (11). In particular, when the ambient space is a Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, quasi-umbilicity of M means that M has a principal curvature of multiplicity $\geq n-1$. It is clear that hypersurfaces in $N^{n+1}(c)$ having at every point two distinct principal curvatures, with multiplicity p and $n-p$, realize (11), where $1 \leq p \leq n-1$. For instance, certain cyclides of Dupin isometrically immersed in a Euclidean space $\mathbb{E}^{n+1}$, $n \geq 4$, have exactly two distinct principal curvatures (cf. [27], Theorem 5).

3. Main results

THEOREM 3.1. *Every Weyl pseudosymmetric hypersurface M isometrically immersed in a semi-Riemannian space of constant curvature $N^{n+1}(c)$, $n \geq 4$, is pseudosymmetric and conversely. Moreover, on the set $U_C \subset M$, (3) and (5) are equivalent.*

P r o o f. Our assertion is an immediate consequence of Lemmas 2.3 and 2.4.

R e m a r k 3.1. From Theorem 3.1 it follows that the 4-dimensional Weyl-semisymmetric manifold defined in [6, Lemme 1.1] cannot be isometrically immersed in a 5-dimensional space of constant curvature. Using some other arguments we have already noted this fact in [25] (see Section 4).

We recall that the Cartan hypersurface in the sphere $S^{n+1}(c)$ is a compact, minimal hypersurface with constant principal curvatures $-(3c)^{1/2}$, 0, $(3c)^{1/2}$ having the same multiplicity. The Cartan hypersurfaces exist only for $n = 3, 6, 12, 24$. More precisely, the Cartan hypersurfaces are tubes of constant radius over the standard Veronese embeddings $i : \mathbb{F}P^2 \to S^{3d+1}(c) \to \mathbb{E}^{3d+2}$, $d = 1, 2, 4, 8$, of the projective plane $\mathbb{F}P^2$ in the sphere $S^{3d+1}(c)$ in a Euclidean space $\mathbb{E}^{3d+2}$, where $\mathbb{F} = \mathbb{R}$ (real numbers), $\mathbb{C}$ (complex numbers), $\mathbb{Q}$ (quaternions) or $\mathbb{O}$ (Cayley numbers), respectively ([1]). As shown in [22, Example 2], the Cartan hypersurface in $S^4(c)$ is a non-semisymmetric, pseudosymmetric manifold satisfying the equation

$$R \cdot R = \frac{\widetilde{\kappa}}{12} Q(g, R).$$

The Cartan hypersurfaces of dimensions: $6, 12, 24$ are non-pseudosymmetric. They realize a weaker curvature condition of pseudosymmetry type. Namely, every Cartan hypersurface in $S^{n+1}(c)$, $n = 6, 12, 24$, is a non-pseudosymmetric, Ricci-pseudosymmetric manifold satisfying the equation [24, Theorem 1]

$$R \cdot S = \frac{\widetilde{\kappa}}{n(n+1)} Q(g, S).$$

A semi-Riemannian manifold (M, g), $\dim M \geq 3$, is said to be *Ricci-pseudosymmetric* ([7], [19]) if at every point of M the following condition is satisfied:

$(*)_3$ the tensors $R \cdot S$ and $Q(g, S)$ are linearly dependent.

The manifold (M, g) is Ricci-pseudosymmetric if and only if

$$R \cdot S = L_S Q(g, S) \tag{12}$$

holds on the set $U_S = \{x \in M : S - (\kappa/n)g \neq 0 \text{ at } x\}$, for some function L_S on U_S. We note that $U_S \subset U_R$. However, on 3-dimensional semi-Riemannian manifolds we have $U_S = U_R$. Evidently, if (1) is satisfied on the subset U_R of a manifold (M, g) then

$$R \cdot S = L_R Q(g, S) \tag{13}$$

holds on $U_S \subset M$. On manifolds of dimensions ≥ 4 the converse statement is not true. On 3-dimensional semi-Riemannian manifolds, (1) and (13) are equivalent [18, Lemma 2]. Recently, in [3, Theorem 3.8] it was proved that every Ricci-pseudosymmetric hypersurface M isometrically immersed in a 5-dimensional semi-Riemannian space of constant curvature is pseudosymmetric. This means that (12) implies on $U_S \subset M$

$$R \cdot R = L_S Q(g, R).$$

Combining this result with Theorem 3.1 we get the following corollary.

COROLLARY 3.1. *Every Weyl pseudosymmetric hypersurface M isometrically immersed in a semi-Riemannian space of constant curvature $N^5(c)$ is Ricci-pseudosymmetric and conversely.*

Remark 3.2. Very recently, in [14] it was stated that the tensor $R \cdot R$ vanishes at a point x of a hypersurface isometrically immersed in a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geq 3$, if and only if either $H^2 = \lambda H$, $\lambda \in \mathbb{R}$, or

$$H = \alpha v \otimes v + \beta w \otimes w, \qquad v, w \in T_x^*(M),\ \alpha, \beta \in \mathbb{R}. \tag{14}$$

In particular, the 3-dimensional Cartan hypersurface realizes (14).

A consequence of Theorem 3.1, Theorem 1 of [10] and the above remark give rise to the following corollary, which generalizes Theorem 3 of [4].

