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CHAINS OF FACTORIZATIONS IN WEAKLY KRULL DOMAINS

BY

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1. Introduction. In a noetherian domain every non-zero non-unit has a factorization into a product of irreducible elements. In general, such a factorization need not be unique. A lot of arithmetical invariants have been introduced to describe the non-uniqueness of factorizations. Most of them concentrate only on lengths of factorizations. However, there are noetherian domains which behave as good as possible when lengths are concerned but whose arithmetic is far from being simple.

The central topic of this paper is an arithmetical invariant, the catenary degree, which is more subtle than invariants which just control the lengths of factorizations. It was introduced in [G-L] and is defined as follows. Let R be a noetherian domain, $0 \neq a \in R$ and z, z' two factorizations of a. We say that there is an N-chain of factorizations from z to z' if a has factorizations $z = z_0, z_1, \ldots, z_k = z'$ such that the distance between two subsequent factorizations z_{i-1} and z_i is bounded by $N \in \mathbb{N}$ for all $1 \leq i \leq k$. The catenary degree $c(R)$ of R is the minimal $N \in \mathbb{N} \cup \{\infty\}$ such that for all $0 \neq a \in R$ and any two factorizations z, z' of a there is an N-chain of factorizations from z to z' (cf. Definition 3.2).

In the theory of non-unique factorizations, Krull domains (including integrally closed noetherian domains) represent the best investigated class of domains. Most results are achieved by a divisor-theoretic approach using the fact that a Krull domain admits a (classical) divisor theory (i.e., a divisor homomorphism into a free abelian monoid). Domains which are not integrally closed admit no divisor theory. In spite of various partial results, their arithmetic is still far less understood than the arithmetic of Krull domains.

Quite recently were weakly Krull domains introduced to literature (cf. [A-M-Z]). These domains are not necessarily integrally closed but include Krull domains and all one-dimensional noetherian domains. Using the close relationship between divisor homomorphisms and generalized valuations (as

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developed in [G-HK]) F. Halter-Koch showed in [HK3] that a domain is weakly Krull if and only if it admits a weak divisor theory (i.e., a divisor homomorphism into a coproduct of primary monoids). This characterization provides the algebraic basis for the main result of the present paper (Theorem 7.3): weakly Krull domains satisfying certain natural finiteness conditions have finite catenary degree.

The property of being a weakly Krull domain is a purely multiplicative one; a domain is weakly Krull if and only if its multiplicative monoid is a weakly Krull monoid. In general, the factorization properties of a domain just depend on the structure of its multiplicative monoid. Hence all notions and most results of this paper are formulated in the context of monoids. Their relevance, however, lies in their ring-theoretic applications. Apart from technical advantages, this semigroup-theoretic procedure makes it possible to describe most clearly the combinatorial structures which are responsible for the investigated phenomena.

The paper is organized as follows. All relevant arithmetical notions are introduced in Section 3. Section 4 deals with (general) block monoids as introduced in [Ge3], which are the crucial combinatorial tool. These technical preparations are developed to such an extent that they meet future requirements. Theorem 5.4 in Section 5 states that weakly Krull monoids satisfying certain finiteness conditions have finite catenary degree. The relevance of these finiteness conditions will become more clear in Section 6, where we give examples of monoids with infinite catenary degree. In Section 7 the semigroup-theoretic result is applied to weakly Krull domains. In particular, the result is valid for orders in holomorphy rings in global fields, and it will serve as a basis for quantitative investigations in these domains (see [Ge5]).

2. Preliminaries on monoids. Throughout this paper, a *monoid* is a commutative and cancellative semigroup with unit element. If not stated otherwise, we will use multiplicative notation. We review some necessary terminology.

For a family $(H_p)_{p\in P}$ of monoids we denote, as usual, by $\prod_{p\in P} H_p$ their direct product, and by

$$\coprod_{p\in P} H_p = \Big\{(a_p)_{p\in P} \in \prod_{p\in P} H_p : a_p = 1 \text{ for almost all } p\in P\Big\}$$

their coproduct. For every $Q \subseteq P$ we view $\coprod_{p\in Q} H_p$ as a submonoid of $\coprod_{p\in P} H_p$. If all H_p are infinite cyclic (i.e. $H_p \simeq (\mathbb{N},+)$) then $\coprod_{p\in P} H_p$ is the free abelian monoid with basis P and will be denoted by $\mathcal{F}(P)$. If $P=\emptyset$, then $\mathcal{F}(P)=\{1\}$.

Every $a \in \mathcal{F}(P)$ has a unique representation

$$a = \prod_{p \in P} p^{v_p(a)}$$

with $v_p(a) \in \mathbb{N}$ and $v_p(a) = 0$ for almost all $p \in P$. Furthermore,

$$\sigma(a) = \sum_{p \in P} v_p(a) \in \mathbb{N}$$

is called the *size* of a.

If D is a monoid, then $D^\times$ denotes the group of invertible elements of D. D is called *reduced* if $D^\times = \{1\}$. $\mathcal{Q}(D)$ denotes a quotient group of D, and we always assume $D \subseteq \mathcal{Q}(D)$. *The complete integral closure* $\widehat{D}$ of D is defined as

$$\begin{aligned}\widehat{D} = \{x \in \mathcal{Q}(D) :& \\ &\text{there exists some } c \in D \text{ such that } \ cx^n \in D \text{ for all } n \in \mathbb{N}_+\}.\end{aligned}$$

By definition, we have $D \subseteq \widehat{D} \subseteq \mathcal{Q}(D)$.

A subset $D' \subseteq D$ is called *divisor closed* if for all $a, b \in D$ with $a \mid b$ and $b \in D'$ we have $a \in D'$.

Let H and D be submonoids of some abelian group. We call

$$\mathfrak{f}_{D/H} = \{f \in H : fD \subseteq H\}$$

the *conductor* of D in H. If $H \subseteq D$ and $\mathfrak{f}_{D/H} \neq \emptyset$, then $\mathcal{Q}(H) = \mathcal{Q}(D)$.

We define congruence modulo H in D by

$$x \equiv y \bmod H \quad \text{if} \quad x^{-1}y \in \mathcal{Q}(H).$$

The *factor monoid* of D with respect to congruence modulo H is denoted by D/H. For $a \in D$ $[a] \in D/H$ denotes the class containing a. If H is a group, then $[a] = \{ax : x \in H\} = aH$. In particular, we set $D_{\mathrm{red}} = D/D^\times$.

$H \subseteq D$ is called *saturated* if $a, b \in H$, $c \in D$ and $a = bc$ imply that $c \in H$ (equivalently, $H = D \cap \mathcal{Q}(H)$). If $H \subseteq D$ is divisor closed, then it is saturated.

