COLLOQUIUM MATHEMATICUM

VOL. 72 1997 NO. 2

BOUNDED PICARD GROUPS

BY

GENE ABRAMS AND
JEREMY HAEFNER (COLORADO SPRINGS, COLORADO)

We recall (see e.g. [4]) that a ring A is said to have *local units* in case for each finite subset $\mathcal{F}$ of A there is an idempotent e of A such that $\mathcal{F}\subset eAe$. In particular, any unital ring has local units. In [1] we investigated the Picard group $\mathrm{Pic}(A)$ of a ring A with local units. Specifically, we showed that there is a strong connection between $\mathrm{Pic}(A)$ and $\mathrm{Pic}(B)$, where B is the unital ring $\mathrm{End}(_AA)$. We used these results to show, among other things, that $\mathrm{Pic}(R)$ is isomorphic to $\mathrm{Pic}(\mathrm{RFM}(R))$ for any unital ring R, where $\mathrm{RFM}(R)$ denotes the (unital) ring of countably infinite row-finite matrices with entries from R.

In this note we focus primarily on the differences between Picard groups for rings with local units, and their unital brethren. To wit, we consider the left module structure of elements of $\mathrm{Pic}(A)$. For a unital ring R, every $P\in\mathrm{Pic}(R)$ has $_RP$ finitely generated projective, so that there exists a split epimorphism $_RR^{(n)}\to P$ for some integer n. In general, this property does not extend to rings with local units; we say a ring with local units A has *bounded Picard group* if this property does hold for all elements of $\mathrm{Pic}(A)$. Our interest in rings with bounded Picard groups stems from [1, Theorem 1.14], in which an isomorphism is established between a "bounded" subgroup of $\mathrm{Pic}(A)$ and a corresponding subgroup of $\mathrm{Pic}(\mathrm{End}(_AA))$.

Our two main objectives of this note are to show that the groups $\mathrm{Pic}(A)$ and $\mathrm{Pic}(\mathrm{End}(_AA))$ need not be isomorphic, and to show that the boundedness property is not in general a Morita invariant. We conclude the note by mentioning some situations in which boundedness is in fact an invariant.

Throughout this paper A will denote a ring with local units with set of idempotents E; B will denote $\mathrm{End}(_AA)$. For each $a\in A$ we have $\varrho_a\in B$ via $(x)\varrho_a=xa$; the map $\varrho:A\to B$ via $a\mapsto\varrho_a$ gives an embedding of A in B as a right ideal.

A left A-module M is called *unitary* in case $AM=M$; the category A-mod is defined to be the collection of unitary left A-modules, together with usual homomorphisms. Unless otherwise indicated, the word *module*

1991 *Mathematics Subject Classification*: 16D20, 16D40, 16W20, 16S50.

(resp. *bimodule*) will always mean *unitary module* (resp. *unitary bimodule*). All module homomorphisms will be written opposite the scalars.

The rings A and A' are said to be *Morita equivalent* in case the module categories A-mod and A'-mod are equivalent. The best source for information about Morita equivalent rings with local units is [4].

Analogously to the definition for unital rings, the group $\mathrm{Pic}(A)$ consists of those A-A-bimodules P (which by definition are both left and right unitary) for which there exists a (left and right unitary) A-A-bimodule Q having $P \otimes_A Q \cong A$ and $Q \otimes_A P \cong A$ as A-A-bimodules. By [4, Theorem 2.2], $\mathrm{Pic}(A)$ is precisely the group of category autoequivalences of A-mod. As such, any two Morita equivalent rings with local units necessarily have isomorphic Picard groups. Additional terminology and examples regarding Picard groups for rings with local units can be found in [1], while [5] is a good source of information about Picard groups for unital rings.

If M is a left A-module and n is any integer then $\mathrm{add}_n(M)$ denotes the collection of those left A-modules which are direct summands of a direct sum of at most n copies of M. We denote by $\mathrm{add}(M)$ the union $\bigcup_{n\in\mathbb{N}} \mathrm{add}_n(M)$; that is, $\mathrm{add}(M)$ is the collection of left A-modules which are direct summands of a direct sum of some finite number of copies of M.

1. Rings with bounded Picard groups. In light of the motivation and discussion presented above, we are now in a position to give the definition of the main idea at hand.

