COLLOQUIUM MATHEMATICUM

VOL. 72 1997 NO. 2

VERY SMALL SETS

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0. Introduction. Let us recall that $\mathrm{cov}(\mathcal{M}) = \min\{|\mathcal{F}| : \mathcal{F} \subset \mathcal{M}, \bigcup \mathcal{F} = \mathbb{R}\}$ ($\mathcal{M}$ denotes the σ-ideal of meagre sets). So $|X| < \mathrm{cov}(\mathcal{M})$ iff for every set $B \subset X \times \mathbb{R}$ with $B_x \in \mathcal{M}$ for each $x \in X$ we have $\bigcup_{x\in X} B_x \neq \mathbb{R}$. If we consider only "nice" families of sections, for example Borel sets B, we get a wider class of sets X. Let us denote it by $\mathrm{Cov}(\mathcal{M})$. We can generalize this notion to any σ-ideal.

Let $\mathcal{J} \subset P(\mathbb{R})$ be a proper σ-ideal with a Borel basis. We define

$$\mathrm{Cov}(\mathcal{J}) = \Big\{X \subset \mathbb{R} : \forall_{B\subset\mathbb{R}\times\mathbb{R},\mathrm{Borel}}\Big(\forall_{x\in\mathbb{R}} B_x \in \mathcal{J} \Rightarrow \bigcup_{x\in X} B_x \neq \mathbb{R}\Big)\Big\}.$$

Let us recall that X is a *strong measure zero set* iff for every meagre set F, $X + F \neq \mathbb{R}$. It is known (see [AR]) that X is strong measure zero iff for every F_σ-set $B \subset \mathbb{R}\times\mathbb{R}$ with $B_x \in \mathcal{M}$ for each $x \in \mathbb{R}$ we have $\bigcup_{x\in X} B_x \neq \mathbb{R}$. It is easy to see that $\mathrm{Cov}(\mathcal{M}) \subset$ strong measure zero sets (see [R]). Let us recall that X is *strongly meagre* iff for every null set F, $X + F \neq \mathbb{R}$. It is easy to see that $\mathrm{Cov}(\mathcal{N}) \subset$ strongly meagre sets (see [R]). For non-invariant σ-ideals it does not make sense to generalize definitions of strong measure zero sets and strongly meagre sets using the algebraic structure of the real line. So we can treat $\mathrm{Cov}(\mathcal{J})$ as a natural generalization of strong measure zero and strongly meagre sets.

We can also define similar classes of sets for some other cardinal coefficients. We define

$$\mathrm{Add}(\mathcal{J}) = \Big\{X \subset \mathbb{R} : \forall_{B\subset\mathbb{R}\times\mathbb{R},\mathrm{Borel}}\Big(\forall_{x\in\mathbb{R}} B_x \in \mathcal{J} \Rightarrow \bigcup_{x\in X} B_x \in \mathcal{J}\Big)\Big\},$$

$$\mathrm{Cof}(\mathcal{J}) = \Big\{X \subset \mathbb{R} : \forall_{B\subset\mathbb{R}\times\mathbb{R},\mathrm{Borel}}\Big(\forall_{x\in\mathbb{R}} B_x \in \mathcal{J} \Rightarrow \{B_x : x \in X\} \text{ is not a basis for } \mathcal{J}\Big)\Big\},$$

1991 *Mathematics Subject Classification*: Primary 04A15; Secondary 03E50.

The third author was partially supported by the Emmy Noether Institute in Mathematics of Bar Ilan University, Israel, and the Alexander von Humboldt Foundation, Germany when he was visiting FU Berlin.

$$\mathrm{Non}(\mathcal{J}) = \{X \subset \mathbb{R} : \forall_{f:X\to\mathbb{R},\mathrm{Borel}}\, f[X] \in \mathcal{J})\}.$$

We have $[\mathbb{R}]^{\le\omega} \subset \mathrm{Add}(\mathcal{J}) \subset \mathrm{Cov}(\mathcal{J}) \cap \mathrm{Non}(\mathcal{J})$ and $\mathrm{Cov}(\mathcal{J}) \subset \mathrm{Cof}(\mathcal{J})$. For $\mathcal{J} = \mathcal{M}$ or $\mathcal{J} = \mathcal{N}$ we have $\mathrm{Non}(\mathcal{J}) \subset \mathrm{Cof}(\mathcal{J})$ (see [PR]).