Next we consider monoid homomorphisms. Such a homomorphism $\varphi : H \to D$ induces a unique group homomorphism $\mathcal{Q}(\varphi) : \mathcal{Q}(H) \to \mathcal{Q}(D)$. Further,

$$\mathrm{Cl}(\varphi) = \mathcal{Q}(D/\varphi H)$$

is called the *class group* of $\varphi : H \to D$. It will be written additively. Obviously we have

$$\mathrm{Cl}(\varphi) \simeq \mathcal{Q}(D)/\mathcal{Q}(\varphi)(\mathcal{Q}(H)).$$

In particular, if H is a submonoid of D, then the class group of the embedding $\varphi : H \hookrightarrow D$ will be called the *class group* of $H \subseteq D$.

A monoid homomorphism $\varphi : H \to D$ is said to be a *divisor homomorphism* if $a, b \in H$ and $\varphi(a) \,|\, \varphi(b)$ imply $a \,|\, b$.

A monoid D is said to be *primary* if $D \neq D^\times$ and if $a, b \in D$ and $b \notin D^\times$ imply that $a \,|\, b^n$ for some $n \in \mathbb{N}_+$. For various equivalent conditions for being primary and some historical remarks cf. [Ge4; Lemma 1].

Let $(D_p)_{p \in P}$ be a family of primary monoids and set $D = \coprod_{p \in P} D_p$. Then the monoids D_p are called the *primary components* of D. For a family $(a^{(i)})_{i \in I}$ of elements $a^{(i)} = (a_p^{(i)})_{p \in P} \in D$ and an element $a = (a_p)_{p \in P} \in D$, we call a a *strict greatest common divisor* and write

$$a = \bigwedge (a^{(i)})_{i \in I}$$

if the following two conditions are satisfied for all $p \in P$:

(i) $a_p \,|\, a_p^{(i)}$ for all $i \in I$;

(ii) $a_p^{(i)} \,|\, a_p$ for at least one $i \in I$.

If D is factorial, then the strict greatest common divisor coincides with the usual greatest common divisor (cf. [G-HK; Definition 4.5]).

DEFINITION 2.1. Let H be a monoid.

1. A divisor homomorphism

$$\varphi : H \to D = \coprod_{p \in P} D_p$$

into a coproduct of reduced primary monoids D_p is called a *weak divisor theory* if for all $a \in D$ there exist $u_1, \ldots, u_m \in H$ such that $a = \bigwedge_{i=1}^m \varphi u_i$. If $D_p \simeq (\mathbb{N}, +)$ for all $p \in P$, then φ is said to be a *divisor theory*.

2. H is called a (*weakly*) *Krull monoid* if it admits a (weak) divisor theory.

Weakly Krull monoids were introduced in [HK3]. The weak divisor theory of a weakly Krull monoid is uniquely determined (up to isomorphism). This uniqueness implies that the group $\mathrm{Cl}(H) = D/\varphi H$ just depends on H. $\mathrm{Cl}(H)$ is called the (*divisor*) *class group* of H (cf. [HK3; Section 2]). The main examples we cite are the multiplicative monoids of weakly Krull domains; these will be discussed in Section 7.

Let $\varphi : H \to D$ be a weak divisor theory. Since φ is a divisor homomorphism, $\varphi(H) \subseteq D$ is saturated and the induced homomorphism $\varphi_{\mathrm{red}} : H_{\mathrm{red}} \to D_{\mathrm{red}}$ is injective (cf. [G-HK; Lemma 2.6]). Hence it means no restriction to suppose that $H \subseteq D$ is a saturated submonoid. We shall adopt this viewpoint in the sequel. Indeed, if D is free abelian and $H \subseteq D$ saturated, then H is a Krull monoid. However, there are monoids H saturated in a coproduct of primary monoids which are not weakly Krull (cf. [HK3; Proposition 2.13]).

Let G be an abelian group. As usual, we say that elements $g_1, \ldots, g_r$ are *linearly independent* if each equation $\sum_{i=1}^{r} n_i g_i = 0$ with integer coefficients $n_i \in \mathbb{Z}$ implies $n_1 g_1 = \ldots = n_r g_r = 0$. If G is a bounded torsion group, then $\exp(G)$ denotes the exponent of G.

3. On the arithmetic of monoids. We briefly recall some arithmetical invariants of monoids and some basic notions from the theory of non-unique factorizations. For their relevance and properties the reader is referred to the cited literature.

Let H be a monoid. We denote by $\mathcal{U}(H)$ the set of irreducible elements of H. The *factorization monoid* $\mathcal{Z}(H)$ of H is defined as the free abelian monoid with basis $\mathcal{U}(H_{\mathrm{red}})$. Thus,

$$\mathcal{Z}(H) = \mathcal{F}(\mathcal{U}(H_{\mathrm{red}}))$$

and the elements $z \in \mathcal{Z}(H)$ are written in the form

$$z = \prod_{u \in \mathcal{U}(H_{\mathrm{red}})} u^{v_u(z)}.$$

Let $\pi : \mathcal{Z}(H) \to H_{\mathrm{red}}$ be the canonical homomorphism. We say that H is *atomic* if π is surjective.

Suppose that H is atomic, and let $a \in H$ be given. The elements of

$$\mathcal{Z}_H(a) = \mathcal{Z}(a) = \pi^{-1}(aH^\times) \subseteq \mathcal{Z}(H)$$

are called *factorizations* of a and

$$L_H(a) = L(a) = \{\sigma(z) : z \in \mathcal{Z}(a)\} \subseteq \mathbb{N}$$

denotes the *set of lengths* of a. For a subset $H' \subseteq H$ the *elasticity* $\varrho(H')$ of H is defined as (cf. [HK4])

$$\varrho(H') = \sup\left\{\frac{\sup L(a)}{\min L(a)} : a \in H' \setminus H^\times\right\} \in \mathbb{N}_+ \cup \{\infty\}.$$

An atomic monoid H is said to be (cf. [HK2]):

1. *factorial* if $\#\mathcal{Z}(a) = 1$ for all $a \in H$,
2. *half-factorial* if $\#L(a) = 1$ for all $a \in H$,
3. an FF-*monoid* (finite-factorization monoid) if $\#\mathcal{Z}(a) < \infty$ for all $a \in H$,
4. a BF-*monoid* (bounded-factorization monoid) if $\#L(a) < \infty$ for all $a \in H$.

The most thoroughly studied invariants, as sets of lengths and the elasticity, consider only lengths of factorizations. However, there are even half-factorial monoids with bad factorization properties. In [A-A-Z; Example 4.1] an example of a noetherian domain is given whose multiplicative monoid is half-factorial but not even an FF-monoid. Such phenomena make it indispensable to look more closely at factorizations.

