ON DECOMPOSITION OF POLYHEDRA INTO A CARTESIAN PRODUCT OF 1-DIMENSIONAL AND 2-DIMENSIONAL FACTORS

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In 1938 K. Borsuk proved [1] that the decomposition of a polyhedron into a Cartesian product of 1-dimensional factors is topologically unique (up to a permutation of the factors). We prove a little more general

Theorem 1. If a connected polyhedron $K$ (of arbitrary dimension) is homeomorphic to a Cartesian product $A_1 \times \ldots \times A_n$, where $A_i$'s are prime compacta of dimension at most 1, then there is no other topologically different system of prime compacta $Y_1, \ldots, Y_k$ of dimension at most 2 such that $Y_1 \times \ldots \times Y_k$ is homeomorphic to $K$.

A space $X$ is said to be prime if it has more than one point and only $X$ and the singleton as Cartesian factors.

In Theorem 1 the dimension of $Y_i$ cannot be greater than 2 (see the examples in [3]–[5]). The 3-dimensional factor of a 6-dimensional torus (in [5]) is not a polyhedron, but the 4-dimensional factors of $I^5$ (in [3] and [4]) are polyhedra non-homeomorphic to a cube. I do not know if Theorem 1 is true when we assume that the sets $Y_i$ are polyhedra of dimension at most 3.

The decomposition of a polyhedron into a Cartesian product of 1- and 2-dimensional factors is not unique. See the examples in [7].

In [7] we have proved that the decomposition of a compact 3-dimensional polyhedron into a Cartesian product is unique if no factor is an arc. In this paper we present a generalization of that theorem. We prove the following

Theorem 2. If a compact connected polyhedron $K$ has two decompositions into Cartesian products

$$K = \top{X} \times A_1 \times \ldots \times A_k = \top{Y} \times B_1 \times \ldots \times B_k,$$

where $\dim A_i = \dim B_i = 1$ for $i = 1, \ldots, k$ and $\dim X = \dim Y = 2$, and all the factors are prime, then for each $i = 1, \ldots, k$ there is $b(i) = 1, \ldots, k$

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such that $A_i = B_{h(i)}$, the correspondence $i \to b(i)$ being one-to-one, whereas
$X \top = Y$ if none of $A_i$’s is an arc.

By Kosiński’s theorem [2] each 2-dimensional Cartesian factor of a polyhedron is polyhedron. Let us recall ([5], [6]) the following

**Definition.** If $P$ is a $k$-dimensional polyhedron, then we define inductively the sets $n_i P$ for $i = 0, 1, \ldots, k$:

(i) $n_0 P = P$.

(ii) $n_i P$ is the set of those points of $n_{i-1} P$ which have no neighborhood in $n_{i-1} P$ homeomorphic to $\mathbb{R}^{k-i+1}$ or $\mathbb{R}^{k-i+1}_+$. We denote the set $n_1 P$ by $n P$.

The proofs of Theorems 1 and 2 are based on investigation of the non-Euclidean parts of Cartesian products of compact connected polyhedra. They use methods similar to those used in [5]–[7]. We need two lemmas to prove both the theorems. In Lemma 1, we investigate the structures of the non-Euclidean parts $n_i K = n_i(X_1 \times \ldots \times X_k)$ of products of polyhedra. These polyhedra are unions of some Cartesian products. In Lemma 2, we find that every homeomorphism $F : X_1 \times \ldots \times X_k \to Y_1 \times \ldots \times Y_n$ of products of polyhedra maps components of the decomposition of $n_i K$ appearing in Lemma 1 onto components of the analogous decomposition of $n_i L$. This result does not give the theorems at once but it is the main tool in the proofs.