COROLLARY 3.2. *Let M be a hypersurface isometrically immersed in a semi-Euclidean space $\mathbb{E}_s^{n+1}$, $n \geq 4$. Then M is a Weyl pseudosymmetric manifold if and only if for every point $x \in M$, (11) or (14) holds at x.*

Remark 3.3. Semi-Riemannian manifolds realizing curvature conditions $(*)_1$, $(*)_2$ or $(*)_3$ are called manifolds of pseudosymmetry type. We refer to [11], [13] and [28] for survey articles on such manifolds.

REFERENCES

[1] T. E. Cecil and P. J. Ryan, *Tight and Taut Immersions of Manifolds*, Pitman, Boston, 1988.

[2] F. Defever, R. Deszcz, P. Dhooghe, L. Verstraelen and Ş. Yaprak, *On Ricci-pseudosymmetric hypersurfaces in spaces of constant curvature*, Results in Math. 27 (1995), 227–236.

[3] F. Defever, R. Deszcz, Z. Şentürk, L. Verstraelen and Ş. Yaprak, *P. J. Ryan's problem in semi-Euclidean space*, to appear.

[4] J. Deprez, R. Deszcz and L. Verstraelen, *Pseudo-symmetry curvature conditions on hypersurfaces of Euclidean spaces and on Kählerian manifolds*, Ann. Fac. Sci. Toulouse 9 (1988), 183–192.

[5] —, —, —, *Examples of pseudo-symmetric conformally flat warped products*, Chinese J. Math. 17 (1989), 51–65.

[6] A. Derdziński, *Exemples de métriques de Kaehler et d'Einstein autoduales sur le plan complexe*, in: Géométrie riemannienne en dimension 4 (Séminaire Arthur Besse 1978/79), Cedic/Fernand Nathan, Paris, 1981, 334–346.

[7] R. Deszcz, *On Ricci-pseudosymmetric warped products*, Demonstratio Math. 22 (1989), 1053–1065.

[8] —, *Examples of four-dimensional Riemannian manifolds satisfying some pseudo-symmetry curvature conditions*, in: Geometry and Topology of Submanifolds 2, World Sci., Singapore, 1990, 134–143.

[9] –, *On four-dimensional Riemannian warped product manifolds satisfying certain pseudosymmetry curvature conditions*, Colloq. Math. 62 (1991), 103–120.

[10] —, *Pseudosymmetry curvature conditions imposed on the shape operators of hypersurfaces in the affine spaces*, Results in Math. 20 (1991), 600–621.

[11] —, *On pseudosymmetric spaces*, Bull. Soc. Math. Belg. Sér. A 44 (1992), 1–34.

[12] —, *Curvature properties of a certain compact pseudosymmetric manifold*, Colloq. Math. 65 (1993), 139–147.

[13] —, *On pseudosymmetric manifolds*, Dept. Math., Agricultural Univ. Wrocław, Preprint No. 34, 1995.

[14] —, *On certain classes of hypersurfaces in spaces of constant curvature*, in: Geometry and Topology of Submanifolds 8, World Sci., Singapore, 1996, in print.

[15] R. Deszcz, S. Ewert-Krzemieniewski and J. Policht, *On totally umbilical submanifolds of conformally birecurrent manifolds*, Colloq. Math. 55 (1988), 79–96.

[16] R. Deszcz and W. Grycak, *On some class of warped product manifolds*, Bull. Inst. Math. Acad. Sinica 15 (1987), 311–322.

[17] —, —, *On manifolds satisfying some curvature conditions*, Colloq. Math. 57 (1989), 89–92.

[18] —, —, *On certain curvature conditions on Riemannian manifolds*, ibid. 58 (1990), 259–268.

[19] R. Deszcz and M. Hotloś, *Remarks on Riemannian manifolds satisfying a certain curvature condition imposed on the Ricci tensor*, Prace Nauk. Pol. Szczec. 11 (1989), 23–34.

[20] —, —, *Notes on pseudo-symmetric manifolds admitting special geodesic mappings*, Soochow J. Math. 15 (1989), 19–27.

[21] R. Deszcz and L. Verstraelen, *Hypersurfaces of semi-Riemannian conformally flat manifold*, in: Geometry and Topology of Submanifolds 3, Singapore, 1991, 131–147.

[22] R. Deszcz, L. Verstraelen and Ş. Yaprak, *Pseudosymmetric hypersurfaces in 4-dimensional spaces of constant curvature*, Bull. Inst. Math. Acad. Sinica 22 (1994), 167–179.

[23] —, —, —, *Hypersurfaces with pseudosymmetric Weyl tensor in conformally flat manifolds*, Dept. Math. Agricultural Univ. Wrocław, preprint No. 32, 1995.

[24] R. Deszcz and Ş. Yaprak, *Curvature properties of Cartan hypersurfaces*, Colloq. Math. 67 (1994), 91–98.

[25] —, —, *Curvature properties of certain pseudosymmetric manifolds*, Publ. Math. Debrecen 45 (1994), 333–345.

[26] W. Grycak, *Riemannian manifolds with a symmetry condition imposed on the 2-nd derivative of the conformal curvature tensor*, Tensor (N.S.) 46 (1987), 287–290.

[27] Ü. Pinkall, *Dupin hypersurfaces*, Math. Ann. 270 (1985), 427–440.

[28] L. Verstraelen, *Comments on pseudo-symmetry in the sense of Ryszard Deszcz*, in: Geometry and Topology of Submanifolds 6, World Sci., Singapore, 1994, 199–209.

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