In [PR] it was shown that all inequalities from Cichoń's diagram can be replaced by inclusions of respective classes of sets. For example, we have $\mathrm{Add}(\mathcal{N}) \subset \mathrm{Add}(\mathcal{M})$. In [R] an example is given, under CH, of a set in $\mathrm{Add}(\mathcal{N})$ of size continuum. It is known that every Lusin set is in $\mathrm{Cov}(\mathcal{M})$ (see [R]) and every Sierpiński set is in $\mathrm{Cov}(\mathcal{N})$ (see [P]).

In this paper we investigate those classes in the general case. In Section 1, under CH, we construct a set of size continuum which is in $\mathrm{Cov}(\mathcal{J})$ and $\mathrm{Non}(\mathcal{J})$ for any CCC σ-ideal. This construction uses a method introduced by Todorčević in [GM]. This also strengthens the result of Todorčević (unpublished) and the third author (see [R1]) that under MA there is a set of size continuum which is in $\mathrm{Non}(\mathcal{N}) \cap \mathrm{Non}(\mathcal{M})$. We also show that there is a CCC σ-ideal $\mathcal{J}$ such that there are no uncountable sets in $\mathrm{Add}(\mathcal{J})$ and there is a σ-ideal $\mathcal{J}$ such that there are no uncountable sets in $\mathrm{Cov}(\mathcal{J})$.

In Section 2, we show under CH that every $\mathcal{I}$-Lusin set is a union of two sets from $\mathrm{Cov}(\mathcal{J})$ if we have a kind of Fubini's theorem for the pair of ideals $\mathcal{I}, \mathcal{J}$. We also show that this can be partially reversed.

Throughout this paper we consider only σ-ideals which have a Borel basis and contain singletons. We say that a σ-ideal has CCC if there is no uncountable family of disjoint Borel sets which do not belong to the σ-ideal.

1. Very small sets

THEOREM 1.1. *Assume the Continuum Hypothesis. There is a set $X \subset \mathbb{R}$ of size continuum such that for every CCC σ-ideal $\mathcal{J}$,*

(i) $X \in \mathrm{Cov}(\mathcal{J})$,

(ii) $X \in \mathrm{Non}(\mathcal{J})$,

(iii) *for every Borel function $f : X \to \mathbb{R}$ there is a countable set $A \subset \mathbb{R}$ such that $f|_{X\setminus f^{-1}(A)}$ is a Borel isomorphism onto its image.*

LEMMA 1.2. *For every $\mathcal{J}$ with CCC and every Borel set $B \subset \mathbb{R}\times\mathbb{R}$ such that $B_x \in \mathcal{J}$ for each $x \in \mathbb{R}$ we have $\{y : \mathbb{R} \setminus B^y$ is uncountable$\} \in \mathcal{J}^{\mathrm{c}}$.*

Proof. Let $A = \{y : (\mathbb{R}^2 \setminus B)^y$ is countable$\}$. By the Mazurkiewicz–Sierpiński Theorem (see [K]) A is coanalytic. Suppose $A \notin \mathcal{J}$. Then by CCC there is a Borel set $E \subset A$ with $E \notin \mathcal{J}$ (Marczewski, see [K]). Then the set $(\mathbb{R}\times E) \cap (\mathbb{R}^2 \setminus B)$ can be represented as a union of countably many graphs of Borel functions $f_n : E \to \mathbb{R}$ with $(\mathbb{R}\times E) \cap B = \bigcup_n \mathrm{graph}(f_n)^{-1}$ (Lusin–Novikov Theorem, see [K]). By CCC there is $x \in \mathbb{R}$ such that for each n, $f_n^{-1}(x) \in \mathcal{J}$. But $E \subset \bigcup_n f_n^{-1}(x) \cup B_x$. Contradiction.

