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TAME ALGEBRAS WITH STRONGLY SIMPLY CONNECTED GALOIS COVERINGS

BY

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Throughout, by an algebra we mean a basic connected, finite-dimensional associative K-algebra with 1 over an algebraically closed field K. By a module over an algebra A we mean a right A-module of finite K-dimension.

From Drozd's remarkable Tame and Wild Theorem [14] the class of algebras may be divided into two disjoint classes. One class consists of tame algebras for which the indecomposable modules occur, in each dimension d, in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory is as complicated as the study of finite-dimensional vector spaces together with two non-commuting endomorphisms, for which the classification is a well-known unsolved problem. Hence, we can hope to classify the modules only for tame algebras. Here, we are concerned with the representation theory of tame algebras having simply connected Galois coverings.

Among tame algebras we may distinguish the class of representation-finite algebras, having only finitely many isomorphism classes of indecomposable modules. This class of algebras is presently rather well understood (see [3], [8], [9], [10]). In particular, we know that every representation-finite algebra A admits a standard form $\bar{A}$ [10], which is the best possible degeneration of A, such that A and $\bar{A}$ have the same number of isomorphism classes of indecomposable modules, and $\bar{A}$ admits a (strongly) simply connected Galois covering. This leads to the criterion of Bongartz for finite representation type [8], and reduces the study of modules over arbitrary representation-finite algebras to that for the corresponding simply connected algebras.

The representation theory of tame representation-infinite algebras is only emerging. At present the most accessible seem to be the (tame) algebras of polynomial growth [26], for which there exists a positive integer m such that the number of one-parameter families of indecomposable modules is bounded, in each dimension d, by d^m. It contains the class of domestic al-

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gebras for which there is a constant bound on the number of one-parameter families in each dimension. Important classes of polynomial growth algebras are tilted algebras of Euclidean type and tubular algebras [24]. We know also that all tame tilted algebras are domestic [18], and hence of polynomial growth. A representation theory of arbitrary strongly simply connected algebras of polynomial growth has been established by the author in [31].

Recently Geiss proved in [16] that if an algebra A admits a tame degeneration B (in the variety of algebras of a given dimension) then A is also tame. Hence, a convenient way to determine whether a given algebra A is tame consists in finding a suitable tame degeneration of A. We expect that every algebra A of polynomial growth admits a canonical degeneration $\bar{A}$ (a standard form of A) which is of polynomial growth, admits a simply connected Galois covering, and such that the representation theories of A and $\bar{A}$ are very close.

The main objective of this paper is to establish criteria for the polynomial growth (respectively, domestic type) of algebras having strongly simply connected Galois coverings. Applying the Galois covering techniques developed in [13] (see also [15]), we prove in Theorem 2.4 that, if an algebra A admits a Galois covering $F : R \to R/G = A$ with R strongly simply connected and without hypercritical and *pg*-critical convex subcategories, then A is tame and every indecomposable A-module is either the push-down $F_\lambda(Z)$ of an indecomposable finite-dimensional R-module Z or is of the form $V \otimes_{K[T,T^{-1}]} F_\lambda(M_L)$ where V is a finite-dimensional indecomposable module over the algebra $K[T,T^{-1}]$ of Laurent polynomials and M_L an infinite-dimensional locally finite-dimensional indecomposable R-module given by a line L in R with the stabilizer $G_L = \mathbb{Z}$. Moreover, applying the main results of [31], we prove in Theorem 2.6 that, if an algebra A admits a strongly simply connected Galois covering $R \to R/G = A$, then A is of polynomial growth (respectively, domestic) if and only if R does not contain a convex subcategory which is hypercritical or *pg*-critical (respectively, hypercritical, *pg*-critical or tubular) and the number of G-orbits of G-periodic lines in R is finite.

The results presented in the paper have been announced in [29] and [30].

1. Galois coverings of algebras. Following [9] by a *locally bounded category* we mean a K-category R which is isomorphic to a factor category KQ/I where Q is a locally finite quiver and I is an admissible ideal in the path category KQ of Q. Thus an algebra A will be considered as a locally bounded category with finitely many objects, briefly a *finite category*. A locally bounded category $R = KQ/I$ with Q having no oriented cycles is said to be *triangular*. The Auslander–Reiten quiver of a locally bounded category R will be denoted by Γ_R [9].

Throughout this section we denote by R a fixed connected locally bounded category. Recall that an R-module M is called *finite-dimensional* (respectively, *locally finite-dimensional*) if $\dim M = \sum_{x\in K} M(x) < \infty$ (respectively, $\dim_K M(x) < \infty$ for any object x of R). We denote by $\mathrm{MOD}\, R$ the category of all right R-modules, by $\mathrm{Mod}\, R$ (respectively, $\mathrm{mod}\ R$) the category of locally finite-dimensional (respectively, finite-dimensional) right R-modules, and by $\mathrm{Ind}\, R$ (respectively, $\mathrm{ind}\, R$) the full subcategory of $\mathrm{Mod}\, R$ (respectively, $\mathrm{mod} R$) formed by all indecomposable objects. The *support* $\mathrm{supp}\, M$ of an R-module M is the full subcategory of R given by all objects x such that $M(x) \neq 0$. A full subcategory C of R is said to be *convex* if any path in the quiver Q of R with source and target from C has all its vertices from C. If R is finite then, following [14], R is said to be *tame* if, for any dimension d, there exist a finite number of $K[x]$-R-bimodules M_i, $1 \leq i \leq n_d$, which are finitely generated and free as left $K[x]$-modules, and all but a finite number of isomorphism classes of indecomposable (right) R-modules of dimension d are of the form $K[x]/(x-\lambda) \otimes_{K[x]} M_i$ for some $\lambda \in K$ and some i. Let $\mu_R(d)$ be the least number of $K[x]$-R-bimodules satisfying the above conditions for d. Then R is said to be of *polynomial growth* (respectively, *domestic*) if there is a positive integer m such that $\mu_R(d) \leq d^m$ (respectively, $\mu_R(d) \leq m$) for any $d \geq 1$ (see [26], [11]). Finally, an arbitrary R is said to be *tame* (respectively, *of polynomial growth*, *domestic*) if so is every finite full subcategory of R.