**Lemma 1.** If $K = X_1 \times \ldots \times X_k$, where $X_i$ are polyhedra of dimension at most 2 for $i = 1, \ldots, k$, then

$$n_i K = \bigcup \{n_{i_1}X_1 \times \ldots \times n_{i_k}X_k : i_p = 0, 1, 2, i_1 + \ldots + i_k = i\}.$$

**Proof.** We can assume that the $X_i$ are connected.

Observe that if $x_i \in nX_i$ and $\dim X_i = 2$, then either each neighborhood of $x_i$ in $X_i$ contains a subset homeomorphic to $T \times I$ (where $T = \top$ cone$\{1, 2, 3\}$ and $I$ is an arc) or $x_i$ locally cuts $X_i$. If $x_i \in n_2 X_i$, then either each neighborhood of $x_i$ in $nX_i$ contains a triod (a set homeomorphic to $T$) or $x_i$ is an isolated local cut point in $X_i$. If $x_i \in nX_i$ and $\dim X_i = 1$, then each neighborhood of $x_i$ in $X_i$ contains a triod.

We proceed by induction.

1. Let $x \in \bigcup \{n_{i_1}X_1 \times \ldots \times n_{i_k}X_k : i = 0, 1, 2, i_1 + \ldots + i_k = 1\}$, say $x \in nX_1 \times nX_2 \times \ldots \times nX_k$. Let $\dim X_1 = 2$. Then either each neighborhood of $x$ in $K$ contains a set $U = (T \times I) \times I^{\dim K-2}$, which is not embeddable in $\mathbb{R}^{\dim K}$, or every small neighborhood of $x$ in $K$ is cut by a set of dimension
smaller than \( \dim K - 1 \). If \( \dim X_1 = 1 \) then each neighborhood of \( x \) in \( K \) contains a set \( U = T \times I^{\dim K - 1} \). So \( x \in nK \).

The inverse inclusion is obvious.

2. Suppose that our formula is true for \( i \leq m \). Let \( x \in \bigcup \{ n_i X_1 \times \ldots \times n_i X_k ; \ i_p = 0, 1, 2, n_i + \ldots + n_k = m + 1 \} \), say \( x \in n_2 X_1 \times \ldots \times n_2 X_p \times n_3 X_1 \times \ldots \times n_3 X_{p+r} \times X_{p+r+1} \times \ldots \times X_k (2p + r = m + 1) \). Assume \( r \neq 0 \). Then we have two possibilities. First, there exists \( l \), \( 1 \leq l \leq r \), such that \( X_{p+l} \) has dimension 2 and \( x_{p+l} \) locally cuts \( X_{p+l} \). Then every small neighborhood of \( x \) in \( n_m K \) is cut by a set of dimension smaller than \( \dim K = (m + 1) \).

Second, each neighborhood of \( y \) in \( n_m K \) contains a subset homeomorphic to \( \{ z_1 \} \times \ldots \times \{ z_p \} \times T \times I^{\dim K - m - 1} \), which is not embeddable in \( \mathbb{R}^{\dim K - m} \). If \( r = 0 \) then \( x \in n_2 X_1 \times \ldots \times n_2 X_{p-1} \times n_3 X_1 \times \ldots \times n_3 X_k \subset n_m K \) (because \( n_2 X_p \subset n_n K \)). We again have two possibilities. Either \( x_p \) is an isolated local cut point or each neighborhood of \( x \) in \( n_m K \) contains a subset homeomorphic to \( \{ z_1 \} \times \ldots \times \{ z_{p-1} \} \times T \times I^{\dim K - m - 1} \), which is not embeddable in \( \mathbb{R}^{\dim K - m} \). Hence \( x \in n_{m+1} K \).

The inverse inclusion is obvious.

**Lemma 2.** Let \( K = X_1 \times \ldots \times X_k \) and \( L = Y_1 \times \ldots \times Y_n \) where \( X_i, Y_i \) are prime polyhedra of dimension at most 2. If \( F : K \rightarrow L \) is a homeomorphism and \( i_p = 0, 1, 2 \) for \( p = 1, \ldots, k \) then \( F(n_i X_1 \times \ldots \times n_i X_k) = n_{j_1} Y_1 \times \ldots \times n_{j_n} Y_n \) for a system \((j_1, \ldots, j_n)\) of numbers such that \( j_p = 0, 1, 2 \) for \( p = 1, \ldots, n \) and \( i_1 + \ldots + i_k = j_1 + \ldots + j_n \). (In the proofs of Theorems 1 and 2 we need the case \( n_2 X_1 = \emptyset \) for \( i > 1 \) only.)