P r o o f o f T h e o r e m 1.1. We construct an Aronszajn tree A of perfect trees $T\subset 2^{\le\omega}$ ordered by reverse inclusion. We write $T\le_n T_1$ iff $T|n=T_1|n$ and T_1 is subtree of T. Let A_α be the αth level of A. We will construct the tree A with the following properties:

$$\forall_\alpha\forall_{T,W\in A_\alpha,T\ne W}[T]\cap[W]=\emptyset \quad\text{and}\quad \forall_\alpha\forall_{\beta>\alpha}\forall_{T\in A_\alpha}\forall_n\exists_{W\in A_\beta}T\le_n W.$$

The last condition enables us to extend our tree at limit stages. We can assume additionally that $[T]$ is meagre in $\bigcap_{T\subset W,W\in A,W\ne T}[W]$. Then let $x_\alpha\in\bigcup_{T\in A_\alpha}[T]\setminus\bigcup_{T\in A_{\alpha+1}}[T]$ and let $X=\{x_\alpha:\alpha<\omega_1\}$.

Observe that till now we only use ZFC. The tree A is a special Aronszajn tree (we define $A_n=\{T\in A:[T]\cap U_n=\emptyset$ and $\bigcap_{T\subset W,W\in A,W\ne T}[W]\cap U_n\ne\emptyset\}$, $\{U_n:n\in\omega\}$ is a countabe basis; Todorčević). Then $X\in\mathcal{J}$ for every $\mathcal{J}$ with CCC. To see this observe that since there are only countably many T in A with $[T]\notin\mathcal{J}$, there is α such that $\bigcup_{T\in A_\alpha}[T]\in\mathcal{J}$. Observe that all but countably many elements of X are in $\bigcup_{T\in A_\alpha}[T]$.

To have a set X with the properties listed in the theorem under CH we will choose levels of A satisfying some additional conditions. For even levels we order all Borel sets B_α on the plane with all sections in a CCC σ-ideal $\mathcal{J}$. For odd levels we order all Borel functions $f_\alpha:\mathbb{R}\to\mathbb{R}$.

Define $T(s)=\{t\in T:s\subset t\}$. Let $\alpha<\mathbf{c}$.

First we define $A'=\{P_{T,n}:T\in A_\beta,\ n\in\omega,\ \beta<\alpha\}$ such that $T\le_n P_{T,n}$ and $\forall_{T,W\in A',T\ne W}[T]\cap[W]=\emptyset$ and $\forall_{W\in A'}\forall_{\beta<\alpha}\exists_{T\in A_\beta}W\subset T$.

Even level. By Lemma 1.2 there is $y\in\mathbb{R}$ such that for each $P_{T,n}\in A'$ and $s\in 2^n$ the set $[P_{T,n}(s)]\setminus(B_\alpha)^y$ is uncountable and $y\notin\bigcup_{\beta<\alpha}(B_\alpha)_{x_\beta}$. Then we choose a subtree $S_{T,n}(s)$ of $P_{T,n}(s)$ with $[S_{T,n}(s)]\subset[P_{T,n}(s)]\setminus(B_\alpha)^y$. Let $A_\alpha=\{\bigcup_{s\in 2^n}S_{T,n}(s):T\in A_\beta,\ n\in\omega,\ \beta<\alpha\}$.

Odd level. For each $P_{T,n}(s)$, if $f_\alpha|[P_{T,n}(s)]$ is countable-to-one then we choose a subtree $S_{T,n}(s)$ of $P_{T,n}(s)$ such that $f_\alpha|[S_{T,n}(s)]$ is a Borel isomorphism. If $f_\alpha|[P_{T,n}(s)]$ is not countable-to-one then we choose $S_{T,n}(s)$ to be a subtree of $P_{T,n}(s)$ such that $|f_\alpha[[S_{T,n}(s)]]|=1$ (Lusin–Novikov Theorem). Additionally we can choose these trees to have disjoint images for trees for which f_α is one-to-one. Let $A_\alpha=\{\bigcup_{s\in 2^n}S_{T,n}(s):T\in A_\beta,\ n\in\omega,\ \beta<\alpha\}$.