In the sequel, G denotes a group of K-linear automorphisms of R acting freely on the objects of R. For a full subcategory D of R we denote by gD the full subcategory of R formed by all objects gx, $x \in D$. Then we denote by G_D the stabilizer $\{g \in G : gD = D\}$ of D. The group G acts on $\mathrm{MOD}\, R$ by the translations $g(-)$ which assign to each R-module M the R-module ${}^gM = M \circ g^{-1}$. For each R-module M, we denote by G_M the stabilizer $\{g \in G : {}^gM \simeq M\}$ of M. A module $Y \in \mathrm{Ind}\, R$ is called *weakly G-periodic* [13, (2.3)] if $\mathrm{supp}\, Y$ is infinite and $(\mathrm{supp}\, Y)/G_Y$ is finite. Clearly, in such a case, G_Y is nontrivial.

Assume now that G acts freely on the isoclasses in $\mathrm{ind}\, R$. Let $F : R \to R/G$ be the Galois covering, which assigns to each object x of R its G-orbit Gx, $F_\bullet : \mathrm{MOD}\, R/G \to \mathrm{MOD}\, R$ the *pull-up functor* associated with F, and $F_\lambda : \mathrm{MOD}\, R \to \mathrm{MOD}\, R/G$ the *push-down functor*, left adjoint to $F_\bullet$ (see [9, (3.2)]). Since G acts freely on the isoclasses in $\mathrm{ind}\, R$, F_λ induces an injection from the set $(\mathrm{ind}\, R/\simeq)/G$ of G-orbits of isoclasses in $\mathrm{ind}\, R$ into the set $(\mathrm{ind}\, R/G)/\simeq$ of isoclasses in $\mathrm{ind}\, R/G$ [15, (3.5)]. Let $\mathrm{ind}_1\, R/G$ be the full subcategory of $\mathrm{ind}\, R/G$ consisting of all modules isomorphic to $F_\lambda M$ for some $M \in \mathrm{ind}\, R$, and $\mathrm{ind}_2\, R/G$ the full subcategory of $\mathrm{ind}\, R$ formed by the remaining modules. It was shown in [13, (2.2) and (2.3)] that X from $\mathrm{ind}\, R/G$ belongs to $\mathrm{ind}_1\, R/G$ (respectively, to $\mathrm{ind}_2\, R/G$) if and only if $F_\bullet X$ is a direct sum of indecomposable finite-dimensional R-modules (respectively, weakly G-periodic R-modules).

The category R is called *G-exhaustive* if $\operatorname{ind} R/G = \operatorname{ind}_1 R/G$. From [13, (2.5)], we know that R is G-exhaustive provided it is *locally support-finite*, that is, for each object $x \in R$, the full subcategory of R consisting of the objects of all $\operatorname{supp} M$, where $M \in \operatorname{ind} R$ is such that $M(x) \neq 0$, is finite. Clearly, this class of categories contains locally representation-finite categories playing a crucial role in the study of representation-finite algebras (see [9], [15]).

A *line* in R is a convex subcategory L of R which is isomorphic to the path category of a linear quiver (of type $\mathbb{A}_n$, $\mathbb{A}_\infty$ or ${}_\infty\mathbb{A}_\infty$). A line L is said to be *G-periodic* if G_L is nontrivial. Clearly, in this case the quiver of L is of type

$${}_\infty\mathbb{A}_\infty : \ldots - \bullet - \bullet - \bullet - \ldots$$

We denote by $\mathcal{L}$ the set of all G-periodic lines in R and by $\mathcal{L}_0$ a fixed set of representatives of the G-orbits in $\mathcal{L}$. With each $L \in \mathcal{L}$ we associate a canonical weakly G-periodic R-module M_L by setting $M_L(x) = K$ for $x \in Q_L$, $M_L(x) = 0$ for $x \notin Q_L$ and $M_L(\gamma) = \mathrm{id}_K$ for each arrow γ in Q_L. A weakly G-periodic R-module isomorphic to a module M_L, for some $L \in \mathcal{L}$, is called *linear*. Let $L \in \mathcal{L}_0$. Then $G_{M_L} = G_L = \mathbb{Z}$, and hence the group algebra KG_L is isomorphic to the algebra $K[T, T^{-1}]$ of Laurent polynomials. Then the canonical action of G_L on L gives a left $K[T, T^{-1}]$-module structure on $F_\lambda(M_L)$ such that, for each object a in R/G, the $K[T, T^{-1}]$-module $F_\lambda(M_L)(a)$ is free of finite rank (see [13, (3.6)]). Therefore, we obtain a functor

$$\Phi^L = - \otimes_{K[T,T^{-1}]} F_\lambda(M_L) : \operatorname{mod} K[T, T^{-1}] \longrightarrow \operatorname{mod} R/G$$

where $\operatorname{mod} K[T, T^{-1}]$ denotes the category of finite-dimensional modules over $K[T, T^{-1}]$.

We then get the following consequence of [13, Theorem 3.6].

THEOREM 1.1. *Assume that G acts freely on the isoclasses in $\operatorname{ind} R$ and every weakly G-periodic R-module is linear. Then the functors Φ^L, $L \in \mathcal{L}_0$, induce an equivalence of categories*

$$\Phi : \coprod_{L \in \mathcal{L}_0} \operatorname{mod} K[T, T^{-1}] \xrightarrow{\sim} (\operatorname{mod} R/G)/[\operatorname{mod}_1 R/G]$$

where $[\operatorname{mod}_1 R/G]$ is the ideal in $\operatorname{mod} R/G$ of all morphisms factorized through a module $F_\lambda(Z)$, $Z \in \operatorname{mod} R$. In particular, $\operatorname{ind}_2 R/G$ consists of modules of the form $\Phi^L(V)$, $L \in \mathcal{L}_0$, $V \in \operatorname{ind} K[T, T^{-1}]$. Moreover, R/G is tame if and only if R is tame.