DEFINITION 1.1. The ring with local units A is said to have *bounded Picard group* in case for each $P \in \mathrm{Pic}(A)$ we have ${}_AP \in \mathrm{add}(A)$. That is, A has bounded Picard group in case for each element P of the Picard group of A there exists some integer n for which there is a split epimorphism $A^n \to P$ of left A-modules. We say A has *unbounded Picard group* otherwise. For an integer N we say that A has *N-bounded Picard group* in case for each $P \in \mathrm{Pic}(A)$ we have ${}_AP \in \mathrm{add}_N(A)$.

As mentioned above, every unital ring has bounded Picard group. We showed in [1] that for any unital ring R, the ring $\mathrm{FM}(R)$ (consisting of countably infinite square matrices over R, each having at most finitely many nonzero entries) has bounded Picard group; we indicate in the present article another proof of this result. We give many additional examples of classes of rings having bounded Picard group in [2]. In contrast, we presented in [1, Example 1.15] a description of a class of rings having unbounded Picard groups. After again describing these rings here, we show in Proposition 1.8 below that we may in fact construct such a ring A for which $\mathrm{Pic}(A)$ and $\mathrm{Pic}(B)$ are nonisomorphic.

LEMMA 1.2. *Let I be any set, and for each $i \in I$ let R_i be a unital ring. Suppose that $P_i \in \mathrm{Pic}(R_i)$ for each $i \in I$. Let $A = \bigoplus_{i\in I} R_i$, and let P denote the A-A-bimodule $\bigoplus_{i\in I} P_i$. Then $P \in \mathrm{Pic}(A)$.*

Proof. For each $i \in I$ let Q_i denote P_i^{-1} in $\mathrm{Pic}(R_i)$, let Q denote the A-A-bimodule $\bigoplus_{i\in I} Q_i$, and let e_i denote the element of A which is 1_{R_i} in the ith coordinate and zero elsewhere. We show that $P^{-1} = Q$. We note that if $i \neq j$ then by the definition of the action of A on P we have $P_i \otimes_A Q_j = P_i \cdot e_i \otimes_A Q_j = P_i \otimes_A e_i \cdot Q_j = P_i \otimes_A 0 = 0$. Using this observation along with the fact that tensor products commute with direct sums (see e.g. [3, Theorem 19.10]) we have

$$\begin{aligned} P \otimes_A Q = \Big(\bigoplus_{i\in I} P_i\Big) \otimes_A \Big(\bigoplus_{j\in I} Q_j\Big) &\cong \bigoplus_{i\in I}\bigoplus_{j\in I} (P_i \otimes_A Q_j) \\ &\cong \bigoplus_{i\in I} (P_i \otimes_A Q_i) \cong \bigoplus_{i\in I} R_i \cong A. \end{aligned}$$

Similarly one can show that $Q \otimes_A P \cong A$. ∎

For the entirety of the discussion up to and including Proposition 1.8, we shall reserve the symbols k_i, A, B, R_i, C, and C_i for the following specific rings.

NOTATION 1.3. Let $\{k_i\}_{i\in\mathbb{N}}$ be any set of pairwise nonisomorphic fields for which each automorphism group $\mathrm{Aut}(k_i)$ is trivial. (For instance, let $k_i = \mathbb{Z}_{p_i}$, the field of p_i elements, where p_i denotes the ith prime.) Let

$$R_i = M_i(k_i) \oplus k_i \quad (M_i \text{ denotes } i \times i \text{ matrices}),$$
$$A = \bigoplus_{i\in\mathbb{N}} R_i, \quad B = \mathrm{End}({}_A A) = \prod_{i\in\mathbb{N}} R_i,$$
$$C = \mathrm{Center}(B), \quad C_i = \mathrm{Center}(R_i).$$

LEMMA 1.4. *For $i \in \mathbb{N}$, each ring R_i has $\mathrm{Pic}(R_i) \cong \mathbb{Z}_2$ (the cyclic group of order 2). Specifically, the nontrivial element of $\mathrm{Pic}(R_i)$ is represented by viewing*

$$R_i = \begin{pmatrix} & & & 0 \\ & M_i(k_i) & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & k_i \end{pmatrix} \quad \text{and} \quad P_i = \begin{pmatrix} & & & k_i \\ & 0 & & \vdots \\ & & & k_i \\ k_i & \cdots & k_i & 0 \end{pmatrix},$$

where R_i is viewed inside the matrix ring $M_{i+1}(k_i)$. Moreover, each P_i requires at least i generators as a left R_i-module.