Properties (i) and (iii) follow from the construction. Let $f:X\to Y$ be Borel with $f[X]=Y$. There is a countable $A\subset Y$ such that $f|_{X\setminus f^{-1}(A)}$ is a Borel isomorphism. If Y is not in some CCC σ-ideal then there is a CCC σ-ideal in $B(Y\setminus A)$. Then we can transport this CCC σ-ideal onto a subset of X by the Borel isomorphism. Contradiction.

We say that a σ-ideal $\mathcal{J}$ with a Borel basis is *perfectly dense* if it contains all singletons and for every perfect set P there is a perfect set $Q\subset P$ with $Q\in\mathcal{J}$.

THEOREM 1.3. *Assume CH. Let* $\{\mathcal{J}_\alpha : \alpha < \mathbf{c}\}$ *be a family of perfectly dense* σ*-ideals. Then there is a set* $X \subset \mathbb{R}$ *of size continuum which belongs to every* $\mathcal{J}_\alpha$.

Proof. The proof is very similar to the proof of Theorem 1.1.

COROLLARY 1.4. *Assume CH. Let* $\mathcal{J}$ *be a perfectly dense* σ*-ideal. Then there is a set* $X \subset \mathbb{R}$ *of size continuum such that* $X \in \mathrm{Non}(\mathcal{J})$.

Proof. Observe that for each Borel function $f : \mathbb{R} \to \mathbb{R}$, $\{f^{-1}(B) : B \in \mathcal{J}\}$ is a perfectly dense σ-ideal. So the family of all Borel functions gives us a family of perfectly dense σ-ideals of size continuum. Thus we can apply Theorem 1.3.

Remark 1.5. It is easy to see that if a σ-ideal with a Borel basis is not perfectly dense, i.e. its restriction to a perfect set is the σ-ideal of countable sets, then there are no uncountable sets in $\mathrm{Non}(\mathcal{J})$.

COROLLARY 1.6. *Assume CH. Let* $\mathcal{J}$ *be a* σ*-ideal such that for every Borel set* B *on the plane with all sections in* $\mathcal{J}$ *the family* $\{A : \bigcup_{x\in A} B_x \in \mathcal{J}\}$ *is perfectly dense. Then there is a set* X *of size continuum such that* $X \in \mathrm{Add}(\mathcal{J})$.

Observe that the ideals $\mathcal{N}$ and $\mathcal{M}$ satisfy the assumption of Corollary 1.6. However, not every CCC σ-ideal satisfies the conclusion of Corollary 1.6. This shows that in Theorem 1.1 we cannot get additivity for all CCC σ-ideals.

FACT 1.7. $\mathrm{Add}(\mathcal{M} \times \mathcal{N}) = [\mathbb{R}^2]^{\le\omega}$.

Proof. In [CP] it is shown that there is a Borel set in $(\mathbb{R}^2)^2$ with all sections in $\mathcal{M} \times \mathcal{N}$ such that their union over any uncountable set is not in $\mathcal{M} \times \mathcal{N}$.

FACT 1.8. *There is a* σ*-ideal* $\mathcal{J}$ *in* 2^ω *and a Borel set* $B \subset 2^\omega \times 2^\omega$ *with* $B_x \in \mathcal{J}$ *for each* $x \in 2^\omega$ *and* $\bigcup_{x\in X} B_x = 2^\omega$ *for each uncountable* $X \subset 2^\omega$. *So* $\mathrm{Cov}(\mathcal{J}) = [2^\omega]^{\le\omega}$.

