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ON DECOMPOSITION OF POLYHEDRA INTO A CARTESIAN PRODUCT OF 1-DIMENSIONAL AND 2-DIMENSIONAL FACTORS

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In 1938 K. Borsuk proved [1] that the decomposition of a polyhedron into a Cartesian product of 1-dimensional factors is topologically unique (up to a permutation of the factors). We prove a little more general

THEOREM 1. *If a connected polyhedron* K (*of arbitrary dimension*) *is homeomorphic to a Cartesian product* $A_1 \times \ldots \times A_n$, *where* A_i*'s are prime compacta of dimension at most* 1, *then there is no other topologically different system of prime compacta* $Y_1, \ldots, Y_k$ *of dimension at most* 2 *such that* $Y_1 \times \ldots \times Y_k$ *is homeomorphic to* K.

A space X is said to be *prime* if it has more than one point and only X and the singleton as Cartesian factors.

In Theorem 1 the dimension of Y_i cannot be greater than 2 (see the examples in [3]–[5]). The 3-dimensional factor of a 6-dimensional torus (in [5]) is not a polyhedron, but the 4-dimensional factors of I^5 (in [3] and [4]) are polyhedra non-homeomorphic to a cube. I do not know if Theorem 1 is true when we assume that the sets Y_i are polyhedra of dimension at most 3.

The decomposition of a polyhedron into a Cartesian product of 1- and 2-dimensional factors is not unique. See the examples in [7].

In [7] we have proved that the decomposition of a compact 3-dimensional polyhedron into a Cartesian product is unique if no factor is an arc. In this paper we present a generalization of that theorem. We prove the following

THEOREM 2. *If a compact connected polyhedron* K *has two decompositions into Cartesian products*

$$K \underset{\text{top}}{=} X \times A_1 \times \ldots \times A_k \underset{\text{top}}{=} Y \times B_1 \times \ldots \times B_k,$$

where $\dim A_i = \dim B_i = 1$ *for* $i = 1, \ldots, k$ *and* $\dim X = \dim Y = 2$, *and all the factors are prime*, *then for each* $i = 1, \ldots, k$ *there is* $b(i) = 1, \ldots, k$

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such that $A_i \underset{\text{top}}{=} B_{b(i)}$, *the correspondence* $i \to b(i)$ *being one-to-one*, *whereas* $X \underset{\text{top}}{=} Y$ *if none of* A_i*'s is an arc.*

By Kosiński's theorem [2] each 2-dimensional Cartesian factor of a polyhedron is polyhedron. Let us recall ([5], [6]) the following

DEFINITION. If P is a k-dimensional polyhedron, then we define inductively the sets n_iP for $i = 0, 1, \ldots, k$:

(i) $n_0P = P$.

(ii) n_iP is the set of those points of $n_{i-1}P$ which have no neighborhood in $n_{i-1}P$ homeomorphic to $\mathbb{R}^{k-i+1}$ or $\mathbb{R}_+^{k-i+1}$.

We denote the set n_1P by nP.

The proofs of Theorems 1 and 2 are based on investigation of the non-Euclidean parts of Cartesian products of compact connected polyhedra. They use methods similar to those used in [5]–[7]. We need two lemmas to prove both the theorems. In Lemma 1, we investigate the structures of the non-Euclidean parts $n_iK = n_i(X_1 \times \ldots \times X_k)$ of products of polyhedra. These polyhedra are unions of some Cartesian products. In Lemma 2, we find that every homeomorphism $F : X_1 \times \ldots \times X_k \to Y_1 \times \ldots \times Y_n$ of products of polyhedra maps components of the decomposition of n_iK appearing in Lemma 1 onto components of the analogous decomposition of n_iL. This result does not give the theorems at once but it is the main tool in the proofs.

LEMMA 1. *If* $K = X_1 \times \ldots \times X_k$, *where* X_i *are polyhedra of dimension at most* 2 *for* $i = 1, \ldots, k$, *then*

$$n_iK = \bigcup\{n_{i_1}X_1 \times \ldots \times n_{i_k}X_k : i_p = 0, 1, 2,\ i_1 + \ldots + i_k = i\}.$$

Proof. We can assume that the X_i are connected.