Proof. We first note that the basic ring of each R_i is $k_i \oplus k_i$ and so $\mathrm{Pic}(R_i) \cong \mathrm{Pic}(k_i \oplus k_i)$. An easy dimension argument shows that every element of $\mathrm{Pic}(k_i \oplus k_i)$ is isomorphic to $k_i \oplus k_i$ as left modules. Hence by the commutativity of k_i and [5, Theorem 55.13] we have $\mathrm{Pic}(k_i \oplus k_i) \cong \mathrm{Aut}(k_i \oplus k_i)$.

Since by hypothesis $\mathrm{Aut}(k_i)$ is trivial, it is easy to show that $\mathrm{Aut}(k_i \oplus k_i)$ has only two elements, the nontrivial element being the automorphism which interchanges coordinates.

Thus we know that each $\mathrm{Pic}(R_i)$ is a group of two elements. It is easy to verify that P_i is an R_i-R_i-bimodule by usual matrix multiplication. Furthermore, $P_i \otimes_{R_i} P_i \cong R_i$ as R_i-R_i-bimodules via multiplication, so that $P_i \in \mathrm{Pic}(R_i)$. Finally, we note that the existence of the i independent entries along the bottom row of P_i ensures that P_i requires at least i generators as a left R_i-module. ∎

As one consequence of the above discussion we get

COROLLARY 1.5. *The ring A has unbounded Picard group.*

Proof. Let P_i be the non-identity element of $\mathrm{Pic}(R_i)$ for each $i \in \mathbb{N}$, and set $P = \bigoplus_{i\in\mathbb{N}} P_i$; by Lemma 1.2, $P \in \mathrm{Pic}(A)$. However, ${}_AP$ cannot be the epimorphic image of a direct sum of finitely many copies of A, as each P_i requires at least i generators. ∎

LEMMA 1.6. *Let e_i denote the element of the ring B which is 1_{R_i} in the ith coordinate and zero elsewhere. Then the ring $C = \mathrm{Center}(B)$ has the properties: $C = \prod_{i\in\mathbb{N}} C_i$; $e_i \in C$ for each i; and for any $\varphi \in \mathrm{Aut}(C)$ and $i \in \mathbb{N}$ we have $(e_i)\varphi = e_i$. Consequently, $\mathrm{Aut}(C) = \prod_{i\in\mathbb{N}} \mathrm{Aut}(C_i)$.*

Proof. The first two statements are clear.

As $\mathrm{Center}(M_i(k_i)) \cong k_i$ (these are just the scalar matrices), we have $C_i \cong k_i \oplus k_i$, which yields $C \cong \prod_{i\in\mathbb{N}} k_i \oplus k_i$.

Let u_i (resp. h_i) denote the element of C which is $(1_{M_i(k_i)}, 0)$ (resp. $(0, 1_{k_i})$) in the ith coordinate and zero in all other coordinates. It is easy to show that any primitive idempotent of C is either of the form u_i or h_i for some i. (We note that even though u_i is not primitive in $M_i(k_i)$, it clearly is primitive in C_i.) Since automorphisms of any ring must preserve primitive idempotents, for each i we have $(h_i)\varphi = h_j$ or u_j for some j. But if v_j denotes either h_j or u_j we have

$$k_i \cong h_i C h_i \cong (h_i C h_i)\varphi = (h_i)\varphi(C)\varphi(h_i)\varphi = v_j C v_j \cong k_j,$$

so that $i = j$ as the $\{k_i\}_{i\in\mathbb{N}}$ were chosen to be pairwise nonisomorphic. Thus $(h_i)\varphi = h_i$ or u_i. Similarly one can show that $(u_i)\varphi = u_i$ or h_i. As $e_i = u_i + h_i$ this then yields that $(e_i)\varphi = e_i$ for all i, and the result follows.