We shall show in the next section that the conditions of Theorem 1.1 are satisfied for R strongly simply connected without hypercritical and *pg*-critical convex subcategories, and G acting freely on the objects of R.

Here, we shall discuss when $\Lambda = R/G$ is of polynomial growth (respectively, domestic). For each dimension d, denote by $\mu^2_\Lambda(d)$ the least number of $L \in \mathcal{L}_0$ such that every $X \in \operatorname{ind}_2 R/G$ of dimension d is isomorphic to some $\Phi^L(V)$ for $L \in \mathcal{L}_0$ and $V \in \operatorname{ind} K[T, T^{-1}]$.

PROPOSITION 1.2. *Assume that $\Lambda = R/G$ is finite, G acts freely on the isoclasses in* $\operatorname{ind} R$ *and every weakly G-periodic R-module is linear. Then the following conditions are equivalent*:

(i) *There exists $c \in \mathbb{N}$ such that $\mu^2_\Lambda(d) \le c$ for any $d \ge 1$.*
(ii) *There exists $m \in \mathbb{N}$ such that $\mu^2_\Lambda(d) \le d^m$ for any $d \ge 1$.*
(iii) *$\mathcal{L}_0$ is finite.*

Proof. The implications (i)⇒(ii) and (iii)⇒(i) are obvious. We show that (ii) implies (iii). Suppose that $\mathcal{L}_0$ is infinite and $\mu^2_\Lambda(d) \le d^m$ for a fixed $m \ge 1$ and all $d \ge 1$. Let $L_1, L_2, L_3, \ldots$ be pairwise different lines from $\mathcal{L}_0$. Since $\Lambda = R/G$ is finite, there exists an object $x \in R$ such that the G-orbit Gx intersects infinitely many lines L_i, $i \ge 1$. Hence, replacing, if necessary, $\mathcal{L}_0$ by another set of representatives of G-orbits in $\mathcal{L}$, we may assume that x belongs to all lines L_i, $i \ge 1$. Observe that, for each $r \ge 1$, there are only finitely many G-periodic lines L passing through x and such that $|L/G_L| \le r$. Moreover, since R is locally bounded and $\Lambda = R/G$ is finite, there is a common bound on the length of nonzero paths in R (that is, paths in Q_R which do not belong to I). This implies that there are two lines $L' = L_i$ and $L'' = L_j$ such that the intersection $L' \cap L''$ contains a convex subcategory of the form

$$\bullet \to y \leftarrow \ldots \leftarrow t \to \ldots \to z \leftarrow \bullet$$

Let $g \in G_{L'}$ be such that $gy \notin L''$ and L' contains a full convex line of the form

$$v:\ y \leftarrow \ldots \leftarrow t \to \ldots \to z \leftarrow \ldots \to gy.$$

Similarly, let $h \in G_{L''}$ be such that $hy \notin L'$ and L'' contains a full convex line of the form

$$w:\ y \leftarrow \ldots \leftarrow t \to \ldots \to z \leftarrow \ldots \to hy.$$

Denote by a the larger of the numbers of points in v and in w. Take a prime number q such that

$$2^q - 2 > 2^{q-1} > q^{m+3} \quad \text{and} \quad q > a^{m+1}.$$

For a positive integer n, denote by v^n the composition $v(gv)\ldots(g^{n-1}v)$ of lines $v, gv, \ldots, g^{n-1}v$, that is, the convex connected subline of L' with targets y and $g^n y$. Similarly, for a positive integer r, denote by w^r the composition

$w(hw)\ldots(h^{r-1}w)$ of the lines $w, hw, \ldots, h^{r-1}w$, which is a convex connected subline of L'' with targets y and $h^r y$.

For any sequence $(n_1, r_1, n_2, r_2, \ldots, n_t, r_t)$ of integers $n_i, r_i \in \mathbb{N}$ with $t \geq 1$, $\sum_{i=1}^{t}(n_i + r_i) = q$, $\sum_{i=1}^{t} n_i > 0$, $\sum_{i=1}^{t} r_i > 0$, we denote by u the following composition of lines:

$$u = v^{n_1}(g^{n_1}w^{r_1})(h^{r_1}g^{n_1}v^{n_2})(g^{n_2}h^{r_1}g^{n_1}w^{r_2}) \ldots (g^{n_t}h^{r_{t-1}}\ldots h^{r_1}g^{n_1}w^{r_t}),$$

and by $L(u)$ the infinite line in R consisting of the objects $f^i c$, $i \in \mathbb{Z}$, $c \in u$, where $f = h^{r_t}g^{n_t}\ldots h^{r_1}g^{n_1}$. Observe that $L(u)$ is a G-periodic line in R with $G_{L(u)}$ generated by f. Let $L(u)$ and $L(u')$ be two such G-periodic lines in R given by the sequences $(n_1, r_1, \ldots, n_t, r_t)$ and $(n'_1, r'_1, \ldots, n'_s, r'_s)$, respectively. Then $L(u)$ and $L(u')$ belong to the same G-orbits in $\mathcal{L}$ if and only if either

$$(n'_1, r'_1, \ldots, n'_s, r'_s) = (b_i, r_i, n_{i+1}, r_{i+1}, \ldots, n_t, r_t, n_1, r_1, \ldots, n_{i-1}, r_{i-1}, c_i, 0)$$

for some $i \geq 1$ and $b_i, c_i \in \mathbb{N}$ with $b_i + c_i = n_i$, or

$$(n'_1, r'_1, \ldots, n'_s, r'_s) = (0, d_i, n_{i+1}, r_{i+1}, \ldots, n_t, r_t, n_1, r_1, \ldots, n_{i-1}, r_{i-1}, n_i, e_i)$$

for some $i \geq 1$ and $d_i, e_i \in \mathbb{N}$ with $d_i + e_i = r_i$.