Proof. Let $f : 2^\omega \to (2^\omega)^\omega$ be a homeomorphism. Let $p_n : (2^\omega)^\omega \to 2^\omega$ be the projection onto the nth coordinate and let $f_n = p_n f$. Let $B = 2^\omega \times 2^\omega \setminus \bigcup_{n\in\omega} \mathrm{graph}(f_n)^{-1}$. Then the union of any countable family of sections of B does not cover the Cantor set and the union of any uncountable family of sections does.

Remark 1.9. The fact above shows that we cannot show Theorem 1.1 for an arbitrary σ-ideal. Note that in the proof we use the property from Lemma 1.2: for every Borel set $B \subset \mathbb{R} \times \mathbb{R}$ such that $B_x \in \mathcal{J}$ for each $x \in X$ we have $\{y : \exists_{D\subset\mathbb{R}}\ D \text{ is perfect and } y \notin \bigcup_{x\in D} B_x\} \in \mathcal{J}^{\mathrm{c}}$.

It easy to see that $X \in \mathcal{J}$ for each CCC σ-ideal $\mathcal{J}$ iff there is no CCC σ-ideal in $B(X)$. Uncountable examples of sets with this property were known in ZFC; for example, a selector from the constituents of a non-Borel coanalytic set (see [M]). We will see that such a set can be mapped continuously onto the reals so it does not belong to $\mathrm{Non}(\mathcal{J})$ for any $\mathcal{J}$.

FACT 1.10. *Assume CH. There is a selector from the constituents of a coanalytic set which can be mapped continuously onto the reals.*

Proof. Let A be a non-Borel coanalytic set. Set $C = A \times \mathbb{R}$. Let $C = \bigcup_{\alpha<\omega_1} B_\alpha$ be a partition into Borel constituents. Let $\mathbb{R} = \{y_\alpha : \alpha < \omega_1\}$. Define $(x_\alpha, y_\alpha) \in C \setminus \bigcup\{B_\gamma : \exists_{\beta<\alpha}(x_\beta, y_\beta) \in B_\gamma\}$. Since $(\bigcup\{B_\gamma : \exists_{\beta<\alpha}(x_\beta, y_\beta) \in B_\gamma\})^{y_\alpha}$ is Borel, $(C \setminus \bigcup\{B_\gamma : \exists_{\beta<\alpha}(x_\beta, y_\beta) \in B_\gamma\})^{y_\alpha} \neq \emptyset$. Then any selector X from $\{B_\alpha : \alpha < \omega_1\}$ containing $\{(x_\alpha, y_\alpha) : \alpha < \omega_1\}$ has $\mathrm{pr}_2(X) = \mathbb{R}$.

In particular, the set from Fact 1.10 is not strong measure zero.

2. $\mathcal{J}$-Lusin sets. We say that X is a *$\mathcal{J}$-Lusin set* if X is uncountable and for each $G \in \mathcal{J}$, $X \cap G$ is countable.

It is known that $\mathcal{M}$-Lusin $\subset \mathrm{Cov}(\mathcal{M})$ and $\mathcal{N}$-Lusin $\subset \mathrm{Cov}(\mathcal{N})$ (see [R], [P]).

Observe that $\mathcal{J}$-Lusin $\cap\, \mathrm{Non}(\mathcal{J}) = \emptyset$. We will investigate relationships between $\mathcal{J}$-Lusin sets and $\mathrm{Cov}(\mathcal{J})$ and $\mathrm{Cof}(\mathcal{J})$. We will see that to have a $\mathcal{J}$-Lusin set which is in $\mathrm{Cov}(\mathcal{J})$ we need a kind of Fubini's theorem for $\mathcal{J}$.

The next fact was suggested to the authors by J. Pawlikowski.

FACT 2.1. *Let $A = \{((x, y), (z, w)) \in (\mathbb{R}^2)^2 : w \in G + x\}$, where G is a Borel comeagre null set. Then $\forall_{(x,y)\in\mathbb{R}^2} A_{(x,y)} \in \mathcal{M} \times \mathcal{N}$ and $\forall_{(z,w)\in\mathbb{R}^2} A^{(z,w)} \in (\mathcal{M} \times \mathcal{N})^{\mathrm{c}}$.*

Proof. Easy.