For the final statement, note that for any $\varphi \in \mathrm{Aut}(C)$ we have $(C_i)\varphi = (e_i C e_i)\varphi = (e_i)\varphi(C)\varphi(e_i)\varphi = e_i C e_i = C_i$, so that any automorphism of C leaves each of the factors C_i invariant. ∎

COROLLARY 1.7. *Let X be any element of the group $\mathrm{Pic}(B)$. Then for each $i \in \mathbb{N}$ and each $x \in X$ we have $e_i x = x e_i$. In particular, $e_i X = e_i X e_i = X e_i$.*

P r o o f. As each e_i is central in B we may apply [5, Lemma 55.7] to conclude that there exists $\varphi \in \mathrm{Aut}(C)$ such that $e_i x = x \cdot (e_i)\varphi$. But any $\varphi \in \mathrm{Aut}(C)$ has $(e_i)\varphi = e_i$ for all $i \in \mathbb{N}$ by the previous lemma, so that $e_i x = x e_i$ as desired. The second part is immediate. ∎

We are now in a position to show that $\mathrm{Pic}(A)$ is not isomorphic to $\mathrm{Pic}(B)$.

PROPOSITION 1.8. *Let A and B be the rings defined above. Then* $\mathrm{Pic}(A)$ *is uncountable*, *while* $\mathrm{Pic}(B)$ *is countable. In particular*, $\mathrm{Pic}(A) \not\cong \mathrm{Pic}(B)$.

P r o o f. By Lemma 1.2, any A-A-bimodule of the form $\bigoplus_{i\in\mathbb{N}} Q_i$ is an element of $\mathrm{Pic}(A)$, where $Q_i \in \mathrm{Pic}(R_i)$ for each i. By the definition of the A-module action, two left A-modules of the form $\bigoplus_{i\in\mathbb{N}} Q_i$ and $\bigoplus_{i\in\mathbb{N}} Q_i'$ are isomorphic if and only if $Q_i \cong Q_i'$ as left R_i-modules for each $i \in \mathbb{N}$. Thus $\mathrm{Pic}(A)$ is uncountable, since by Lemma 1.4 there are two choices for Q_i for each $i \in \mathbb{N}$, and any collection of distinct choices produces distinct elements of $\mathrm{Pic}(A)$.

We claim on the other hand that $\mathrm{Pic}(B)$ is countable. To see this, it suffices to show that any element of $\mathrm{Pic}(B)$ is of the form $\prod_{i\in\mathbb{N}} P_i$, where each $P_i \in \mathrm{Pic}(R_i)$, and there exists some integer s with the property that $P_i = R_i$ for all $i > s$.

We begin by noting that each R_i is self-injective (as each is semisimple); this in turn yields that B is self-injective, so that ${}_BP$ is injective for any $P \in \mathrm{Pic}(B)$. Furthermore, for each $i \in \mathbb{N}$ we see that e_iP is a B-submodule of P, and that $\bigoplus_{i\in\mathbb{N}} e_iP \le P$. Also, the map $P \to \prod_{i\in\mathbb{N}} e_iP$ is injective, so that $\bigoplus_{i\in\mathbb{N}} e_iP \le P \le \prod_{i\in\mathbb{N}} e_iP$. But $\bigoplus_{i\in\mathbb{N}} e_iP$ is an essential submodule of $\prod_{i\in\mathbb{N}} e_iP$; with the injectivity of P, we conclude that in fact $P = \prod_{i\in\mathbb{N}} e_iP$.

We claim that each e_jP is in $\mathrm{Pic}(R_j)$. Let $Q = P^{-1}$ in $\mathrm{Pic}(B)$. Arguing as above, using the fact that B is also right-self-injective, we have $Q = \prod_{j\in\mathbb{N}} Qe_j$ as right B-modules. But finitely generated projective modules commute with direct products (on the appropriate sides); in addition, any bimodule structure is clearly preserved. Applying this observation first to the finitely generated projective left B-module P, and then to the finitely generated projective right B-modules Qe_j, we have the series of B-B-bimodule isomorphisms

$$\begin{aligned} B &\cong Q \otimes_B P = \Big(\prod_{j\in\mathbb{N}} Qe_j\Big) \otimes_B P \cong \prod_{j\in\mathbb{N}} (Qe_j \otimes_B P) \\ &\cong \prod_{j\in\mathbb{N}} \Big(Qe_j \otimes_B \prod_{i\in\mathbb{N}} e_iP\Big) \cong \prod_{j\in\mathbb{N}} \prod_{i\in\mathbb{N}} (Qe_j \otimes_B e_iP) \\ &\cong \prod_{j\in\mathbb{N}} \prod_{i\in\mathbb{N}} (Qe_j \otimes_B e_iP) \cong \prod_{j\in\mathbb{N}} (Qe_j \otimes_B e_jP) \end{aligned}$$