Observe that if $x_i \in nX_i$ and $\dim X_i = 2$, then either each neighborhood of x_i in X_i contains a subset homeomorphic to $T \times I$ (where $T \underset{\text{top}}{=} \text{cone}\{1, 2, 3\}$ and I is an arc) or x_i locally cuts X_i. If $x_i \in n_2X_i$, then either each neighborhood of x_i in nX_i contains a triod (a set homeomorphic to T) or x_i is an isolated local cut point in X_i. If $x_i \in nX_i$ and $\dim X_i = 1$, then each neighborhood of x_i in X_i contains a triod.

We proceed by induction.

1. Let $x \in \bigcup\{n_{i_1}X_1 \times \ldots \times n_{i_k}X_k : i = 0, 1, 2,\ i_1 + \ldots + i_k = 1\}$, say $x \in nX_1 \times X_2 \times \ldots \times X_k$. Let $\dim X_1 = 2$. Then either each neighborhood of x in K contains a set $U \underset{\text{top}}{=} (T \times I) \times I^{\dim K - 2}$, which is not embeddable in $\mathbb{R}^{\dim K}$, or every small neighborhood of x in K is cut by a set of dimension

smaller than $\dim K-1$. If $\dim X_1=1$ then each neighborhood of x in K contains a set $U \underset{\text{top}}{=} T\times I^{\dim K-1}$. So $x\in nK$.

The inverse inclusion is obvious.

2. Suppose that our formula is true for $i\le m$. Let $x\in\bigcup\{n_{i_1}X_1\times\ldots\times n_{i_k}X_k\colon i_p=0,1,2,\ i_1+\ldots+i_k=m+1\}$, say $x\in n_2X_1\times\ldots\times n_2X_p\times nX_{p+1}\times\ldots\times nX_{p+r}\times X_{p+r+1}\times\ldots\times X_k$ $(2p+r=m+1)$. Assume $r\ne0$. Then we have two possibilities. First, there exists l, $1\le l\le r$, such that X_{p+l} has dimension 2 and x_{p+l} locally cuts X_{p+l}. Then every small neighborhood of x in n_mK is cut by a set of dimension smaller than $\dim K-(m+1)$. Second, each neighborhood of x in n_mK contains a subset homeomorphic to $\{z_1\}\times\ldots\times\{z_p\}\times T\times I^{\dim K-m-1}$, which is not embeddable in $\mathbb{R}^{\dim K-m}$.

If $r=0$ then $x\in n_2X_1\times\ldots\times n_2X_{p-1}\times nX_p\times X_{p+1}\times\ldots\times X_k\subset n_mK$ (because $n_2X_p\subset nX_p$). We again have two possibilities. Either x_p is an isolated local cut point or each neighborhood of x in n_mK contains a subset homeomorphic to $\{z_1\}\times\ldots\times\{z_{p-1}\}\times T\times I^{\dim K-m-1}$, which is not embeddable in $\mathbb{R}^{\dim K-m}$. Hence $x\in n_{m+1}K$.

The inverse inclusion is obvious.

LEMMA 2. *Let $K=X_1\times\ldots\times X_k$ and $L=Y_1\times\ldots\times Y_n$ where X_i,Y_i are prime polyhedra of dimension at most* 2. *If $F:K\to L$ is a homeomorphism and $i_p=0,1,2$ for $p=1,\ldots,k$ then $F(n_{i_1}X_1\times\ldots\times n_{i_k}X_k)=n_{j_1}Y_1\times\ldots\times n_{j_n}Y_n$ for a system $(j_1,\ldots,j_n)$ of numbers such that $j_p=0,1,2$ for $p=1,\ldots,n$ and $i_1+\ldots+i_k=j_1+\ldots+j_n$.* (In the proofs of Theorems 1 and 2 we need the case $n_2X_i=\emptyset$ for $i>1$ only.)

Proof. The proof is similar to the proofs of Lemmas 3.2 of [5] and 2.1 of [6].

Let $i_1+\ldots+i_k=m$. If $m=m_0$ is a maximal number such that $n_mK\ne\emptyset$, then the lemma holds. By induction, we can assume that the lemma holds for $i_k+\ldots+i_k>m$.