Let H be an atomic monoid. For two factorizations $z, z' \in \mathcal{Z}(H)$ we call

$$d(z, z') = \max\left\{\sigma\left(\frac{z}{\gcd(z, z')}\right), \sigma\left(\frac{z'}{\gcd(z, z')}\right)\right\} \in \mathbb{N}$$

the *distance* between z and z'. This means that, if $z = u_1 \dots u_l v_1 \dots v_m$ and $z' = u_1 \dots u_l w_1 \dots w_n$ with $u_i, v_j, w_k \in \mathcal{U}(H_{\mathrm{red}})$ such that $\{v_j : 1 \le j \le m\} \cap \{w_k : 1 \le k \le n\} = \emptyset$, then $d(z, z') = \max\{m, n\}$. Thus $d(z, z') = 0$ if and only if $z = z'$. If $z, z' \in \mathcal{Z}(a)$ for some $a \in H$ and $z \ne z'$, then $d(z, z') \ge 2$.

The following lemma is trivial but throws a first light on the situation in non-factorial monoids.

LEMMA 3.1. *Let H be an atomic monoid. If H is not factorial, then for every $n \in \mathbb{N}_+$ there exists some element $a \in H$ and factorizations $z, z' \in \mathcal{Z}(a)$ with $d(z, z') \ge n$.*

Proof. Suppose that H is not factorial. Then there exists some element $c \in H$ having two distinct factorizations $y, y' \in \mathcal{Z}(c)$. So for every $n \in \mathbb{N}_+$ we have $y^n, {y'}^n \in \mathcal{Z}(c^n)$ and

$$d(y^n, {y'}^n) = nd(y, y') \ge 2n. \ \blacksquare$$

Hence in all non-factorial monoids there are elements having completely different factorizations. Thus the best we can expect is that these factorizations are somehow connected. This is made precise in the following definition.

DEFINITION 3.2. Let H be an atomic monoid.

1. Let $a \in H$, $z, z' \in \mathcal{Z}(a)$ and $N \in \mathbb{N} \cup \{\infty\}$; we say that there is an *N-chain* (*of factorizations*) *from z to z'* if there exist factorizations $z = z_0, z_1, \dots, z_k = z' \in \mathcal{Z}(a)$ such that $d(z_{i-1}, z_i) \le N$ for $1 \le i \le k$.

2. The *catenary degree*

$$c_H(H') = c(H') \in \mathbb{N} \cup \{\infty\}$$

of a subset $H' \subseteq H$ is the minimal $N \in \mathbb{N} \cup \{\infty\}$ such that for every $a \in H'$ and any two factorizations $z, z' \in \mathcal{Z}(a)$ there exists an N-chain from z to z'. For simplicity, we write $c(a)$ instead of $c(\{a\})$.

The main aim of this paper is to prove that weakly Krull monoids satisfying certain natural finiteness conditions have finite catenary degree (cf. Theorem 5.4).

Remarks. Let H be an atomic monoid and let $a \in H$.

1. We have $c(a) = 0$ if and only if $\#\mathcal{Z}(a) = 1$. Thus H is factorial if and only if $c(H) = 0$.

2. By definition, we always have $c(a) \neq 1$. If $c(a) = 2$, then $\#L(a) = 1$. Therefore $c(H) = 2$ implies that H is half-factorial. However, there are half-factorial monoids with infinite catenary degree (see [G-L; Remark 2 after Definition 2]).

3. If $c(a) = 3$, then $L(a) = \{y, y+1, \dots, y+k\}$ for some $y, k \in \mathbb{N}_+$.

4. Suppose that $a = \prod_{i=1}^n a_i$ with $\max L(a_i) \le N$ for some $N \in \mathbb{N}_+$. Further, let $z_i, z_i' \in \mathcal{Z}(a_i)$ for $1 \le i \le n$ and $z = \prod_{i=1}^n z_i,\ z' = \prod_{i=1}^n z_i'$. Then there exists an N-chain from z to z'. Indeed, setting $y_j = \prod_{i=1}^j z_i' \prod_{i=j+1}^n z_i$ for $0 \le j \le n$, we have $y_0 = z,\ y_n = z'$ and $d(y_j, y_{j+1}) = d(z_{j+1}', z_{j+1}) \le N$.

5. Let $\varphi : H \to D$ be a monoid epimorphism onto an atomic monoid D with $\varphi(\mathcal{U}(H)) \subseteq \mathcal{U}(D) \cup D^\times$. Then φ has a natural extension to $\varphi : \mathcal{Z}(H) \to \mathcal{Z}(D)$ and for $z, z' \in \mathcal{Z}(H)$ we have $d(\varphi z, \varphi z') \le d(z, z')$. Furthermore, $c(\varphi H') \le c(H')$ for all subsets $\emptyset \neq H' \subseteq H$.

We introduce a new arithmetical invariant which will be crucial for our further investigations.

Definition 3.3. Let D be an atomic monoid and $D' \subseteq D$ a non-empty subset.

1. For $u \in D$ let $w_D(D', u)$ be defined as the minimum of all $w \in \mathbb{N}_+ \cup \{\infty\}$ having the following property: if $a_1, \dots, a_n \in D \setminus D^\times$ with $\prod_{i=1}^n a_i \in D'$ such that $u \mid \prod_{i=1}^n a_i$, then there exists a subset $J \subseteq \{1, \dots, n\}$ with $\#J \le w$ and $u \mid \prod_{i \in J} a_i$.

2. For a subset $U \subseteq D$ we set

$$w_D(D', U) = \sup\{w_D(D', u) : u \in U\} \in \mathbb{N}_+ \cup \{\infty\}.$$

The following two situations will be of special importance:

(i) $U = \mathcal{U}(D)$ and $D' \subseteq D$ a divisor closed subset,

(ii) $D' = D$ and $U = \mathcal{U}(H)$ for a saturated submonoid $H \subseteq D$.

Remarks. Let D be an atomic monoid.

1. For every $u \in D$ we have $w_D(D, u) = w_{D_{\mathrm{red}}}(D_{\mathrm{red}}, uH^\times)$ and hence $w_D(D, \mathcal{U}(D)) = w_{D_{\mathrm{red}}}(D_{\mathrm{red}}, \mathcal{U}(D_{\mathrm{red}}))$.

2. If $D'' \subseteq D' \subseteq D$ and $U \subseteq V \subseteq D$ are subsets, then by definition

$$w_D(D'', U) \le w_D(D', V).$$

3. Let $\beta \in \mathbb{N}_+$ and $D' = \{a \in D : \sup L(a) \le \beta\}$. Then $D' \subseteq D$ is divisor closed and $w_D(D', \mathcal{U}(D)) \le \beta$.

4. Let $u \in D$ be a product of primes, say $u = p_1 \dots p_r$, and let $D' \subseteq D$ be a divisor closed subset containing u. Then $w_D(D', u) = r$; in particular, if $u \in D$ is prime, then $w_D(D, u) = 1$. Conversely, if for some $u \in D \setminus D^\times$ we have $w_D(D, u) = 1$, then u is a prime element.