(as $Qe_j \otimes_B e_iP = 0$ for $i \ne j$). On multiplication by e_j this yields R_j-R_j-

bimodule isomorphisms ${}_{R_j}R_{j R_j} \cong e_jQe_j \otimes_{R_j} e_jPe_j$. But by Corollary 1.7 we have $e_jQe_j = Qe_j$ and $e_jPe_j = e_jP$. Thus we have ${}_{R_j}R_{j R_j} \cong Qe_j \otimes_{R_j} e_jP$. A similar computation yields that ${}_{R_j}R_{j R_j} \cong e_jP \otimes_{R_j} Qe_j$. We conclude that $e_jP \in \mathrm{Pic}(R_j)$ as desired.

Thus we have shown that any $P \in \mathrm{Pic}(B)$ has the form $P = \prod_{i\in\mathbb{N}} e_iP$, and each $e_iP \in \mathrm{Pic}(R_i)$. As ${}_BP$ is finitely generated, there exists an integer s for which P is generated by s elements as a left B-module. By the definition of the module action, this means that the element e_iP of $\mathrm{Pic}(R_i)$ is generated by s elements as a left R_i-module. But for $i > s$ the only element of $\mathrm{Pic}(R_i)$ which can be generated by at most s elements is R_i. Thus $e_iP = R_i$ for all $i > s$, and we are done. ■

2. Morita invariance. With Proposition 1.8 put to rest, we move on to our second goal, the verification that boundedness is not a Morita invariant of the ring. We work somewhat harder than is necessary, in that we present *two* pairs of rings A and A' which are Morita equivalent, where A' has bounded Picard group, but A does not. The specific ring A studied above will serve as its namesake in the first such pair; here the corresponding A' has bounded Picard group, but is not unital. We then provide a second pair of the desired type in which A' is unital.

NOTATION 2.1. We continue to let A and k_i denote the rings described in Notation 1.3. For $i \in \mathbb{N}$ we denote $k_i \oplus k_i$ by R_i', and we let A' denote the ring $\bigoplus_{i\in\mathbb{N}} R_i'$.

PROPOSITION 2.2. *Let A and A' be the rings given in Notation* 2.1.

1. *A' has bounded Picard group*; *in fact*, *A' has* 1-*bounded Picard group.*
2. *The rings A and A' are Morita equivalent.*

Consequently, *the property "bounded Picard group" is not a Morita invariant of the ring.*

Proof. 1. This follows directly from [2], as the ring A' is a basic semiperfect ring with local units.

2. Any left R_i-module of the form $k_i^{(i)} \oplus k_i$ is clearly a progenerator for R_i'-mod. Thus A' and $\varinjlim_{i\in\mathbb{N}} \mathrm{End}_{R_i'}(k_i^{(i)} \oplus k_i) = \bigoplus_{i\in\mathbb{N}} \mathrm{End}_{R_i'}(k_i^{(i)} \oplus k_i)$ are Morita equivalent, by [4, Theorem 2.5]. But this latter ring is clearly isomorphic to A. ■

We now proceed to produce the promised example of a pair of Morita equivalent rings A and A' for which A has unbounded Picard group, and A' is unital. To this end, the following useful notation will remain in effect through Proposition 2.4.

NOTATION 2.3. Let k be any ring with identity, let $K = \prod^{\mathbb{N}} k$ (countable direct product), let $A' = \prod^{\mathbb{N}} K$ and let $U = \coprod^{\mathbb{N}} K$ (countable direct sum). We observe that U is a ring with local units; in addition, U is a 2-sided ideal of the unital ring A'. Furthermore, ${}_{A'}A'^n \cong {}_{A'}A'$ and ${}_U U^n \cong {}_U U$ for any integer n. We let P denote the left A'-module $U \oplus A'$. For convenience, we denote the symbols $\prod^{\mathbb{N}}$ and $\coprod^{\mathbb{N}}$ simply by $\prod$ and $\coprod$, respectively.

We first verify that P is a locally projective generator for A'-mod. Clearly P is a generator for A'-mod since A' is a direct summand. Now write $U = \varinjlim_{e \in E} Ue$, where E is a set of local units of U. Each Ue is a finitely generated, projective A'-module and is a direct summand of U. But it is easy to check that $P = \varinjlim_{e\in E}(Ue \oplus A')$, which verifies the assertion.