Consequently, there exist $(2^q - 2)/q$ G-periodic lines $L(u)$ lying in pairwise different G-orbits in $\mathcal{L}$. Denote by B_u the canonical weakly G-periodic R-module associated with $L(u)$. Then, for any $\lambda \in K^*$, the Λ-module

$$M(\lambda, u) = K[T, T^{-1}]/(T - \lambda) \otimes_{K[T,T^{-1}]} F_\lambda(B_u)$$

is indecomposable, belongs to $\operatorname{ind}_2 R/G$ and $\dim M(\lambda, u) \leq aq$. Moreover, $M(\lambda, u) \simeq M(\lambda', u')$ if and only if $\lambda = \lambda'$ and $L(u)$ and $L(u')$ belong to the same G-orbit in $\mathcal{L}$. Therefore, we infer that

$$\sum_{d \leq aq} \mu_\Lambda^2(d) \geq (2^q - 2)/q.$$

Hence there exists $s \leq aq$ such that $\mu_\Lambda^2(s) \geq (2^q - 2)/aq^2$. On the other hand, $\mu_\Lambda^2(s) \leq s^m \leq (aq)^m$ and so $2^q - 2 \leq a^{m+1}q^{m+2}$. But, by our choice of q, we have $2^q - 2 > 2^{q-1} > q^{m+3} > a^{m+1}q^{m+2}$, a contradiction. Consequently, (ii) implies (iii).

The following corollary is a direct consequence of the above proposition and [13, (3.6)].

COROLLARY 1.3. *Under the conditions of the above proposition the following equivalences hold.*

(i) *$\Lambda = R/G$ is of polynomial growth if and only if R is of polynomial growth and $\mathcal{L}_0$ is finite.*

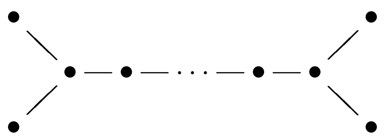
(ii) *$\Lambda = R/G$ is domestic if and only if R is domestic and $\mathcal{L}_0$ is finite.*

2. Tame algebras with simply connected Galois coverings. In this section we are concerned with algebras having strongly simply connected Galois coverings. Following [1] a triangular algebra is called *simply connected* if, for any presentation $A \simeq KQ/I$ as a bound quiver algebra, the fundamental group $\pi_1(Q, I)$ of (Q, I) is trivial. Moreover, following [28], an algebra A is called *strongly simply connected* if every convex subcategory of A is simply connected. It is shown in [28, (4.1)] that an algebra A is strongly simply connected if and only if the first Hochschild cohomology group $H^1(C, C)$ vanishes for any convex subcategory C of A, and if and only if every convex subcategory C of A has the separation property of Bautista, Larrión and Salmerón [4]. Finally, a triangular locally bounded category $R = KQ/I$ is said to be *strongly simply connected* if the following two conditions are satisfied: (1) For any two vertices x and y in Q there are only finitely many paths in Q from x to y (R is interval-finite in the sense of [10]); (2) Every finite convex subcategory C of R is simply connected.

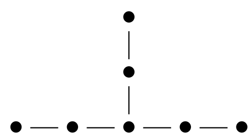
We shall now exhibit some important classes of strongly simply connected algebras playing a crucial role in our investigations.

Observe that a hereditary algebra is simply connected if and only if it is the path algebra of a tree. Let Δ be a finite connected quiver whose underlying graph $\bar{\Delta}$ is a tree, and $H = K\Delta$. Then it is well-known that H is representation-infinite and tame if and only if $\bar{\Delta}$ is one of the Euclidean graphs

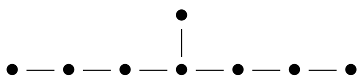
$\widetilde{\mathbb{D}}_n$
$n \geq 4$



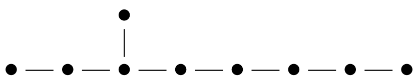
$\widetilde{\mathbb{E}}_6$



$\widetilde{\mathbb{E}}_7$

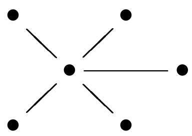


$\widetilde{\mathbb{E}}_8$

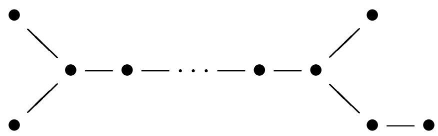


Hence, $H = K\Delta$ is wild if and only if $\bar{\Delta}$ contains one of the following graphs:

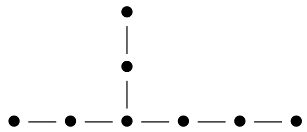
T_5



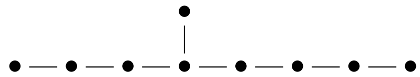
$\widetilde{\widetilde{\mathbb{D}}}_n$



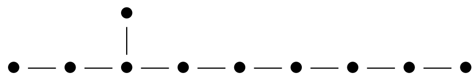
$\widetilde{\widetilde{\mathbb{E}}}_6$



$\widetilde{\mathbb{E}}_7$



$\widetilde{\mathbb{E}}_8$



where in the case of $\widetilde{\mathbb{D}}_n$ the number of vertices is $n+2$, $4 \le n \le 8$.