An atomic monoid is factorial if and only if all its irreducible elements are prime. Hence D is factorial if and only if $w_D(D,\mathcal{U}(D)) = 1$.

PROPOSITION 3.4. *Let D be an atomic monoid.*

1. *If D_{red} is finitely generated, then $w_D(D,\mathcal{U}(D)) < \infty$.*
2. *If $D = \coprod_{i\in I} D_i$ and $D_i' \subseteq D$ are non-empty subsets, then*

$$w_D\Big(\coprod_{i\in I} D_i',\mathcal{U}(D)\Big) = \sup_{i\in I} w_{D_i}(D_i',\mathcal{U}(D_i)).$$

Proof. 1. By the previous remark we may assume without restriction that D is finitely generated. Let $\mathcal{U}(D) = \{u_1,\ldots,u_s\}$ and let $i \in \{1,\ldots,s\}$ be given. It suffices to show that $w_D(D,u_i) < \infty$. For this we consider the set

$$A_i = \Big\{\mathbf{k} = (k_1,\ldots,k_s) \in \mathbb{N}^s : u_i \mid \prod_{\nu=1}^{s} u_\nu^{k_\nu}\Big\} \subseteq \mathbb{N}^s.$$

By [C-P; Theorem 9.18] the set M_i of minimal points of A_i is finite and we set

$$w = \max\Big\{\sum_{\nu=1}^{s} k_\nu : \mathbf{k} \in M_i\Big\}.$$

Let $a_1,\ldots,a_n \in D\backslash D^\times$ be given with $u_i \mid \prod_{\nu=1}^n a_\nu$. Now, if $\prod_{\nu=1}^n a_\nu = \prod_{\nu=1}^s u_\nu^{l_\nu}$, then there exists some $\mathbf{k} \in M_i$ with $\mathbf{k} \le \mathbf{l}$ and $u_i \mid \prod_{\nu=1}^s u_\nu^{k_\nu}$. Hence there exists a subset $J \subseteq \{1,\ldots,n\}$ with $u_i \mid \prod_{j\in J} a_j$ and $\#J \le \sum_{\nu=1}^s k_\nu \le w$.

2. Clearly,

$$w_{D_i}(D_i',u) = w_D\Big(\coprod_{j\in I} D_j',u\Big)$$

for every $i \in I$ and every $u \in \mathcal{U}(D_i)$. Since $\mathcal{U}(D) = \bigcup_{i\in I}\mathcal{U}(D_i)$ we infer that

$$\begin{aligned} w_D\Big(\coprod_{j\in I} D_j',\mathcal{U}(D)\Big) &= \sup_{i\in I} w_D\Big(\coprod_{j\in I} D_j',\mathcal{U}(D_i)\Big) \\ &= \sup_{i\in I} w_{D_i}(D_i',\mathcal{U}(D_i)). \ \blacksquare \end{aligned}$$

PROPOSITION 3.5. *Let D be an atomic monoid, $D' \subseteq D$ a divisor closed subset and $u,v \in D'$.*

1. $\sup L(u) \le w_D(D',u)$.
2. $w_D(D',uv) \le w_D(D',u) + w_D(D',v)$.
3. $\sup L(u) \le \min L(u)\cdot w_D(D',\mathcal{U}(D))$.
4. $\varrho(D') \le w_D(D',\mathcal{U}(D))$.

Proof. 1. We show that $k \le w_D(D', u)$ for every $k \in L(u)$. Let $u = v_1 \dots v_k$ with each $v_j \in \mathcal{U}(D)$. Then $u \mid v_1 \dots v_k$ and hence $u \mid \prod_{j\in J} v_j$ for some $J \subseteq \{1, \dots, k\}$ with $\#J \le w_D(D', u)$. But this implies that $v_1 \dots v_k \mid \prod_{j\in J} v_j$ and thus $J = \{1, \dots, k\}$. Therefore we obtain $k = \#J \le w_D(D', u)$.

2. Let $a_1, \dots, a_n \in D\backslash D^\times$ be given with $\prod_{i=1}^n a_i \in D'$ such that $uv \mid \prod_{i=1}^n a_i$. Then without restriction of generality it follows that $u \mid \prod_{i=1}^k a_i$ with $k \le w_D(D', u)$. If we set $\prod_{i=1}^k a_i = ua_0$, then $v \mid a_0a_{k+1} \dots a_n$. Again we may assume that $v \mid a_0a_{k+1} \dots a_{k+l}$ with $l \le w_D(D', v)$. Therefore

$$uv \mid ua_0a_{k+1} \dots a_{k+l} = \prod_{i=1}^{k+l} a_i,$$

which implies the assertion.

3. Let $u = v_1 \dots v_k$ with $v_j \in \mathcal{U}(D)$ and $k = \min L(u)$. Using parts 1 and 2 we infer that

$$\sup L(u) \le w_D(D', u) \le \sum_{i=1}^k w_D(D', v_i) \le \min L(u) \cdot w_D(D', \mathcal{U}(D)).$$

4. This follows from part 3. ∎

Corollary 3.6. *Let D be an atomic monoid and $H \subseteq D$ a saturated atomic submonoid.*

1. *If $D' \subseteq D$ is a divisor closed subset, then $H' = H \cap D' \subseteq H$ is divisor closed and*

$$w_H(H', \mathcal{U}(H)) \le \sup_{u\in\mathcal{U}(H)} \sup L_D(u) \cdot w_D(D', \mathcal{U}(D)).$$

2. *If D is free abelian, then*

$$w_H(H, \mathcal{U}(H)) \le \sup\{\sigma(u) : u \in \mathcal{U}(H)\}.$$

Furthermore, if $H \hookrightarrow D$ is a divisor theory with class group $G, G_0 \subseteq G$ the set of classes containing primes and $G_0 = -G_0$, then equality holds in the above formula.

Proof. 1. Obviously, $H' \subseteq H$ is a divisor closed subset. Let $u \in \mathcal{U}(H)$ be given.

First we show that $w_H(H', u) \le w_D(D', u)$. Let $a_1, \dots, a_n \in H\backslash H^\times$ with $\prod_{i=1}^n a_i \in H'$ such that $u \mid \prod_{i=1}^n a_i$ in H. Since $H \subseteq D$ is saturated, we have $H^\times = D^\times \cap H$. Therefore $a_1, \dots, a_n \in D\backslash D^\times$, $\prod_{i=1}^n a_i \in D'$ and $u \mid \prod_{i=1}^n a_i$ in D. So there exists a subset $J \subseteq \{1, \dots, n\}$ with $\#J \le w_D(D', u)$ such that $u \mid \prod_{i\in J} a_i$ in D. Thus $u \mid \prod_{i\in J} a_i$ in H and $w_H(H', u) \le w_D(D', u)$.