Now let $\alpha \in \mathrm{Aut}(A')$ be defined by setting $[(a_{ij})_j]_i^\alpha = [(a_{ji})_j]_i$. We use α to define the left A'-module ${}_\alpha P$, which as an abelian group is P, and whose left A'-action is given by setting $a * p = a^\alpha p$ for $a \in A'$, $p \in P$. We define similarly the modules ${}_\alpha U$ and ${}_\alpha A'$, and note that ${}_\alpha P = {}_\alpha U \oplus {}_\alpha A'$.

We show that ${}_\alpha P \notin \mathrm{add}(P)$. For just suppose $\gamma : P^n \to {}_\alpha P$ is a split epimorphism. Since $P^n = U^n \oplus A'^n \cong U \oplus A' \cong P$, we may assume $\gamma : P \to {}_\alpha P$ is a split epimorphism as left A'-modules. In particular, $\gamma\pi : P \to {}_\alpha U$ is a split epimorphism; we will set $\gamma = \gamma\pi$. Now write $\gamma = [f\ \varrho_x] : U \oplus A' \to {}_\alpha U$, where $f : U \to {}_\alpha U$ and $x \in U$. Thus, ${}_\alpha U = (U)f + A' * x = (U)f + (A')^\alpha x$. Moreover, $(U)f = (UU)f = U * (U)f = U^\alpha (U)f$.

Since $x \in U$, we can write $x = ((x_{1i}), \ldots, (x_{mi}), 0, \ldots)$, where $(x_{ji}) \in K$. Also, since U^α is a 2-sided ideal of A', $U^\alpha(U)f \subset U^\alpha$. But $U^\alpha = \prod(\coprod k)$; that is, $v \in U^\alpha$ if and only if $v = ((v_{ji})_i)_j$ such that $(v_{ji})_i \in \coprod k$. Moreover, $(U)f \subset U = \coprod(\prod k)$, so $(U)f = U^\alpha(U)f \subset \prod(\coprod k) \cap \coprod(\prod k) = \coprod(\coprod k)$.

Now define $u \in {}_\alpha U$ as follows: write $u = ((u_{ji})_i)_j$, and set $u_{ji} = 0$ if $j \neq m+1$ and $u_{ji} = 1$ if $j = m+1$. Now suppose $u = rx + b$, where $r \in A'$ and $b \in (U)f$. From the form of x and u it follows that we may assume that $r = 0$, so that $u = b$. But $b \in (U)f = \coprod(\coprod k)$, so $b = ((b_{ji})_i)_j$, where $(b_{ji})_i = 0$ if $j \neq m+1$ and $(b_{ji})_i \in \coprod k$ if $j = m+1$. However, it is clear that $u \neq b$ and so we have a contradiction. We conclude that ${}_\alpha P \notin \mathrm{add}(P)$ as claimed. We are now in a position to prove

PROPOSITION 2.4. *Let A' be the ring described in Notation 2.3. Let $A = \varinjlim_{e\in E}\mathrm{End}({}_{A'}Ue \oplus A')$. Then A is a ring with unbounded Picard group which is Morita equivalent to the unital ring A'.*

Proof. By [4, Theorem 2.5], $A = \varinjlim_{e\in E}\mathrm{End}({}_{A'}Ue \oplus A')$ and A' are Morita equivalent via the Morita context

$$[A', A, {}_{A'}P_A, {}_AQ_{A'}, \mu : P \otimes Q \to A', \tau : Q \otimes P \to A],$$

where $Q = \varinjlim_{e\in E}\mathrm{Hom}_{A'}(Ue \oplus A', A')$. In addition, this context yields an isomorphism of Picard groups $\mathrm{Pic}(A') \to \mathrm{Pic}(A)$, given by $X \mapsto Q \otimes X \otimes P$.

In particular, ${}_\alpha A' \in \mathrm{Pic}(A')$ (it has inverse ${}_{\alpha^{-1}}A'$), so $Q\otimes {}_\alpha A'\otimes P \in \mathrm{Pic}(A)$. But we note that if $Q\otimes {}_\alpha A'\otimes P \in \mathrm{add}(A) = \mathrm{add}(Q\otimes P)$, then tensoring on the left by P would give ${}_\alpha A'\otimes P \cong {}_\alpha P \in \mathrm{add}(P)$, contrary to the previous observation.