Since F is a homeomorphism, $F(n_mK-n_{m+1}K)=n_mL-n_{m+1}L$. Each component of $n_mK-n_{m+1}K$ is equal to $V_1\times\ldots\times V_k$, where $V_p\in\pi_0(n_{i_p}X_p-n_{i_p+1}X_p)$. (We denote the set of components of Z by π_0Z.) Then $F(V_1\times\ldots\times V_k)=V_1'\times\ldots\times V_n'$, where $V_p'\in\pi_0(n_{j_p}Y_p-n_{j_p+1}Y_p)$.

Let $\dim V_1\times\ldots\times V_k=r$.

First we consider the case when V_1 is a component of X_1-nX_1 and $\dim X_1=2$. Now, let U_1 be also a component of X_1-nX_1 such that $\dim\overline{V}_1\cap\overline{U}_1=1$. Then $F(U_1\times V_2\times\ldots\times V_k)=V_1''\times\ldots\times V_n''$, where $V_p''\in\pi_0(n_{j_p}Y_p-n_{j_{p+1}}Y_p)$ and $\dim F((\overline{V}_1\cap\overline{U}_1)\times\overline{V}_2\times\ldots\times\overline{V}_k)=\dim(\overline{V}_1'\cap\overline{V}_1'')\times\ldots\times(\overline{V}_k'\cap\overline{V}_k'')=r-1$. Only one factor $\overline{V}_{i_1}'\cap\overline{V}_{i_1}''$ has dimension smaller than $\dim V_{i_1}'$ and only one factor $\overline{V}_{i_2}'\cap\overline{V}_{i_2}''$ has dimension smaller than $\dim V_{i_2}''$. If $\dim V_{i_1}'=\dim V_{i_1}''$ then $i_1=i_2$. In the opposite case $\dim V_{i_1}''<\dim V_{i_1}'$ and

$\dim V'_{i_2} < \dim V''_{i_2}$. Then $V''_{i_1} \cap \overline{V}'_{i_1} \neq \emptyset$ and $V'_{i_2} \cap \overline{V}''_{i_2} \neq \emptyset$. Let $V_1 \times \ldots \times V_k = \mathbf{V}$ and $U_1 \times V_2 \times \ldots \times V_k = \mathbf{U}$. Choose $\mathbf{x}' \in F(\mathbf{V})$ and $\mathbf{y}' \in F(\mathbf{U})$ such that their coordinates satisfy $y'_{i_1} \in V''_{i_1} \cap \overline{V}'_i$ and $x'_{i_2} \in V'_{i_2} \cap \overline{V}''_{i_2}$. Then there exists an open arc $(\mathbf{x}'\mathbf{y}') \subset V'_1 \times \ldots \times V'_{i_1} \times \ldots \times V''_{i_2} \times \ldots \times V'_n \subset L$ disjoint from $n_{m+1}L$. But if $\mathbf{x} \in \mathbf{V}$ and $\mathbf{y} \in \mathbf{U}$ then each open arc $(\mathbf{x}\,\mathbf{y}) \subset K$ has a non-empty intersection with $n_{m+1}K$. So $F^{-1}((\mathbf{x}'\mathbf{y}')) \cap n_{m+1}K \neq \emptyset$, which is impossible. So, $i_1 = i_2$ and $V'_p = V''_p$ for $p \neq i_1$.

If $W_1 \in \pi_0(X_1 - nX_1)$ and also $\dim \overline{V}_1 \cap \overline{W}_1 = 1$ then $F(W_1 \times V_2 \times \ldots \times V_k) = V^*_1 \times \ldots \times V^*_n$, where $V^*_p \in \pi_0(n_{j_p}Y_p - n_{j_{p+1}}Y_p)$, $V^*_p = V'_p$ for $p \neq i_2$ and $\dim \overline{V}'_{i_2} \cap \overline{V}^*_{i_2} = 1$. By induction $F(nX_1 \times n_{i_2}X_2 \times \ldots \times n_{i_k}X_k)$ is a Cartesian product of the sets $n_{s_p}Y_p$, where only one s_p is one greater than j_p. The sets $\overline{V}_1 \cap \overline{U}_1$ and $\overline{V}_1 \cap \overline{W}_1$ are contained in nX_1. Therefore, $F(\overline{\mathbf{V}}) \cap F(\overline{\mathbf{U}}) = \overline{V}'_1 \times \ldots \times (\overline{V}'_{i_1} \cap \overline{V}''_{i_1}) \times \ldots \times \overline{V}'_n \subset n_{s_1}Y_1 \times \ldots \times n_{s_n}Y_n$. So, $s_{i_1} = j_{i_1} + 1$. Since $\overline{V}_1 \cap \overline{W}_1 \subset nX_1$, we also have $s_{i_2} = j_{i_2} + 1$. Therefore, $i_1 = i_2$.