**Proof.** The proof is similar to the proofs of Lemmas 3.2 of [5] and 2.1 of [6].

Let \( i_1 + \ldots + i_k = m \). If \( m = m_0 \) is a maximal number such that \( n_m K \neq \emptyset \), then the lemma holds. By induction, we can assume that the lemma holds for \( i_1 + \ldots + i_k > m \).

Since \( F \) is a homeomorphism, \( F(n_m K - n_{m+1} K) = n_m L - n_{m+1} L \) each component of \( n_m K - n_{m+1} K \) is equal to \( V_1 \times \ldots \times V_k \), where \( V_p \in \pi (n_l X_p - n_{l+1} X_p) \). (We denote the set of components of \( Z \) by \( \pi Z \).)

Then \( F(V_1 \times \ldots \times V_k) = V_1' \times \ldots \times V_k' \), where \( V_p' \in \pi (n_j Y_p - n_{j+1} Y_p) \).

Let \( \dim V_1 \times \ldots \times V_k = r \).

First we consider the case when \( V_1 \) is a component of \( X_1 - n_i X_1 \) and \( \dim X_1 = 2 \). Now, let \( U_1 \) be also a component of \( X_1 - n_i X_1 \) such that \( \dim V_1' \cap U_1' = 1 \). Then \( F(U_1 \times V_2 \times \ldots \times V_k) = V_1'' \times \ldots \times V_k'' \), where \( V_p'' \in \pi (n_j Y_p - n_{j+1} Y_p) \) and \( \dim F((V_1' \cap U_1') \times V_2 \times \ldots \times V_k) = \dim (V_1' \cap V_1'') \times \ldots \times (V_k' \cap V_k'') = r - 1 \). Only one factor \( V_i' \cap V_i'' \) has dimension smaller than \( \dim V_i' \) and only one factor \( V_i' \cap V_i'' \) has dimension smaller than \( \dim V_i' \). If \( \dim V_i = \dim V_i' \) then \( i_1 = i_2 \). In the opposite case \( \dim V_i'' < \dim V_i' \) and
dim \( V' \) < dim \( V'' \). Then \( V' \cap V'' \neq 0 \) and \( V' \cap V'' \neq 0 \). Let \( V_1 \times \ldots \times V_k = V \) and \( U_1 \times V_2 \times \ldots \times V_k = U \). Choose \( x' \in F(V) \) and \( y' \in F(U) \) such that their coordinates satisfy \( y'_i \in V''_i \cap V'_i \) and \( x'_i \in V''_i \cap V'_i \). Then there exists an open arc \((x'y') \subset V'_1 \times \ldots \times V'_i \times \ldots \times V''_{i+1} \times \ldots \times V''_n \subset L \) disjoint from \( n_{m+1} L \). But if \( x \in V \) and \( y \in U \) then each open arc \((xy) \subset K \) has a non-empty intersection with \( n_{m+1} K \). So \( F^{-1}((x'y')) \cap n_{m+1} K \neq \emptyset \), which is impossible. So, \( i_1 = i_2 \) and \( V'_p = V''_p \) for \( p \neq i_1 \).

If \( W_1 \in \pi_0(X_1 - nX_1) \) and also \( \overline{V}_1 \cap \overline{W}_1 = 1 \) then \( F(W_1 \times V_2 \times \ldots \times V_k) = V''_1 \times \ldots \times V''_n \), where \( V''_p \in \pi_0(p_1 Y_p - p_{i+1} Y_p), V''_p = V''_p \) for \( p \neq i_2 \) and \( \overline{V}_1 \cap \overline{W}_2 = 1 \). By induction \( F(nX_1 \times i_{i_2} X_2 \times \ldots \times n_{i_1} X_k) \) is a Cartesian product of the sets \( n_{i_1} Y_p \), where only one \( s_p \) is one greater than \( p_1 \). The sets \( \overline{V}_1 \cap \overline{W}_1 \) and \( \overline{V}_1 \cap \overline{W}_2 \) are contained in \( nX_1 \). Therefore, \( F(V) \cap F(U) = V''_1 \times \ldots \times V''_n \), \( F(U) = V''_1 \times \ldots \times V''_n \) still have only the \( i_1 \)-factor different and the remaining ones are the same.