Therefore the bimodule $Q \otimes {}_\alpha A' \otimes P$ belongs to $\mathrm{Pic}(A)$, but does not belong to $\mathrm{add}(A)$. We conclude that A has unbounded Picard group. ∎

We put the finishing touches on the above discussion by giving a concrete description of the ring A. It is straightforward to see that

$$\mathrm{End}({}_{A'}Ue \oplus A') \cong \begin{pmatrix} eUe & eUe \\ eUe & A' \end{pmatrix},$$

and so it follows that

$$A \cong \begin{pmatrix} U & U \\ U & A' \end{pmatrix}.$$

Furthermore, $P = [U\ A']$ while

$$Q \cong \begin{pmatrix} U \\ A' \end{pmatrix}.$$

The Morita context maps μ and τ are given by matrix multiplication. Thus, simple calculations give

$$X = Q\otimes {}_\alpha A'\otimes P \cong \begin{pmatrix} U{}_\alpha U & U{}_\alpha A' \\ A'{}_\alpha A' & A'{}_\alpha A' \end{pmatrix}$$

while

$$X^{-1} \cong \begin{pmatrix} U({}_{\alpha^{-1}}U) & U({}_{\alpha^{-1}}A') \\ A'({}_{\alpha^{-1}}U) & A'({}_{\alpha^{-1}}A') \end{pmatrix}.$$

One easily checks that these are the appropriate invertible bimodules.

We conclude this article on a somewhat reassuring note by showing that there are indeed situations in which boundedness is a Morita invariant. Our main result of this flavor is Proposition 2.6, from which we will be able to deduce the Morita invariance of the property "bounded Picard group" in three situations, involving hypotheses on the equivalence functors, hypotheses on the ring, and hypotheses on the corresponding Picard groups. We first need a lemma.

LEMMA 2.5. *Let A be a ring with local units which has N-bounded Picard group. Then for each pair $Y, Z \in \mathrm{Pic}(A)$ there exists a split epimorphism $Y^N \to Z$.*

Proof. Since $Y^{-1}\otimes Z \in \mathrm{Pic}(A)$, there exists a split epimorphism $A^N \to Y^{-1}\otimes Z$. Upon tensoring this map by the bimodule Y on the left, we get the desired result. ∎

Proposition 2.6. *Let R be a ring having N-bounded Picard group for some positive integer N. Suppose A is a ring with local units which is Morita equivalent to R. Let $-\otimes_R Q_A : \text{mod-}R \to \text{mod-}A$ be an equivalence functor such that ${}_RQ$ is a direct sum of elements of* $\mathrm{Pic}(R)$. *Then A has N-bounded Picard group.*

Proof. Let ${}_AP_R \otimes_R - : R\text{-mod} \to A\text{-mod}$ be the left equivalence functor such that $[R, A, P, Q]$ defines an appropriate Morita context. Since A and R are Morita equivalent we have $\mathrm{Pic}(R) \cong \mathrm{Pic}(A)$, with the isomorphism given by $X \mapsto P \otimes_R X \otimes_R Q$ for $X \in \mathrm{Pic}(R)$. Thus to prove the proposition we need only show that each $P \otimes X \otimes Q$ belongs to $\mathrm{add}_N(A)$.

We first show that for $X \in \mathrm{Pic}(R)$, Q is a direct summand of $(X^{-1}\otimes Q)^N$, considered as left R-modules. By hypothesis ${}_RQ$ is a direct sum of elements from $\mathrm{Pic}(R)$ when considered as a left R-module; thus we can write $Q = \bigoplus_{i\in I} Q_i$ with $Q_i \in \mathrm{Pic}(R)$, for some index set I. By Lemma 2.5, there are split epimorphisms $\alpha_i : (X^{-1} \otimes Q_i)^N \to Q_i$. It follows that the coproduct map $\bigoplus_{i\in I} \alpha_i : \bigoplus_{i\in I}(X^{-1} \otimes Q_i)^N \cong (X^{-1} \otimes Q)^N \to \bigoplus_{i\in I} Q_i = Q$ is a split epimorphism of left R-modules.