Assume that $H = K\Delta$ is representation-infinite ($\overline{\Delta}$ is not a Dynkin graph) and T is a preprojective tilting H-module, that is, $\mathrm{Ext}^1_H(T,T) = 0$ and T is a direct sum of $n = |\Delta_0|$ pairwise nonisomorphic indecomposable H-modules lying in the τ_H-orbits of projective modules. Then $C = \mathrm{End}_H(T)$ is called a *concealed algebra of type* $\overline{\Delta}$. It is known that $\mathrm{gl.dim}\, C \le 2$ and C has the same representation type as H. A concealed algebra of type $\overline{\Delta} = \widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8$ (resp. $\overline{\Delta} = T_5, \widetilde{\widetilde{\mathbb{D}}}_n, \widetilde{\widetilde{\mathbb{E}}}_6, \widetilde{\widetilde{\mathbb{E}}}_7, \widetilde{\widetilde{\mathbb{E}}}_8$) is said to be *critical* (resp. *hypercritical*). The critical (resp. hypercritical) algebras have been classified completely in [7], [17] (resp. [19], [32], [33]). It is known [8] that a simply connected algebra A is representation-finite if and only if A does not contain a critical convex subcategory. It is expected (see [22], [26]) that a simply connected algebra A is tame if and only if A does not contain a hypercritical convex subcategory.

Following [24], by a *tubular algebra* we mean a tubular extension of a tame concealed algebra of tubular type $(2,2,2,2)$, $(3,3,3)$, $(2,4,4)$, or $(2,3,6)$. It is known that every tubular algebra is nondomestic of polynomial growth (see [24, (5.2)] and [26, (3.6)]).

In the representation theory of tame simply connected algebras an important role is played by the polynomial growth critical algebras introduced and investigated by R. Nörenberg and A. Skowroński in [21]. By a *polynomial growth critical algebra* (briefly *pg-critical algebra*) we mean an algebra A satisfying the following conditions:

(i) A is one of the matrix algebras

$$B[X] = \begin{bmatrix} K & X \\ 0 & B \end{bmatrix}, \qquad B[Y,t] = \begin{bmatrix} K & K & \dots & K & K & K & Y \\ & K & \dots & K & K & K & 0 \\ & & \ddots & \vdots & \vdots & \vdots & \vdots \\ & & & K & K & K & 0 \\ & & & & K & 0 & 0 \\ & 0 & & & & K & 0 \\ & & & & & & B \end{bmatrix}$$

where B is a representation-infinite tilted algebra of Euclidean type $\widetilde{\mathbb{D}}_n$,

$n \geq 4$, with a complete slice in the preinjective component of Γ_B, X (respectively, Y) is an indecomposable regular B-module of regular length 2 (respectively, regular length 1) lying in a tube of Γ_B with $n-2$ rays, $t+1$ ($t \geq 2$) is the number of isoclasses of simple $B[Y,t]$-modules which are not B-modules.

(ii) Every proper convex subcategory of A is of polynomial growth.

The *pg*-critical algebras have been classified by quivers and relations in [21]. There are 31 frames of such algebras. In particular, it is known that if A is a *pg*-critical algebra then: (1) A is tame minimal of nonpolynomial growth, (2) gl.dim $A = 2$, (3) A is simply connected, (4) the opposite algebra A^{op} is also *pg*-critical. There are only 16 frames of *pg*-critical algebras which are strongly simply connected.

We may now recall the following criteria for the polynomial growth (respectively, domestic type) of strongly simply connected algebras given in [31, (4.1) and (4.3)].

THEOREM 2.1. *Let A be a strongly simply connected algebra. Then*

(i) *A is of polynomial growth if and only if A does not contain a convex subcategory which is hypercritical or pg-critical.*

(ii) *A is domestic if and only if A does not contain a convex subcategory which is hypercritical*, *pg-critical or tubular.*

We shall need the following lemma.

LEMMA 2.2. *Let B be a strongly simply connected* (*finite*) *locally bounded category of one of the types*: *critical*, *tubular*, *pg-critical*, *or hypercritical. Then any K-linear automorphism of B fixes at least one of its objects.*

Proof. If B is critical, *pg*-critical, or hypercritical, then our claim follows from a direct inspection of the frames of critical, *pg*-critical and hypercritical algebras given in [7], [17], [21], [32], respectively. Assume B is a tubular algebra and g a K-linear automorphism of B. It follows from [24, (5.2)] that Γ_B admits a unique preprojective component $\mathcal{P}$ whose support algebra is a convex critical subcategory C of B. But then $gC = C$, and consequently g fixes an object of C, and hence of B.

PROPOSITION 2.3. *Let R be a strongly simply connected locally bounded category and G a group of K-linear automorphisms of R acting freely on the objects of R. Assume that R does not contain a convex subcategory which is hypercritical or pg-critical. Then G acts freely on the isoclasses in* $\operatorname{ind} R$.

Proof. Let M be a module in $\operatorname{ind} R$ and g an element of G such that ${}^gM \simeq M$. Denote by Λ the convex hull of $\operatorname{supp} M$ in R. Since R is interval-finite, Λ is a finite convex subcategory of R. Hence, by our assumptions on R, Λ is strongly simply connected and does not contain a convex subcategory which is hypercritical or pg-critical. Moreover, $gΛ = Λ$, and M is a $Λ$-module. If $Λ$ is representation-finite, then, by a result due to R. Martínez and J. A. de la Peña [20], ${}^gM \simeq M$ implies $g = 1$, and we are done. Therefore, assume that $Λ$ is representation-infinite. We shall show that then there exists a convex critical subcategory C of $Λ$ such that $gC = C$. Then Lemma 2.2 will imply $g = 1$. We have two cases to consider.