If there exists a sequence of $U_i \in \pi_0(X_1 - nX_1)$ for $i = 1, \ldots, q$ such that $\dim \overline{U}_i \cap \overline{U}_{i+1} = 1$ for $i = 1, \ldots, q-1$ and $U_q = V_1$ then the products $F(V_1 \times \ldots \times V_k) = V'_1 \times \ldots \times V'_n$ and $F(U_1 \times V_2 \times \ldots \times V_k) = V''_1 \times \ldots \times V''_n$ still have only the i_1-factor different and the remaining ones are the same.

If such a sequence does not exist, the points of $\overline{U}_1 \cap \overline{V}_1$ are isolated local cut points of K_1.

Let Z be the set of points of n_mK at which n_mK is locally cut by a set of dimension $r-2$. If Z' is the analogous subset of n_mL, then $F(Z) = Z'$. If $\mathbf{x} \in V_1 \times \ldots \times V_k$ and $\mathbf{y} \in U_1 \times V_2 \times \ldots \times V_k$ then the interior of an arc $\mathbf{x}\,\mathbf{y} \subset n_mK$ has a non-empty intersection with Z. Similarly, if there exist two indices i and j such that $V'_i \neq V''_i$ and $V'_j \neq V''_j$, then there exists an arc $F(\mathbf{x})F(\mathbf{y})$ in n_mL with interior disjoint from Z'.

So, if D is a component of a subset of the locally 2-dimensional part of X_1 such that $V_1 \subset D$, then $F(D \times V_2 \times \ldots \times V_k) = V'_1 \times \ldots \times D' \times \ldots \times V'_n$, where the i_1-factor D' is an appropriate subset of Y_{i_1}.

Similarly, we can show that if J is a component of the 1-dimensional part of X_1 such that $\overline{J} \cap \overline{D} \neq \emptyset$, then $F(J \times V_2 \times \ldots \times V_k) = V'_1 \times \ldots \times J' \times \ldots \times V'_n$, where the i_1-factor J' is an appropriate subset of Y_{i_1}.

The same considerations are true for the homeomorphism $F^{-1} : L \to K$.

So, $F(X_1 \times V_2 \times \ldots \times V_k) = V'_1 \times \ldots \times Y_{i_1} \times \ldots \times V'_n$.

If $\dim V_1 = 1$ then either $\dim X_1 = 1$ and $F(X_1 \times V_2 \times \ldots \times V_k) = V'_1 \times \ldots \times Y_{i_1} \times \ldots \times V'_n$, or $V_1 \subset nX_1$ and $F(nX_1 \times V_2 \times \ldots \times V_k) = V'_1 \times \ldots \times nY_{i_1} \times \ldots \times V'_n$. If $\dim V_1 = 0$ then for $\dim X_1 = 2$ we have $V_1 \subset n_2X_1$ and $F(n_2X_1 \times V_2 \times \ldots \times V_k) = V'_1 \times \ldots \times n_2Y_{i_1} \times \ldots \times V'_n$, while for $\dim X_1 = 1$ we have $V_1 \subset nX_1$ and then $F(nX_1 \times V_2 \times \ldots V_k) = V'_1 \times \ldots \times nY_{i_1} \times \ldots \times V'_n$. The proof uses the same methods as before but is simpler.

P r o o f o f T h e o r e m 1. Let $K = A_1 \times \ldots \times A_n$ and $L = Y_1 \times \ldots \times Y_k$. The polyhedra K and L are homeomorphic.