If such a sequence does not exist, the points of \( \overline{U} \cap \overline{V}_1 \) are isolated local cut points of \( K \).

Let \( Z \) be the set of points of \( n_{m} K \) at which \( n_{m} K \) is locally cut by a set of dimension \( r - 2 \). If \( Z' \) is the analogous subset of \( n_{m} L \), then \( F(Z) = Z' \).

If \( x \in V_1 \times \ldots \times V_k \) and \( y \in U_1 \times V_2 \times \ldots \times V_k \) then the interior of an arc \( xy \subset \overline{V}_1 \times \overline{W}_1 \) has a non-empty intersection with \( Z \). Similarly, if there exist two indices \( i \) and \( j \) such that \( V'_i \neq V''_j \) and \( V'_j \neq V''_i \), then there exists an arc \( F(x)F(y) \) in \( n_{m} L \) with interior disjoint from \( Z' \).

So, if \( D \) is a component of a subset of the locally 2-dimensional part of \( X_1 \) such that \( V_1 \subset D \), then \( F(D \times V_2 \times \ldots \times V_k) = V''_1 \times \ldots \times D' \times \ldots \times V''_n \), where the \( i_1 \)-factor \( D' \) is an appropriate subset of \( Y'_1 \).

Similarly, we can show that if \( J \) is a component of the 1-dimensional part of \( X_1 \) such that \( J \cap D \neq \emptyset \), then \( F(J \times V_2 \times \ldots \times V_k) = V''_1 \times \ldots \times J' \times \ldots \times V''_n \), where the \( i_2 \)-factor \( J' \) is an appropriate subset of \( Y'_1 \).

The same considerations are true for the homeomorphism \( F^{-1} : L \to K \).

If \( V_1 = 1 \) then either \( V'_1 = 1 \) and \( F(X_1 \times V_2 \times \ldots \times V_k) = V''_1 \times \ldots \times V''_n \) or \( V_1 \subset nX_1 \) and \( F(nX_1 \times V_2 \times \ldots \times V_k) = V''_1 \times \ldots \times n_{i_1} Y_1 \times \ldots \times V''_n \). If \( V_1 = 0 \) then for \( V_1 = 2 \) we have \( V_1 \subset n_{i_2} X_1 \) and \( F(n_{i_2} X_1 \times V_2 \times \ldots \times V_k) = V''_1 \times \ldots \times n_{i_2} Y_1 \times \ldots \times V''_n \). While for \( V_1 = 1 \) we have \( V_1 \subset nX_1 \) and \( F(nX_1 \times V_2 \times \ldots \times V_k) = V''_1 \times \ldots \times nY_1 \times \ldots \times V''_n \). The proof uses the same methods as before but is simpler.
The polyhedra $K$ and $L$ are homeomorphic.

If $nK = \emptyset$ then by Lemma 1, $nA_i = \emptyset$ for all $i = 1, \ldots, n$, so $A_i$ are arcs or simple closed curves (say $I$ and $S^1$). Hence, $\pi_1(K) = \mathbb{Z}^r$, where $r$ is the number of $S^1$’s in the product. The group $\pi_1(L) \approx \pi_1(K)$ is abelian, as are all $\pi(Y_i)$, because $\pi_1(L) = \bigoplus_{i=1}^k \pi_1(Y_i)$. Two-dimensional factors $Y_i$ are polyhedra by Kosiński’s theorem [2] and $nY_i = \emptyset$ by Lemma 1 for all $i = 1, \ldots, k$, so they are compact 2-manifolds with boundary. There are only five such manifolds with abelian fundamental groups: $I^2, S^1 \times I, S^1 \times S^1, S^2$ and the projective plane. It is easy to see that $S^2$ and the projective plane cannot be factors and the remaining manifolds are not prime.