Assume first that M is a directing $Λ$-module, that is, M does not lie on an oriented cycle $M = M_0 \to M_1 \to \ldots \to M_r = M_0$, $r \geq 1$, of nonzero nonisomorphisms between indecomposable $Λ$-modules. By the well-known convexity argument of Bongartz [6, (3.2)] we have $\operatorname{supp} M = Λ$. Further, by [24, p. 376], $Λ$ is a tilted algebra. Moreover, we conclude from Theorem 2.1 that $Λ$ is tame. Consequently, $Λ$ is a tame tilted algebra and M is an indecomposable sincere $Λ$-module lying in a connecting component of $\Gamma_Λ$. Then it follows from [23] that $\Gamma_Λ$ admits exactly one preprojective component and exactly one preinjective component, and moreover one of them is of Euclidean type. By symmetry we may assume that $\Gamma_Λ$ admits a preprojective component $\mathcal{P}$ of Euclidean type. The support algebra B of $\mathcal{P}$ is then a convex subcategory of $Λ$ which is a tubular coextension (of Euclidean type) of a critical convex subcategory C of $Λ$. Clearly, the automorphism $g : Λ \to Λ$ induces an automorphism ${}^g(-) : \operatorname{ind} Λ \to \operatorname{ind} Λ$, and then ${}^g\mathcal{P} = \mathcal{P}$. Hence, $gB = B$, and consequently $gC = C$, because C is a unique critical convex subcategory of B.

Assume now that M is a nondirecting $Λ$-module. Since $Λ$ is strongly simply connected of polynomial growth (by Theorem 2.1) and $Λ$ is the convex hull of $\operatorname{supp} M$, it follows from [31, (4.8)] that $Λ$ is a coil enlargement (in the sense of [2]) of a critical convex subcategory C of $Λ$ and M lies in one of the standard coils $\mathcal{C}_\lambda$ of a weakly separating family $\mathcal{C} = (\mathcal{C}_\lambda)_{\lambda \in \mathbb{Q}_1(K)}$ of standard coils of $\Gamma_Λ$. Then ${}^gM \simeq M$ implies that g maps all stable tubes of rank 1 in $\mathcal{C}$ onto stable tubes of rank 1 of $\mathcal{C}$. Since C is the support algebra of any stable tube of rank 1 in $\mathcal{C}$ we get $gC = C$. This finishes the proof.

A group G of K-linear automorphisms of a locally bounded category R is said to be *admissible* if its action on the objects of R is free and has finitely many orbits. In such a case, R/G is a finite category (algebra).

THEOREM 2.4. *Let R be a strongly simply connected locally bounded K-category, G an admissible group of K-linear automorphisms of R and $A = R/G$. Assume that R does not contain a convex subcategory which is hypercritical or pg-critical. Then*

(i) *Every indecomposable finite-dimensional A-module Z is isomorphic either to $F_\lambda(X)$, for some indecomposable finite-dimensional R-module X,* *or to* $V \otimes_{K[T,T^{-1}]} F_\lambda(M_L)$ *for some* $L \in \mathcal{L}_0$ *and some indecomposable finite-dimensional* $K[T,T^{-1}]$*module* V.

(ii) *We have*

$$\Gamma_A = (\Gamma_R/G) \vee \Big(\bigvee_{L\in\mathcal{L}_0} \Gamma_{K[T,T^{-1}]} \Big),$$

where $\Gamma_{K[T,T^{-1}]}$ *is the Auslander–Reiten quiver of the category of finite-dimensional* $K[T,T^{-1}]$*-modules.*

(iii) A *is tame.*

P r o o f. We know from Proposition 2.3 that G acts freely on the isoclasses in $\operatorname{ind} R$. Hence, the push-down functor $F_\lambda : \operatorname{mod} R \to \operatorname{mod} A$ preserves the Auslander–Reiten sequences and induces an injection from the set of G-orbits of isoclasses in $\operatorname{ind} R$ into the set of isoclasses in $\operatorname{ind} A$ (see [15, Section 3]). Therefore, by Theorem 1.1, it is enough to show that every weakly G-periodic R-module is linear. We shall apply the results of [13, Section 4], showing that the weakly G-periodic R-modules are limits of G-periodic sequences of finite-dimensional indecomposable R-modules.

For a full subcategory C of R we denote by $\widehat{C}$ the full subcategory of R formed by all object x such that $R(x,y) \neq 0$ or $R(y,x) \neq 0$ for some object $y \in C$. Clearly, if C is finite, then $\widehat{C}$ is also finite because R is locally bounded. For an R-module Z and a full subcategory C of R, we denote by $Z|C$ the restriction of Z to C. Finally, for $X, Y \in \operatorname{MOD} C$ we write $X \Subset Y$ whenever X is isomorphic to a direct summand of Y.

Fix a family C_n, $n \in \mathbb{N}$, of finite convex subcategories of R such that

(1) For each $n \in \mathbb{N}$, C_{n+1} is the convex hull of $\widehat{C}_n$ in R.
(2) $R = \bigcup_{n\in\mathbb{N}} C_n$.

Since R is connected, locally bounded and interval-finite, such a family exists. We shall identify a C_n-module M with an R-module, by setting $M(x) = 0$ for all objects x of R which are not in C_n.

Let Y be a weakly G-periodic R-module. We show that Y is linear. Let $m \in \mathbb{N}$ be the least number such that $Y|C_m \neq 0$. We define a family $Y_n \in \operatorname{ind} C_n$, $n \in \mathbb{N}$, as follows. Put $Y_n = 0$ for $n < m$ and let Y_m be an arbitrary indecomposable direct summand of $Y|C_m$. Then there exist $Y_{m+1} \in \operatorname{ind} C_{m+1}$ and a splittable monomorphism $u_m : Y_m \to Y_{m+1}|C_m$ such that $Y_{m+1} \Subset (Y|C_{m+1})$. Repeating this procedure we can find, for all $n \geq m$, $Y_n \in \operatorname{ind} C_n$ and splittable monomorphisms $u_n : Y_n \to Y_{n+1}|C_n$ such that $Y_n \Subset (Y|C_n)$. Thus we obtain a sequence $(Y_n, u_n)_{n\in\mathbb{N}}$, called in [13] a fundamental R-sequence produced by Y. Since in our case C_n are convex subcategories of R, it is in fact a sequence of finite-dimensional indecomposable R-modules. The following facts are direct consequences of [13, (4.3), (4.4), (4.5)]:

(α) $Y = \varinjlim Y_n$.