Suppose $u = v_1 \ldots v_d$ with $v_j \in \mathcal{U}(D)$. Using parts 1 and 2 of Proposition 3.5 we infer that

$$w_D(D', u) \le \sum_{i=1}^{d} w_D(D', v_i) \le d w_D(D', \mathcal{U}(D)) \le \sup L_D(u) \cdot w_D(D', \mathcal{U}(D)).$$

2. Suppose that D is free abelian. Then $L_D(u) = \{\sigma(u)\}$ for every $u \in D$ and $w_D(D, \mathcal{U}(D)) = 1$ by Remark 4 after Definition 3.3. This implies

$$w_H(H, \mathcal{U}(H)) \le \sup\{\sigma(u) : u \in \mathcal{U}(H)\}$$

by part 1.

Suppose further that $H \hookrightarrow D$ is a divisor theory and that $G_0 = -G_0$ with G_0 as above. Let $u = p_1 \ldots p_r \in \mathcal{U}(H)$ be given with primes $p_1, \ldots, p_r \in D$.

Case 1: $r = 2$. Since p_1 is a greatest common divisor of elements from H, there is a $v \in H$ with $v = p_1 a$ for some $a \in D$ with $u \nmid v$. For the same reason there is some $w = p_2 b \in H$ with $b \in D$ and $u \nmid w$. Then $u \mid vw$, $u \nmid v$, $u \nmid w$, which implies $w_H(H, u) \ge 2 = \sigma(u)$.

Case 2: $r \ge 3$. By assumption we may choose primes $q_i \in D$ such that $v_i = p_i q_i \in H$ for $1 \le i \le r$. Because $u \in \mathcal{U}(H)$ and $r \ge 3$, we infer that $q_i \notin \{p_1, \ldots, p_r\} \setminus \{p_i\}$ for $1 \le i \le r$. Then $u \mid \prod_{i=1}^{r} v_i$ but $u \nmid \prod_{i \in I} v_i$ for any $I \subsetneq \{1, \ldots, r\}$, which implies $w_H(H, u) \ge r = \sigma(u)$. ■

PROPOSITION 3.7. *Let D be an atomic monoid and $D' \subseteq D$ a divisor closed subset. Then*

$$c(D') \le w_D(D', \mathcal{U}(D)).$$

Proof. We set $w = w_D(D', \mathcal{U}(D))$; if $w = \infty$, nothing has to be done. So suppose $w < \infty$; then for every $a \in D'$,

$$\sup L(a) \le \min L(a) \cdot w < \infty$$

by Proposition 3.5. So we may argue by induction on $\max L(a)$. Obviously, the assertion is true for all $a \in D'$ with $\max L(a) \le w$. Now let $a \in D'$, $z = \prod_{i=1}^{r} u_i \in \mathcal{Z}(a)$, and $z' = \prod_{j=1}^{s} v_j \in \mathcal{Z}(a)$ with $u_i, v_j \in \mathcal{U}(D_{\mathrm{red}})$. If $r \le w$ and $s \le w$ then $d(z, z') \le w$. So we can suppose that $r > w$. After some suitable renumbering, we infer that $v_1 \mid u_1 \ldots u_{r-1}$. Hence, there are $w_1, \ldots, w_t \in \mathcal{U}(D_{\mathrm{red}})$ with $u_1 \ldots u_{r-1} = v_1 w_1 \ldots w_t$. Since $\max L(u_1 \ldots u_{r-1}) < \max L(a)$ and $\max L(w_1 \ldots w_t u_r) < \max L(a)$, there are w-chains from

$$z = (u_1 \ldots u_{r-1}) u_r \quad \text{to} \quad z'' = (v_1 w_1 \ldots w_t) u_r$$

and from

$$z'' = v_1 (w_1 \ldots w_t u_r) \quad \text{to} \quad z' = v_1 (v_2 \ldots v_s). \ ■$$

4. Block monoids. Let G be an abelian group, $G_0 \subseteq G$ an arbitrary subset and T an atomic reduced monoid. A monoid homomorphism

$$\iota : \mathcal{F}(G_0) \times T \to G$$

is called a *content homomorphism* if for every $S = \prod_{g \in G_0} g^{v_g(S)} \in \mathcal{F}(G_0)$ we have $\iota(S) = \sum_{g \in G_0} v_g(S) g \in G_0$. Suppose ι is a content homomorphism; then

$$\mathcal{B} = \mathcal{B}(G_0, T, \iota) = \mathrm{Ker}(\iota) \subseteq \mathcal{F}(G_0) \times T$$

is called the *block monoid over* G_0 *with respect to* ι *and* T. Next,

$$\mathcal{B}(G_0) = \mathcal{B} \cap \mathcal{F}(G_0) = \Big\{ \prod_{g \in G_0} g^{n_g} \in \mathcal{F}(G_0) : \sum_{g \in G_0} n_g g = 0 \Big\}$$

is the (*ordinary*) *block monoid over* G_0.

If $\iota(T) = \{0\}$, then $\mathcal{B} = \mathcal{B}(G_0) \times T$; if $T = \{1\}$, then $\mathcal{B} = \mathcal{B}(G_0)$.

Recall that *Davenport's constant* $\mathcal{D}(G_0)$ of G_0 is defined as

$$\mathcal{D}(G_0) = \sup\{\sigma(U) : U \in \mathcal{U}(\mathcal{B}(G_0))\} \in \mathbb{N}_+ \cup \{\infty\}.$$

If G_0 is finite, then $\mathcal{D}(G_0) < \infty$ ([Ge1; Proposition 2]). If G_0 is a finite abelian group, say $G_0 \simeq \bigoplus_{i=1}^r \mathbb{Z}/n_i\mathbb{Z}$ with $n_1 \mid \ldots \mid n_r$, then $\mathcal{D}(G_0) \geq 1 + \sum_{i=1}^r (n_i - 1)$; equality holds for cyclic groups and for p-groups (cf. [G-S] for a survey).

Block monoids in the above sense were introduced in [Ge3], where they were called T-block monoids. If H is a saturated submonoid of an atomic monoid D, there exists a corresponding block monoid $\mathcal{B}$ whose arithmetic reflects the arithmetic of H. The argument runs as follows.

Let H, D be reduced atomic monoids such that $H \subseteq D$ is saturated with class group G. Let $P \subseteq \mathcal{U}(D)$ be the set of prime elements of D and $T = \{a \in D : p \nmid a \text{ for any } p \in P\}$. Then $D \simeq \mathcal{F}(P) \times T$ (cf. [Ge3; Lemma 2]) and we shall later identify these two monoids. We set $G_0 = \{g \in G : g \cap P \neq \emptyset\}$ and define a content homomorphism

$$\iota : \mathcal{F}(G_0) \times T \to G$$

by $\iota(t) = [t] \in G$ for every $t \in T$. Then $\mathcal{B} = \mathcal{B}(G_0, T, \iota)$ is the block monoid associated with $H \subseteq D$ and the relationship between H and $\mathcal{B}$ is established by the *block homomorphism*

$$\boldsymbol{\beta} : \mathcal{F}(P) \times T \to \mathcal{F}(G_0) \times T$$

which is defined by $\boldsymbol{\beta}(t) = t$ for all $t \in T$ and $\boldsymbol{\beta}(p) = [p] \in G_0$ for all $p \in P$.