If $nK = \emptyset$ then by Lemma 1, $nA_i = \emptyset$ for all $i = 1, \ldots, n$, so A_i are arcs or simple closed curves (say I and S^1). Hence, $\pi_1(K) = \mathbb{Z}^r$, where r is the number of S^1's in the product. The group $\pi_1(L) \approx \pi_1(K)$ is abelian, as are all $\pi(Y_i)$, because $\pi_1(L) = \bigoplus_{i=1}^k \pi_1(Y_i)$. Two-dimensional factors Y_i are polyhedra by Kosiński's theorem [2] and $nY_i = \emptyset$ by Lemma 1 for all $i = 1, \ldots, k$, so they are compact 2-manifolds with boundary. There are only five such manifolds with abelian fundamental groups: $I^2, S^1 \times I, S^1 \times S^1, S^2$ and the projective plane. It is easy to see that S^2 and the projective plane cannot be factors and the remaining manifolds are not prime.

First, we assume that only one factor Y_1 has dimension 2.

Now, we proceed by induction with respect to the number of 1-dimensional factors.

If $n = 2$ the problem is trivial. (If $n \leq 3$, then the problem is easy and it is solved in [7].)

Assume that the problem is solved for $m \leq n$.

If $F : L \to K$ is a homeomorphism, then $F(nL) = nK$, and if $nY_k \neq \emptyset$, then $F(Y_1 \times \ldots \times Y_{k-1} \times nY_k) = A_1 \times \ldots \times A_{n-1} \times nA_n$ (up to a permutation) by Lemma 2. The sets $Y_1 \times \ldots \times Y_{k-1}$ and $A_1 \times \ldots \times A_{n-1}$ are homeomorphic because nY_k and nA_n are finite. The problem is solved by induction.

If $nY_i = \emptyset$ for all $i = 2, \ldots, k$, the problem can be solved by the technique from [5]–[7] and the proof is left to the reader.

Now assume that more than one factor Y_i has dimension 2.

Let $r = \max\{i \in \mathbb{N} : n_i K \neq \emptyset\}$. By Lemma 1 only r factors of the product $A_1 \times \ldots \times A_n$ have nA_i non-empty. Assume $nA_j = \emptyset$ for $j \geq r$. Then $n_r K = nA_1 \times \ldots \times nA_r \times A_{r+1} \times \ldots \times A_n$. Since $n_r K \underset{\text{top}}{=} n_r L$, the set $n_r L$ is homeomorphic to $Z \times A_{r+1} \times \ldots \times A_n$, where Z is finite.

By Lemma 1, $n_r L = n_{i_1} Y_1 \times \ldots \times n_{i_k} Y_k$, where $i_p = 0, 1, 2$. The union from Lemma 1 has only one component in this case because if $n_{i_p+1} Y_p \neq \emptyset$ for one p, then $n_{r+1} L \neq \emptyset$. Each component of $n_r K$ is a Cartesian product of arcs and simple closed curves, so no prime Cartesian factor of a component of $n_r L$ is a 2-manifold with boundary. Hence $nY_i \neq \emptyset$ if $\dim Y_i = 2$, for $i = 1, \ldots, m$.

If we assume $\dim Y_1 = 2$, then only the first factor of $Y_1 \times n_{i_2} Y_2 \times \ldots \times n_{i_k} Y_k$ has dimension 2, and this product is homeomorphic to a Cartesian product of 1-dimensional polyhedra, by Lemma 2. So Y_1 is not prime as in the first part of the proof.

The proof of Theorem 1 is complete.

P r o o f o f T h e o r e m 2. Set $K = X \times A_1 \times \ldots \times A_k$ and $L = Y \times B_1 \times \ldots \times B_k$.

In the first part of the proof we show that $A_1, \ldots, A_k$ are homeomorphic to $B_1, \ldots, B_k$ up to a permutation.

First, we consider the case when one of the nA_i is not empty, say $nA_k \neq \emptyset$. If $F : K \to L$ is a homeomorphism, then $F(nK) = nL$. By Lemmas 1 and 2, either $F(X \times A_1 \times \ldots \times nA_k) = nY \times B_1 \times \ldots \times B_k$ or $F(X \times A_1 \times \ldots \times nA_k) = Y \times B_1 \times \ldots \times nB_i \times \ldots \times B_k$. The first possibility does not occur by Theorem 1 because X does not have a decomposition into 1-dimensional factors.

We have proved in [7] that the assertion holds for $k = 1$. Assume that this part of Theorem 2 is true for $k - 1$ factors of dimension 1.