COROLLARY 2.2. $\mathrm{Cov}(\mathcal{M} \times \mathcal{N}) \subset \mathcal{M} \times \mathcal{N}$.

Proof. Let $X \in \mathrm{Cov}(\mathcal{M} \times \mathcal{N})$. For the set A from Fact 2.1 there is $(z, w) \in \mathbb{R}$ with $X \cap A^{(z,w)} = \emptyset$ so $X \in \mathcal{M} \times \mathcal{N}$.

So no $\mathcal{M} \times \mathcal{N}$-Lusin set is in $\mathrm{Cov}(\mathcal{M} \times \mathcal{N})$. The result above and the facts that $\mathrm{Cov}(\mathcal{N}) \subset \mathrm{Non}(\mathcal{M})$ and $\mathrm{Cov}(\mathcal{M}) \subset \mathrm{Non}(\mathcal{N})$ can be generalized as follows.

COROLLARY 2.3. *Let $\mathcal{I}$ and $\mathcal{J}$ be σ-ideals. Assume that there there is a Borel set B on the plane such that all its vertical sections are in $\mathcal{J}$ and all its horizontal sections are in $\mathcal{I}^{\mathrm{c}}$. Then* $\mathrm{Cov}(\mathcal{J}) \subset \mathrm{Non}(\mathcal{I})$.

Proof. From the proof of Corollary 2.2 we have $\mathrm{Cov}(\mathcal{J}) \subset \mathcal{I}$. Observe that $\mathrm{Cov}(\mathcal{J})$ is closed under Borel images.

Corollary 2.3 can be partially reversed. We say that a pair $(\mathcal{I},\mathcal{J})$ has the *Fubini property* if for every Borel set $B\subset\mathbb{R}\times\mathbb{R}$ with $B_x\in\mathcal{J}$ for each $x\in\mathbb{R}$ we have $\{y: B^y\in\mathcal{I}\}\notin\mathcal{J}$.

THEOREM 2.4. *Assume CH. Every $\mathcal{I}$-Lusin set is the union of two sets Y,Z such that for each $\mathcal{J}$, if the pair $(\mathcal{I},\mathcal{J})$ has the Fubini property then $Y,Z\in\mathrm{Cov}(\mathcal{J})$.*

Proof. Let $\{B_\alpha:\alpha<\omega_1\}$ be a family of all Borel sets on the plane such that there is a σ-ideal $\mathcal{J}$ with $(B_\alpha)_x\in\mathcal{J}$ for each $x\in\mathbb{R}$ and with $(\mathcal{I},\mathcal{J})$ having the Fubini property. Let L be an $\mathcal{I}$-Lusin set.

We define sequences $y_\alpha,z_\alpha,A_\alpha,C_\alpha$ such that $y_\alpha,z_\alpha\in\mathbb{R}$ and A_α,C_α are countable subsets of L. Let

$$D_\alpha=\bigcup_{x\in\bigcup_{\beta<\alpha}(A_\beta\cup C_\beta)}(B_\alpha)_x.$$

Then $D_\alpha\in\mathcal{J}$. So there is $y_\alpha\notin D_\alpha$ such that $(B_\alpha)^{y_\alpha}\in\mathcal{I}$. Define $C_\alpha=(B_\alpha)^y\cap L$. Let $D'_\alpha=D_\alpha\cup\bigcup_{x\in C_\alpha}B_x$ and let $z_\alpha\notin D'_\alpha$ with $(B_\alpha)^{z_\alpha}\in\mathcal{I}$. Define $A_\alpha=(B_\alpha)^{z_\alpha}\cap L$. Then $Y=\bigcup_{\alpha<\omega_1}A_\alpha$ and $Z=L\setminus Y$ have the required properties.

THEOREM 2.5. *If a pair $(\mathcal{I},\mathcal{J})$ has the Fubini property then* $\mathcal{I}$-Lusin $\subset\mathrm{Cof}(\mathcal{J})$.