Of course, the whole procedure is most powerful if D is free abelian (then $\mathcal{B} = \mathcal{B}(G_0)$) and is completely ineffective if D has no primes (then $P = \emptyset$ and $H = \mathcal{B}$).

Lemma 4.1. *Let all notations be as above and set* $G_1 = \{g \in G : g \cap \mathcal{U}(D) \neq \emptyset\}$. *Then* $G_0 \subseteq G_1$ *and we have*

1. *If* $a = \prod_{p\in P} p^{n_p} \prod_{i=1}^{s} t_i \in H$ *with* $t_1,\dots,t_s \in \mathcal{U}(T)$, *then* $A = \prod_{p\in P}[p]^{n_p} \prod_{i=1}^{s}[t_i] \in \mathcal{B}(G_1)$. *Moreover*, *if* $a\in\mathcal{U}(H)$ *then* $A\in\mathcal{U}(\mathcal{B}(G_1))$.

2. $\boldsymbol{\beta}(H) = \mathcal{B}$, $\boldsymbol{\beta}(\mathcal{U}(H)) = \mathcal{U}(\mathcal{B})$ *and* $\boldsymbol{\beta}^{-1}(\mathcal{U}(\mathcal{B})) = \mathcal{U}(H)$.

3. $\boldsymbol{\beta}$ *induces an epimorphism* $\boldsymbol{\beta} : \mathcal{Z}(H) \to \mathcal{Z}(\mathcal{B})$ *such that for every* $a \in H$, $\boldsymbol{\beta}(\mathcal{Z}(a)) = \mathcal{Z}(\boldsymbol{\beta}(a))$. *In particular*, $\sup L_D(a) = \sup L_{\mathcal{F}(G_0)\times T}(\boldsymbol{\beta}(a))$.

4. *We have*

$$\mathcal{D}(G_0) \le \sup_{U\in\mathcal{U}(\mathcal{B})} \sup L_{\mathcal{F}(G_0)\times T}(U) = \sup_{u\in\mathcal{U}(H)} \sup L_D(u) \le \mathcal{D}(G_1).$$

Proof. 1. Obvious.

2 and 3 follow from [Ge3; Proposition 4].

4. We have $\mathcal{B}(G_0) \subseteq \mathcal{B}$ and an element $B \in \mathcal{B}(G_0)$ is irreducible in $\mathcal{B}(G_0)$ if and only if it is irreducible in $\mathcal{B}$. Hence $\mathcal{U}(\mathcal{B}(G_0)) \subseteq \mathcal{U}(\mathcal{B})$ and thus

$$\mathcal{D}(G_0) \le \sup_{U\in\mathcal{U}(\mathcal{B})} \sup L_{\mathcal{F}(G_0)\times T}(U).$$

Part 3 implies that for every $u\in\mathcal{U}(H)$ we have

$$\sup L_D(u) = \sup L_{\mathcal{F}(G_0)\times T}(\boldsymbol{\beta}(u))$$

and by 1 we infer that

$$\sup_{u\in\mathcal{U}(H)} L_D(u) \le \mathcal{D}(G_1). \ \blacksquare$$

The following proposition reveals the usefulness of block monoids for our purpose.

PROPOSITION 4.2. *With all notations as above*, *suppose that* $\emptyset \ne H' \subseteq H$, $\boldsymbol{\beta}(H') = \mathcal{B}'$, $a \in H'$ *and* $\boldsymbol{\beta}(a) = A \in \mathcal{B}'$.

1. *Let* $Z, Z' \in \mathcal{Z}(A)$ *and* $z_0,\dots,z_k \in \mathcal{Z}(a)$ *with* $\boldsymbol{\beta}(z_0) = Z$ *and* $\boldsymbol{\beta}(z_k) = Z'$. *Then* $\boldsymbol{\beta}(z_0),\dots,\boldsymbol{\beta}(z_k) \in \mathcal{Z}(A)$ *and* $d(\boldsymbol{\beta}(z_{i-1}),\ \boldsymbol{\beta}(z_i)) \le d(z_{i-1}, z_i)$ *for* $1 \le i \le k$.

2. *Let* $z, z' \in \mathcal{Z}(a)$ *and* $Z_0,\dots,Z_k \in \mathcal{Z}(A)$ *with* $\boldsymbol{\beta}(z) = Z_0$ *and* $\boldsymbol{\beta}(z') = Z_k$. *Then there exists a chain* $z = z_0,\dots,z_k \in \mathcal{Z}(a)$ *with* $\boldsymbol{\beta}(z_i) = Z_i$ *and* $d(z_{i-1}, z_i) = d(Z_{i-1}, Z_i)$ *for* $1 \le i \le k$. *Furthermore*, *there is a* 2*-chain* $z_k,\dots,z_l \in \mathcal{Z}(a)$ *with* $z_l = z'$ *and* $\boldsymbol{\beta}(z_i) = \boldsymbol{\beta}(z')$ *for* $k \le i \le l$.

3. $c(\mathcal{B}') \le c(H') \le \max\{c(\mathcal{B}'), 2\}$.

Proof. 1. Since $\boldsymbol{\beta} : H \to \mathcal{B}$ is surjective and $\boldsymbol{\beta}(\mathcal{U}(H)) = \mathcal{U}(\mathcal{B})$, the assertion follows from Remark 5 after Definition 3.2.

2. It is sufficient to verify the following two assertions:

ASSERTION 1. *For every* $Z, Z' \in \mathcal{Z}(A)$ *and every* $z \in \mathcal{Z}(a)$ *with* $\boldsymbol{\beta}(z) = Z$ *there exists some* $z' \in \mathcal{Z}(a)$ *with* $\boldsymbol{\beta}(z') = Z'$ *and* $d(Z, Z') = d(z, z')$.

ASSERTION 2. *For every* $z, z' \in \mathcal{Z}(a)$ *with* $\boldsymbol{\beta}(z) = \boldsymbol{\beta}(z')$ *there is a* 2*-chain* $z = z_0,\dots,z_k = z' \in \mathcal{Z}(a)$ *from* z *to* z' *with* $\boldsymbol{\beta}(z_i) = \boldsymbol{\beta}(z)$ *for* $1 \le i \le k$.