We now complete the proof of the proposition. Tensoring this split epimorphism on the left by P yields a left A-split epimorphism $(P \otimes X^{-1} \otimes Q)^N \to P \otimes Q$. But $P \otimes Q \cong A$ from the Morita context, so we have a split epimorphism $(P \otimes X^{-1} \otimes Q)^N \to A$. Upon tensoring this map on the left by $P \otimes X \otimes Q$ we reach the desired conclusion. ∎

We note here that by essentially mimicking the proof of Proposition 2.6, we can produce another proof of [1, Proposition 2.8(1)]: *For any unital ring R the ring* $\mathrm{FM}(R)$ *has bounded Picard group.*

Corollary 2.7. *Let R be a ring having bounded Picard group such that* $\mathrm{Pic}(R)$ *is finite. Let A be a ring with local units which is Morita equivalent to R via an equivalence functor $-\otimes_R Q_A : \text{mod-}R \to \text{mod-}A$ such that ${}_RQ$ is a direct sum of elements of* $\mathrm{Pic}(R)$. *Then both R and A are N-bounded for some positive integer N.*

Proof. Since R has bounded Picard group and $\mathrm{Pic}(R)$ is finite, it follows that R has N-bounded Picard group, where N is the smallest integer n such that $\mathrm{Pic}(R) \subset \mathrm{add}_n(R)$. Now apply Proposition 2.6. ∎

Corollary 2.8. *For local perfect rings, boundedness is a Morita invariant. That is, if A is Morita equivalent to a* (*unital*) *local perfect ring, then A has* 1*-bounded Picard group.*

Proof. (The verification of each of the following statements can be found in [3].) Since any local ring is basic semiperfect, we deduce by [2] that any local ring has 1-bounded Picard group. If we denote the local perfect ring by R, then the module ${}_RQ_A$ which yields the Morita equivalence has the property that ${}_RQ$ is a direct limit of projective modules, hence is flat. But any flat module over a perfect ring is projective, so that ${}_RQ$ is free as R is local. Since $R \in \mathrm{Pic}(R)$, the module ${}_RQ$ thereby satisfies the hypotheses of Proposition 2.6. ∎

As our final application, we show that the smash product $R \# G$ of a strongly graded ring often has N-bounded Picard group. The appropriate definitions and background information regarding graded rings and their associated smash products can be found in [6]. We remind the reader that the graded ring R is called *strongly graded* in case $R_g R_h = R_{gh}$ for all $g, h \in G$; in this case there is a category equivalence between R-gr and R_1-mod, the category of modules over the identity component ring R_1.

PROPOSITION 2.9. *Let R be a ring strongly graded by the infinite group G, and suppose that the identity component ring R_1 has N-bounded Picard group. Then $R\#G$ has N-bounded Picard group. In particular, if* $\mathrm{Pic}(R_1)$ *is finite, then $R \# G$ has N-bounded Picard group for some positive integer N.*

Proof. Since R is strongly graded there is a Morita equivalence $(R_1, R \# G, P, Q)$, where each of P and Q is isomorphic to R as left (resp. right) R_1-modules; see e.g. [6, Theorem 5.5]. But as R_1-modules we have $R \cong \bigoplus_{x \in G} R_x$. Since R is strongly graded, each $R_x \in \mathrm{Pic}(R_1)$. Thus the conditions of Proposition 2.6 are satisfied and 2.9 follows. ∎

REFERENCES

[1] G. Abrams and J. Haefner, *Picard groups and infinite matrix rings*, submitted.
[2] —, —, *Outer induced Picard groups*, in preparation.
[3] F. W. Anderson and K. R. Fuller, *Rings and Categories of Modules*, Grad. Texts in Math. 13, Springer, 1992.
[4] P. N. Ánh and L. Márki, *Morita equivalence for rings without identity*, Tsukuba J. Math. 11 (1987), 1–16.
[5] C. W. Curtis and I. Reiner, *Methods of Representation Theory with Applications to Finite Groups and Orders*, *Vol. II*, Pure Appl. Math., Wiley-Interscience, 1987.
[6] J. Haefner, *Morita theory for infinite groups*, J. Algebra 169 (1994), 552–586.

Department of Mathematics
University of Colorado
Colorado Springs, Colorado 80933
U.S.A.
E-mail: abrams@vision.uccs.edu
haefner@vision.uccs.edu

Received 17 May 1996;
revised 8 July 1996