(β) For each $n \in \mathbb{N}$, there exists $p \geq n$ such that $Y_p|C_n \simeq Y|C_n$.

(γ) For each $g \in G_Y$ and $n \in \mathbb{N}$, there exists $q \geq n$ such that

$$gC_n \subset C_q \quad \text{and} \quad {}^gY_n \Subset (Y_q|gC_n).$$

For $n \geq m$, denote by B_n the support of Y_n. Clearly $B_n \subset C_n$. Moreover, since Y is indecomposable, infinite-dimensional, locally finite-dimensional, and C_{n+1} contains $\widehat{C}_n$, for each $n \in \mathbb{N}$, we deduce from [12, Lemma 2] that, for any $n \geq m$, B_n is not contained in C_{n-1}.

Let $s = m + 14$. Then each of the categories B_n, $n \geq s$, has at least 14 objects. Moreover, fix an element $1 \neq g \in G_Y$.

Assume first that all categories B_n, $n \geq s$, are representation-finite. We know from [31, (4.9)] that such B_n is a strongly simply connected convex subcategory of R, and hence belongs to the 24 families listed by Bongartz in [5, (2.4)]. We know that $Y = \varinjlim Y_n$. Moreover, it follows from (γ) that for any $1 \neq g \in G_Y$ and $n \geq s$ there exists $r \geq n$ such that $gC_n \subset C_r$, ${}^gY_n \Subset (Y_r|gC_n)$, and obviously $Y_n \Subset (Y_r|C_n)$. Using now the structure of indecomposable sincere modules over Bongartz's 24 families of algebras and the fact that G acts freely on the isoclasses in $\operatorname{ind} R$, we deduce that all B_n are lines. Hence, by (β), $L = \operatorname{supp} Y$ is a (convex) line of type ${}_\infty\mathbb{A}_\infty$, and clearly $G_L = G_Y$ is nontrivial. Consequently, Y is a linear R-module.

Assume now that B_p, for some $p \geq m$, is representation-infinite. Then, by [31, (4.9)], $B_p = \operatorname{supp} Y_p$ contains a critical full subcategory D. Take an arbitrary $r \in \mathbb{N}$ and consider the critical full subcategories $D, gD, \ldots, g^r D$ of R. From Lemma 2.2 and our proof of Proposition 2.3 we infer that these categories are pairwise different. From (γ) we see that, for each $0 \leq i \leq r$, there exists $t_i \geq p$ such that $g^i C_p \subset C_{t_i}$ and ${}^{g^i}Y_p \Subset (Y_{t_i}|g^i C_p)$. Take $q \in \mathbb{N}$ such that C_q contains all categories C_{t_i}, $0 \leq i \leq r$. Then $Y_{t_i} \Subset (Y_q|C_{t_i})$ and hence ${}^{g^i}Y_p \Subset (Y_q|g^i C_p)$ for $i = 0, \ldots, r$. Therefore, Y_q is an indecomposable finite-dimensional R-module whose support contains $D, gD, \ldots, g^r D$ as full subcategories. On the other hand, we know from [31, (4.10)] that for any module $X \in \operatorname{ind} R$ the convex hull of $\operatorname{supp} X$ in R contains at most 3 convex critical subcategories. This implies that there is a common bound on the number of critical full subcategories of R which belongs to one G-orbit and are full subcategories of the support of a module from $\operatorname{ind} R$. Hence, taking r large enough, we get a contradiction. This finishes the proof.

As a direct consequence of Theorem 2.4 and [13, (5.2)] we get the following fact.

COROLLARY 2.5. *Assume that R and G are as in the above theorem and $d = \dim_K R/G$. Then the following conditions are equivalent*:

(i) *The push-down functor $F_\lambda : \mathrm{mod} R \to \mathrm{mod} R/G$ is dense.*
(ii) *R is locally support-finite.*
(iii) $\mathrm{ind}\, R = \mathrm{Ind}\, R$.
(iv) *G acts freely on the isoclasses in* $\mathrm{Ind}\, R$.
(v) *R does not contain a line of type $\mathbb{A}_{2d+1}$.*

The following theorem gives a criterion for the polynomial growth (respectively, domestic type) of algebras with strongly simply connected Galois coverings.

THEOREM 2.6. *Let R be a strongly simply connected locally bounded K-category*, *G an admissible group of K-linear automorphisms of R and $A = R/G$. Then*

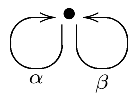
(i) *A is of polynomial growth if and only if R does not contain a convex subcategory which is hypercritical or pg-critical*, *and the number of G-orbits of G-periodic lines in R is finite.*

(ii) *A is domestic if and only if R does not contain a convex subcategory which is hypercritical*, *pg-critical or tubular*, *and the number of G-orbits of G-periodic lines in R is finite.*