First, we assume that only one factor $Y_1$ has dimension 2.

Now, we proceed by induction with respect to the number of 1-dimensional factors.

If $n = 2$ the problem is trivial. (If $n \leq 3$, then the problem is easy and it is solved in [7].)

Assume that the problem is solved for $m \leq n$.

If $F : L \rightarrow K$ is a homeomorphism, then $F(nL) = nK$, and if $nY_k \neq \emptyset$, then $F(Y_1 \times \ldots \times Y_{k-1} \times nY_k) = A_1 \times \ldots \times A_{n-1} \times nA_n$ (up to a permutation) by Lemma 2. The sets $Y_1 \times \ldots \times Y_{k-1}$ and $A_1 \times \ldots \times A_{n-1}$ are homeomorphic because $nY_k$ and $nA_n$ are finite. The problem is solved by induction.

If $nY_i = \emptyset$ for all $i = 2, \ldots, k$, the problem can be solved by the technique from [5]–[7] and the proof is left to the reader.

Now assume that more than one factor $Y_1$ has dimension 2.

Let $r = \max\{i \in \mathbb{N} : n_i K \neq \emptyset\}$. By Lemma 1 only $r$ factors of the product $A_1 \times \ldots \times A_n$ have $nA_i$ non-empty. Assume $nA_j = \emptyset$ for $j \geq r$. Then $n_r K = nA_1 \times \ldots \times nA_r \times A_{r+1} \times \ldots \times A_n$. Since $n_r K = n_r L$, the set $n_r L$ is homeomorphic to $Z \times A_{r+1} \times \ldots \times A_n$, where $Z$ is finite.

By Lemma 1, $n_r L = n_{i_1} Y_1 \times \ldots \times n_{i_p} Y_k$, where $i_p = 0, 1, 2$. The union from Lemma 1 has only one component in this case because if $n_{i_p+1} Y_p \neq \emptyset$ for one $p$, then $n_{r+1} L \neq \emptyset$. Each component of $n_r K$ is a Cartesian product of arcs and simple closed curves, so no prime Cartesian factor of a component of $n_r L$ is a 2-manifold with boundary. Hence $nY_i \neq \emptyset$ if $\dim Y_i = 2$, for $i = 1, \ldots, m$.

If we assume $\dim Y_1 = 2$, then only the first factor of $Y_1 \times n_{i_2} Y_2 \times \ldots \times n_{i_q} Y_k$ has dimension 2, and this product is homeomorphic to a Cartesian product of 1-dimensional polyhedra, by Lemma 2. So $Y_1$ is not prime as in the first part of the proof.

The proof of Theorem 1 is complete.

**Proof of Theorem 2.** Set $K = X \times A_1 \times \ldots \times A_k$ and $L = Y \times B_1 \times \ldots \times B_k$. 


In the first part of the proof we show that $A_1, \ldots, A_k$ are homeomorphic to $B_1, \ldots, B_k$ up to a permutation.

First, we consider the case when one of the $nA_i$ is not empty, say $nA_k \neq \emptyset$. If $F : K \to L$ is a homeomorphism, then $F(nK) = nL$. By Lemmas 1 and 2, either $F(X \times A_1 \times \ldots \times nA_k) = nY \times B_1 \times \ldots \times B_k$ or $F(X \times A_1 \times \ldots \times nA_k) = Y \times B_1 \times \ldots \times nB_1 \times \ldots \times B_k$. The first possibility does not occur by Theorem 1 because $X$ does not have a decomposition into 1-dimensional factors.

We have proved in [7] that the assertion holds for $k = 1$. Assume that this part of Theorem 2 is true for $k - 1$ factors of dimension 1.

Since $nA_k$ and $nB_i$ are finite, $X \times A_1 \times \ldots \times nA_{k-1}$ and $Y \times B_1 \times \ldots \times B_{i-1} \times B_{i+1} \times \ldots \times B_k$ are homeomorphic. Therefore, $A_1, \ldots, A_{k-1}$ and $B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_k$ are homeomorphic, by induction.