Since nA_k and nB_i are finite, $X \times A_1 \times \ldots \times A_{k-1}$ and $Y \times B_1 \times \ldots \times B_{i-1} \times B_{i+1} \times \ldots \times B_k$ are homeomorphic. Therefore, $A_1, \ldots, A_{k-1}$ and $B_1, \ldots, B_{i-1}, B_{i+1}, \ldots, B_k$ are homeomorphic, by induction.

If there exists $j \neq k$ such that $nA_j \neq \emptyset$, we again use induction to show that all the sets A_i and B_i are homeomorphic (up to a permutation).

Assume $nA_1 = \ldots = nA_{k-1} = \emptyset$. Since $F(nK) = nL$ and $F(K - nK) = L - nL$ we conclude that nA_k and nB_i, and $A_k - nA_k$ and $B_i - nB_i$, are homeomorphic. Components of $A_k - nA_k$ are arcs. A point $x \in nA_k$ is an end point of such an arc iff the corresponding point $x' \in nB_i$ is an end point of an arc which is a component of $B_i - nB_i$. So A_k and B_i are also homeomorphic.

If $nA_i = \emptyset$ for all $i = 1, \ldots, k$, then each A_i is homeomorphic to an arcs or a circle, and similarly for each B_i. It is easy to show that the numbers of circles are the same in both cases.

In the second part of the proof we prove that if no Cartesian factor of K is an arc, then X and Y are homeomorphic.

Let $A_i \neq [0,1]$ for all $i = 1, \ldots, k$ and $A_1 = \ldots = A_m = S^1$. Then (up to a permutation of the B_i) the sets $K = X \times S^1 \times \ldots \times S^1 \times nA_{m+1} \times \ldots \times nA_k$ and $L = Y \times S^1 \times \ldots \times S^1 \times nB_{m+1} \times \ldots \times nB_k$ are homeomorphic.

The 1-polyhedra $A_{m+1}, \ldots, A_k$ are neither arcs nor simple closed curves so none of $nA_{m+1}, \ldots, nA_k$ is empty.

Let $F : K \to L$ be a homeomorphism. By Lemma 1, $n_{k-m}K$ is the union of $X \times S^1 \times \ldots \times S^1 \times nA_{m+1} \times \ldots \times nA_k$ and the sets $nX \times S^1 \times \ldots \times S^1 \times n_{i_1}A_{m+1} \times \ldots \times n_{i_{k-m}}A_k$, where one of $i_1, \ldots, i_{k-m}$ is 0 and the remaining indices are 1, and the sets $n_2X \times S^1 \times \ldots \times S^1 \times n_{i_1}A_{m+1} \times \ldots \times n_{i_{k-m}}A_k$, where two of $i_1, \ldots, i_{k-m}$ are 0 and the remaining indices are 1. Similarly, $n_{k-m}L$ is the union of $Y \times S^1 \times \ldots \times S^1 \times nB_{m+1} \times \ldots \times nB_k$ and the sets $nY \times S^1 \times \ldots \times S^1 \times n_{i_1}B_{m+1} \times \ldots \times n_{i_{k-m}}B_k$, where one of $i_1, \ldots, i_{k-m}$ is 0 while the remaining indices are 1, and the sets $n_2Y \times S^1 \times \ldots \times S^1 \times n_{i_1}B_{m+1} \times \ldots \times n_{i_{k-m}}B_k$, where two of $i_1, \ldots, i_{k-m}$ are 0 and the remaining indices are 1. We have $F(n_{k-m}K) = n_{k-m}L$. By Lemma 2, $F(X \times S^1 \times \ldots \times S^1 \times nA_{m+1} \times \ldots \times nA_k)$ is one of the above sets whose union is the set $n_{k-m}L$.

Now $F(X \times S^1 \times \ldots \times S^1 \times nA_{m+1} \times \ldots \times nA_k) = Y \times S^1 \times \ldots \times S^1 \times nB_{m+1} \times \ldots \times nB_k$ by Theorem 1, because X and Y are not products of 1-polyhedra.

Since $nA_{m+1} \times \ldots \times nA_k$ and $nB_{m+1} \times \ldots \times nB_k$ are finite sets, $X \times S^1 \times \ldots \times S^1$ and $Y \times S^1 \times \ldots \times S^1$ are homeomorphic. Similarly to Proposition 4.2 of [5], we conclude that X and Y are homeomorphic.

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