Proof. Let $L\in\mathcal{I}$-Lusin and let $B\subset\mathbb{R}^2$ be a Borel set with all sections in $\mathcal{J}$. Let $y\in\mathbb{R}$ be such that $B^y\in\mathcal{I}$. Then $L\cap B^y$ is countable. Let $z\in\bigcup_{x\in L\cap B^y}B_x$. Then no B_x for $x\in L$ covers $\{y,z\}$.

If it is consistent that there is a measurable cardinal then the following is consistent:

(∗) Martin's Axiom holds and there exists $\kappa<\mathbf{c}$ such that $P(\kappa)$ contains a proper uniform ω_1-saturated, κ-additive ideal $\mathcal{K}$.

Assume (∗). We can treat κ as a subset of $\mathbb{R}$. We can define $\mathcal{L}=\{B\in B(\mathbb{R}): B\cap\kappa\in\mathcal{K}\}$. Then $\kappa\in\mathrm{Cov}(\mathcal{J})$ for each CCC σ-ideal and $\kappa\notin\mathcal{L}$. Observe that $\mathcal{L}$ is CCC.

We can construct such a set of size $\mathbf{c}$. We say that X is a *generalized $\mathcal{J}$-Lusin set* if $|X|=\mathbf{c}$ and for each $B\in\mathcal{J}$, $|B\cap X|<\mathbf{c}$. In [FJ] the authors showed that under (∗) a generalized $\mathcal{L}$-Lusin set satisfies many definitions of smallness. The following facts generalize some of them.

THEOREM 2.6. *Assume (∗). Then there is a generalized $\mathcal{L}$-Lusin set X such that for each CCC σ-ideal $\mathcal{J}$, $X\in\mathrm{Cov}(\mathcal{J})$.*

Proof. We order all Borel sets B_α on the plane with all sections in a CCC σ-ideal $\mathcal{J}$, and all Borel sets D_α from $\mathcal{L}$. On each stage we choose x_α,y_α. Let $y_\alpha\notin\bigcup_{\beta\le\alpha}(B_\alpha)_{x_\beta}\cup\bigcup_{x\in\kappa}(B_\alpha)_x$. Observe that $(B_\alpha)^{y_\alpha}\in\mathcal{L}$

because $(B_\alpha)^{y_\alpha} \cap \kappa = \emptyset$. Let $x_\alpha \notin \bigcup_{\beta\le\alpha}(D_\beta \cup (B_\beta)^{y_\beta})$. Let $X = \{x_\alpha : \alpha < \mathbf{c}\}$ Then $y_\alpha \notin \bigcup_{x\in X}(B_\alpha)_x$.

THEOREM 2.7. *Assume* $(*)$. *For each CCC σ-ideal $\mathcal{J}$ with* $\mathrm{add}(\mathcal{J}) = \mathbf{c}$ *and each generalized $\mathcal{L}$-Lusin set X*, $X \in \mathrm{Add}(\mathcal{J})$.

Proof. Let $B \subset \mathbb{R}^2$ be such that $B_x \in \mathcal{J}$ for each $x \in \mathbb{R}$. Let C be a Borel set such that $\bigcup_{x\in\kappa} B_x \subset C \in \mathcal{J}$. Let $D = \{x : B_x \subset C\}$. Since D is coanalytic, by CCC there are Borel sets E, F such that $F \in \mathcal{L}$ and $E\setminus F \subset D \subset E\cup F$ (Marczewski, see [K]). We have $\kappa \subset D$ so $\mathbb{R}\setminus(E\cup F) \in \mathcal{L}$. Thus $|X \cap (\mathbb{R} \setminus (E \setminus F))| < \mathbf{c}$. So $\bigcup_{x\in X} B_x \subset C \cup \bigcup_{x\in X\cap(\mathbb{R}\setminus(E\setminus F))} B_x \in \mathcal{J}$.

The authors would like to thank Janusz Pawlikowski for fruitful remarks.

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Received 11 January 1995