If there exists $j \neq k$ such that $nA_j \neq \emptyset$, we again use induction to show that all the sets $A_1$ and $B_i$ are homeomorphic (up to a permutation).

Assume $nA_1 = \ldots = nA_{k-1} = \emptyset$. Since $F(nK) = nL$ and $F(K - nK) = L - nL$ we conclude that $nA_k$ and $nB_i$, and $A_k - nA_k$ and $B_i - nB_i$, are homeomorphic. Components of $A_k - nA_k$ are arcs. A point $x \in nA_k$ is an end point of such an arc iff the corresponding point $x' \in nB_i$ is an end point of an arc which is a component of $B_i - nB_i$. So $A_k$ and $B_i$ are also homeomorphic.

If $nA_i = \emptyset$ for all $i = 1, \ldots, k$, then each $A_i$ is homeomorphic to an arc or a circle, and similarly for each $B_i$. It is easy to show that the numbers of circles are the same in both cases.

In the second part of the proof we prove that if no Cartesian factor of $K$ is an arc, then $X$ and $Y$ are homeomorphic.

Let $A_1 \neq [0,1]$ for all $i = 1, \ldots, k$ and $A_1 = \ldots = A_m = S^1$. Then (up to a permutation of the $B_i$) the sets $K = X \times S^1 \times \ldots \times S^1 \times nA_{m+1} \times \ldots \times nA_k$ and $L = Y \times S^1 \times \ldots \times S^1 \times nB_{m+1} \times \ldots \times nB_k$ are homeomorphic.

The 1-polyhedra $A_{m+1}, \ldots, A_k$ are neither arcs nor simple closed curves so none of $nA_{m+1}, \ldots, nA_k$ is empty.

Let $F : K \to L$ be a homeomorphism. By Lemma 1, $n_{k-m}K$ is the union of $X \times S^1 \times \ldots \times S^1 \times nA_{m+1} \times \ldots \times nA_k$ and the sets $nX \times S^1 \times \ldots \times S^1 \times n_iA_{m+1} \times \ldots \times n_{i,\ldots, k-m}A_{k}$, where one of $i_1, \ldots, i_{k-m}$ is 0 and the remaining indices are 1, and the sets $n2X \times S^1 \times \ldots \times S^1 \times n_iA_{m+1} \times \ldots \times n_{i,\ldots, k-m}A_{k}$, where two of $i_1, \ldots, i_{k-m}$ are 0 and the remaining indices are 1. Similarly, $n_{k-m}L$ is the union of $Y \times S^1 \times \ldots \times S^1 \times nB_{m+1} \times \ldots \times nB_k$ and the sets $nY \times S^1 \times \ldots \times S^1 \times n_iB_{m+1} \times \ldots \times n_{i,\ldots, k-m}B_{k}$, where one of $i_1, \ldots, i_{k-m}$ is 0 while the remaining indices are 1, and the sets $n2Y \times S^1 \times \ldots \times S^1 \times n_iB_{m+1} \times \ldots \times n_{i,\ldots, k-m}B_{k}$, where two of $i_1, \ldots, i_{k-m}$ are 0 and the remaining indices are 1. We have $F(n_{k-m}K) = n_{k-m}L$. By Lemma 2, $F(X \times S^1 \times \ldots \times S^1 \times nA_{m+1} \times \ldots \times nA_k)$ is one of the above sets whose union is the set $n_{k-m}L$. 

Now \( F(X \times S^1 \times \ldots \times S^1 \times nA_{m+1} \times \ldots \times nA_k) = Y \times S^1 \times \ldots \times S^1 \times nB_{m+1} \times \ldots \times nB_k \) by Theorem 1, because \( X \) and \( Y \) are not products of 1-polyhedra.

Since \( nA_{m+1} \times \ldots \times nA_k \) and \( nB_{m+1} \times \ldots \times nB_k \) are finite sets, \( X \times S^1 \times \ldots \times S^1 \) and \( Y \times S^1 \times \ldots \times S^1 \) are homeomorphic. Similarly to Proposition 4.2 of [5], we conclude that \( X \) and \( Y \) are homeomorphic.

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