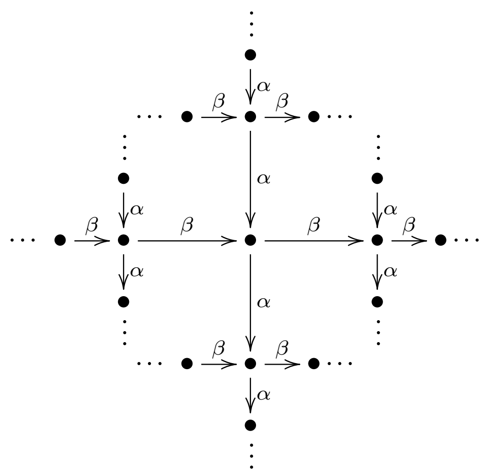
P r o o f. Let M be an indecomposable finite-dimensional R-module whose support is a convex subcategory of R of one of the types: hypercritical, *pg*-critical, or tubular. Then it follows from Lemma 2.2 that ${}^gM \not\simeq M$ for any $1 \neq g \in G$, and hence the push-down $F_\lambda(M)$ is an indecomposable A-module. Assume now that A is of polynomial growth (respectively, domestic). From the above remark and Theorem 2.1 we infer that R does not contain a convex subcategory which is hypercritical or *pg*-critical (respectively, hypercritical, *pg*-critical or tubular). Moreover, by Proposition 2.3, Theorem 2.4 and Corollary 1.3, we then infer that the number of G-orbits of G-periodic lines in R is finite. Conversely, assume that R does not contain a convex subcategory which is hypercritical or *pg*-critical (respectively, hypercritical, *pg*-critical or tubular) and the number of G-orbits of G-periodic lines in R is finite. Then it follows from Theorems 2.1 and 2.4, Proposition 2.3, Corollary 1.3 that A is of polynomial growth (respectively, domestic). This finishes the proof.

3. Examples and remarks. (3.1) In the notation of the above theorem, Bongartz's criterion [8] for finite representation type can be formulated as follows: A is of finite representation type if and only if R does not contain a convex subcategory which is critical or a G-periodic line. The latter is equivalent to the nonexistence of a line of Dynkin type $\mathbb{A}_{2d+1}$, where $d = \dim_K A$.

(3.2) Let A be the algebra $K[x,y]/(x^3, y^3, xy)$. Then $A = KQ/I$ where Q is the quiver

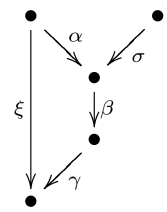


and I is generated by α^3, β^3, $\alpha\beta$, $\beta\alpha$. Further, A admits a strongly simply connected Galois covering $F : R \to R/G = A$ where $R = K\widetilde{Q}/\widetilde{I}$ is the locally bounded K-category given by the quiver $\widetilde{Q}$ of the form

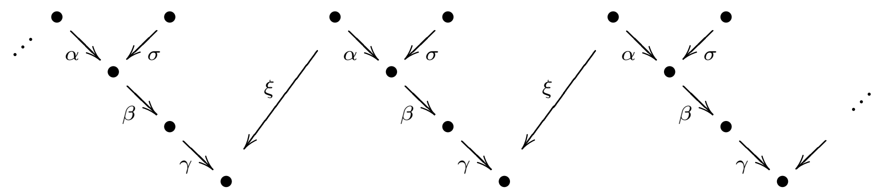


and $\widetilde{I}$ is generated by all paths α^3, β^3, $\alpha\beta$, $\beta\alpha$, and where G is the free (nonabelian) group of K-linear automorphisms of R generated by the α-shift and β-shift. It is well-known that the support of any module from $\operatorname{ind} R$ is a line, and hence every finite convex subcategory of R is representation-finite. In particular, R has no convex subcategory which is hypercritical or pg-critical, and so R satisfies the assumptions of Theorem 2.4. On the other hand, it is easy to see that R admits infinitely many G-orbits of G-periodic lines. Therefore, A is tame but not of polynomial growth. In fact, all special biserial algebras are tame and have strongly simply connected Galois coverings with all finite subcategories being representation-finite (see [13, (5.2)]).

(3.3) Let A be the bound quiver algebra KQ/I where Q is the quiver



and I is generated by the path $\sigma\beta\gamma$. Then A admits a strongly simply connected Galois covering $F : R \to R/G = A$, where $R = k\widetilde{Q}/\widetilde{I}$ is the locally bounded category given by the quiver $\widetilde{Q}$ of the form

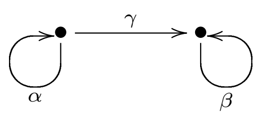


and $\widetilde{I}$ is generated by all paths $\sigma\beta\gamma$, and where G is the infinite cyclic group generated by the obvious shift g of R. Then again every finite convex subcategory of R is representation-finite. Moreover, R admits exactly one G-periodic line. Hence, A is representation-infinite domestic (even one-parametric).

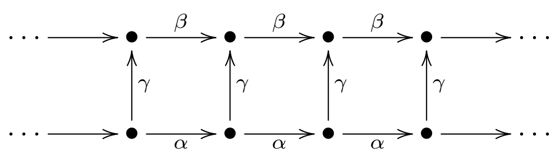
(3.4) Let A be the algebra

$$\begin{bmatrix} K[x]/(x^4) & K[x]/(x^4) \\ 0 & K[x]/(x^4) \end{bmatrix}.$$

Then $A = KQ/I$ where Q is the quiver



and I is generated by α^4, β^4, $\alpha\gamma - \gamma\beta$. Then A admits a strongly simply connected Galois covering $F : R \to R/G = A$ where $R = K\widetilde{Q}/\widetilde{I}$ is the locally bounded K-category given by the quiver $\widetilde{Q}$



and $\widetilde{I}$ is generated by all elements of the form α^4, β^4, $\alpha\gamma - \gamma\beta$, and where G is the infinite cyclic group generated by the obvious shift g of R. It was shown in [25] that R is locally support-finite of polynomial growth (hence without hypercritical and pg-critical convex subcategories) and contains convex subcategories which are tubular. Hence, in this case, the push-down functor $F_\lambda : \mathrm{mod} R \to \mathrm{mod} A$ is dense and A is nondomestic of polynomial growth. In fact, all tame triangular matrix algebras $\left(\begin{smallmatrix} \Lambda & \Lambda \\ 0 & \Lambda \end{smallmatrix}\right)$ over Nakayama algebras Λ are of polynomial growth and have such nice strongly simply connected Galois coverings [25].

(3.5) We refer to [27] for a description of polynomial growth selfinjective algebras having simply connected Galois coverings.

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