Proof of Assertion 1. Suppose $Z = YB_1 \dots B_r$, $Z' = YC_1 \dots C_s$ with $Y \in \mathcal{Z}(\mathcal{B})$, $B_i, C_j \in \mathcal{U}(\mathcal{B})$, $\{B_1, \dots, B_r\} \cap \{C_1, \dots, C_s\} = \emptyset$, $z = yb_1 \dots b_r$ with $y \in \mathcal{Z}(H)$, $b_i \in \mathcal{U}(H)$, $\boldsymbol{\beta}(y) = Y$, $\boldsymbol{\beta}(b_i) = B_i$ and $d(Z, Z') = \max\{r, s\}$. Clearly, we may choose $c_j \in \boldsymbol{\beta}^{-1}(C_j)$ such that $\prod_{j=1}^s c_j = \prod_{i=1}^r b_i$. Then $z' = yc_1 \dots c_s \in \mathcal{Z}(a)$ and $d(z, z') = d(Z, Z')$.

Proof of Assertion 2. Let $z = \prod_{i=1}^m u_i \in \mathcal{Z}(a)$ and $z' = \prod_{j=1}^n u'_j \in \mathcal{Z}(a)$ be given with

$$u_i = \prod_{\nu=1}^{r_i} p_{i,\nu} \cdot t_i \in \mathcal{U}(H), \qquad u'_j = \prod_{\nu=1}^{r'_j} p'_{j,\nu} \cdot t'_j \in \mathcal{U}(H),$$

where $p_{i,\nu}$, $p'_{j,\nu} \in P$ and $t_i, t'_j \in T$. Since $\boldsymbol{\beta}(z) = \boldsymbol{\beta}(z') \in \mathcal{Z}(\mathcal{B})$, we infer $n = m$. After a suitable renumbering it follows that, for $1 \le i \le m$,

$$\boldsymbol{\beta}(u_i) = \boldsymbol{\beta}(u'_i)$$

and hence

$$r_i = r'_i, \quad t_i = t'_i \quad \text{and} \quad \boldsymbol{\beta}(p_{i,\nu}) = \boldsymbol{\beta}(p'_{i,\nu}).$$

Because $z, z' \in \mathcal{Z}(a)$ we obtain

$$\prod_{i=1}^m \prod_{\nu=1}^{r_i} p_{i,\nu} = \prod_{i=1}^m \prod_{\nu=1}^{r_i} p'_{i,\nu}.$$

Thus, there is some permutation

$$\varrho : Q = \{p_{i,\nu} : 1 \le \nu \le r_i,\ 1 \le i \le m\} \to Q$$

such that $\varrho(p_{i,\nu}) = p'_{i,\nu}$ for $1 \le \nu \le r_i$ and $1 \le i \le m$.

Let $\tau : P \to P$ be a permutation with $[\tau(p)] = [p] \in G$ for all $p \in P$. For $b = \prod_{p \in P} p^{n_p} \cdot t \in \mathcal{F}(P) \times T$ we set $\tau(b) = \prod_{p \in P} \tau(p)^{n_p} \cdot t$. Then $\boldsymbol{\beta}(b) = \boldsymbol{\beta}(\tau(b))$ and hence $b \in \mathcal{U}(H)$ if and only if $\tau(b) \in \mathcal{U}(H)$. Thus τ has an extension $\tau : \mathcal{U}(H) \to \mathcal{U}(H)$ and a unique extension to a monoid homomorphism $\tau : \mathcal{Z}(H) = \mathcal{F}(\mathcal{U}(H)) \to \mathcal{Z}(H)$. If τ is a transposition, then clearly $d(x, \tau(x)) \le 2$ for every $x \in \mathcal{Z}(H)$. If $P' \subseteq P$ is finite, $\tau(P') = P'$ and $b = \prod_{p \in P'} p \cdot t \in \mathcal{F}(P') \times T$, then $\tau(x) \in \mathcal{Z}(b)$ for every $x \in \mathcal{Z}(b)$.

To complete the proof of Assertion 2, we extend $\varrho : Q \to Q$ to $\varrho : P \to P$ by $\varrho(p) = p$ for all $p \in P \backslash Q$. Then $\varrho(z) = z'$. We write ϱ as a product of transpositions

$$\varrho = \varrho_k \circ \dots \circ \varrho_1$$

such that $[\varrho_j(q)] = [q]$ for all $q \in Q$ and all $j \in \{1, \dots, k\}$. If $z_0 = z$ and $z_j = \varrho_j(z_{j-1})$ for $1 \le j \le k$, then $z_k = \varrho(z) = z', d(z_j, z_{j-1}) \le 2$ and $\boldsymbol{\beta}(z_j) = \boldsymbol{\beta}(z)$ for $1 \le j \le k$.

3. The left inequality follows from 1, and the right inequality follows from 2. ∎

Let G be an abelian group and $G_0 \subseteq G$ a non-empty subset. We will write $c(G_0)$ instead of $c(\mathcal{B}(G_0))$. The rest of this section is devoted to the study of $c(G_0)$.

PROPOSITION 4.3. *Let G be an abelian group and $\emptyset \neq G_0 \subseteq G$.*

1. $c(G_0) \le w_{\mathcal{B}(G_0)}(\mathcal{B}(G_0), \mathcal{U}(\mathcal{B}(G_0))) \le \mathcal{D}(G_0)$.
2. *If* $\#G \le 2$, *then* $\mathcal{B}(G)$ *is factorial*, *whence* $c(G) = 0$.
3. *Suppose* $2 < \#G < \infty$ *and let* r *denote the maximal* p-*rank of* G. *Then*

$$\max\{r+1, \exp(G)\} \le c(G) \le w_{\mathcal{B}(G)}(\mathcal{B}(G), \mathcal{U}(\mathcal{B}(G))) = \mathcal{D}(G).$$

Proof. 1. The left inequality follows from Proposition 3.7 and the right inequality from Corollary 3.6.

2. Obvious.

3. By [HK1; §2, Beispiel 6], $\mathcal{B}(G) \hookrightarrow \mathcal{F}(G)$ is a divisor theory such that each class contains exactly one prime divisor. Hence Corollary 3.6 implies $w_{\mathcal{B}(G)}(\mathcal{B}(G), \mathcal{U}(\mathcal{B}(G))) = \mathcal{D}(G)$.

It remains to verify that $\max\{r+1, \exp(G)\} \le c(G)$. Since $\#G \ge 3$ we have $\max\{r+1, \exp(G)\} \ge 3$. Suppose $\exp(G) = n \ge 3$ and let $g \in G$ with $\mathrm{ord}(g) = n$. Then

$$A = (g^n)((-g)^n) = (-g \cdot g)^n \in \mathcal{B}(G)$$

has exactly two factorizations whose distance equals n.

Suppose $r \ge 2$ and $g_1, \ldots, g_r \in G$ are linearly independent. Setting $g_0 = -\sum_{i=1}^r g_i$ it follows that

$$A = \Big(\prod_{i=0}^r g_i\Big)\Big(\prod_{i=0}^r -g_i\Big) = \prod_{i=0}^r (-g_i \cdot g_i) \in \mathcal{B}(G)$$

has exactly two factorizations with distance $r+1$. ∎

The previous result shows in particular that $c(G) = \mathcal{D}(G)$ for cyclic groups and for elementary 2-groups G with $\#G > 2$. However, it is possible that $c(G) < \mathcal{D}(G)$.

5. Weakly Krull monoids with finitely primary components. Finitely primary monoids were introduced in [HK4] and further studied in [Ge4]. Their relevance lies in their appearance in ring theory, as will be seen in Section 7. For other examples see [Ge4].

In the sequel we use all notations concerning the complete integral closure and the conductor of monoids as introduced in Section 2. Furthermore, for $s \in \mathbb{N}_+$ let $\mathbb{N}^s$ denote the additive monoid $(\mathbb{N}^s, +)$.

DEFINITION 5.1. A monoid D is said to be *finitely primary* (*of rank* $s \in \mathbb{N}_+$) if one of the following two equivalent conditions is satisfied:

1. D is primary, $\widehat{D} \simeq \mathbb{N}^s \times \widehat{D}^\times$ and $f_{\hat{D}^\times/D} \neq \emptyset$,
2. D is a submonoid of a finitely generated factorial monoid F containing s pairwise non-associated prime elements $p_1, \ldots, p_s$ such that the following holds:
   (a) $D^\times = D \cap F^\times$,
   (b) there exists an $\alpha \in \mathbb{N}_+$ such that for every $a = \varepsilon p_1^{k_1} \ldots p_s^{k_s} \in F$ (with $\varepsilon \in F^\times$ and $k_i \in \mathbb{N}$), $\min\{k_i : 1 \leq i \leq s\} \geq \alpha$ implies that $a \in D$,
   (c) if $a = \varepsilon p_1^{k_1} \ldots p_s^{k_s} \in D \backslash D^\times$ (where $\varepsilon \in F^\times$ and $k_i \in \mathbb{N}$), then $\min\{k_i : 1 \leq i \leq s\} \geq 1$.

The equivalence of the two conditions was proved in [Ge4; Theorem 1] where it was also shown that $\widehat{D} = F$. If some $\alpha \in \mathbb{N}_+$ satisfies 2(b), then α is called an *exponent* of D. If $a = \varepsilon p_1^{k_1} \ldots p_s^{k_s} \in F$ with all notation as above, then set

$$v_{p_\nu}(a) = k_\nu \quad \text{for all } 1 \leq \nu \leq s.$$

We shall frequently use the fact that for $a \in D$,

$$\max L_D(a) \leq \min\{v_{p_\nu}(a) : 1 \leq \nu \leq s\}$$

(cf. [Ge4; Lemma 6] for the details).

PROPOSITION 5.2. *Let D be a finitely primary monoid of rank s and exponent α.*

1. *If $s \geq 2$, then $w_D(D,\mathcal{U}(D)) = \infty$.*
2. *If $s = 1$, then $c(D) \leq w_D(D,\mathcal{U}(D)) \leq 3\alpha/2$.*

Proof. 1. By [HK4; Theorem 4] we have $\varrho(D) = \infty$ and thus Proposition 3.5 implies the assertion.

2. By Proposition 3.7 it is sufficient to show that $w_D(D,\mathcal{U}(D)) \leq 3\alpha/2$. Let $p \in \widehat{D}$ be a prime element. Suppose that $\varepsilon p \in D$ for some $\varepsilon \in \widehat{D}^\times$. Since εp is prime in D, it follows that D is factorial by [Ge4; Proposition 5] and hence $w_D(D,\mathcal{U}(D)) = 1$ by Remark 4 after Definition 3.3. Now suppose $v_p(a) \geq 2$ for all $a \in D$. Let $u = \varepsilon p^l \in \mathcal{U}(H)$ be given; we show that $w_D(D,u) \leq [3\alpha/2] = \lambda$.

For this it suffices to verify that u divides any product consisting of λ factors. For $1 \leq i \leq \lambda$, let $a_i = \varepsilon_i p^{l_i} \in H$ be given with $\varepsilon_i \in \widehat{D}^\times$ and $l_i \geq 2$. Then

$$b = u^{-1} \prod_{i=1}^{\lambda} a_i = \Big(\varepsilon^{-1} \prod_{i=1}^{\lambda} \varepsilon_i\Big) p^{\sum_{i=1}^{\lambda} l_i - l},$$

and hence $b \in D$, since

$$\sum_{i=1}^{\lambda} l_i - l \geq 2\lambda - (2\alpha - 1) \geq \alpha. \ ■$$

PROPOSITION 5.3. *Let $H \subseteq D = \coprod_{i \in I} D_i$ be a saturated submonoid with finitely primary monoids D_i. Then H is a BF-monoid. If all $\widehat{D}_i^{\times}$ are finite, then H is an FF-monoid.*

Proof. Without restriction of generality we can suppose that D is reduced (cf. [G-HK; Lemma 2.6], [Ge4; Corollary 1] and [HK2]).

By [Ge4; Proposition 6] all D_i are BF-monoids and hence D is a BF-monoid. This implies that $H \subseteq D$ is a BF-monoid by [HK2; Theorem 3].

If all $\widehat{D}_i^{\times}$ are finite, then all D_i are FF-monoids by [Ge4; Proposition 6] and therefore D is an FF-monoid. Since H is a submonoid of the reduced FF-monoid D, it is an FF-monoid by [HK2; Corollary 3]. ■

THEOREM 5.4. *Let $H \subseteq D = \coprod_{i \in I} D_i$ be a saturated submonoid with bounded class group G. Suppose that all D_i are finitely primary of some fixed exponent $\alpha \in \mathbb{N}_+$ and that $\mathcal{D}(G_1) < \infty$ with $G_1 = \{g \in G : g \cap \mathcal{U}(D) \neq \emptyset\}$. Then H has finite catenary degree. More precisely, we have*

$$c(H) \leq (\alpha + \beta)\mathcal{D}(G_1)[4\beta + \alpha + (\alpha + \beta)\mathcal{D}(G_1)(2\alpha - 1)],$$

where $\beta = \alpha \exp(G)$.

Starting with the preliminaries of the proof, we introduce some notation which will remain valid throughout this section.

Let $H \subseteq D = \coprod_{i \in I} D_i$ be a saturated submonoid with bounded class group G, where all D_i are finitely primary of some exponent $\alpha \in \mathbb{N}_+$. Suppose that $i \in I$, each D_i is finitely primary of rank $s_i \in \mathbb{N}_+$, and $p_{i,1}, \ldots, p_{i,s_i}$ are pairwise non-associated primes of $\widehat{D}_i$. For any $k_1, \ldots, k_{s_i} \in \mathbb{N}_+$ we have

$$\prod_{\nu=1}^{s_i} p_{i,\nu}^{k_\nu \beta} = \Big(\prod_{\nu=1}^{s_i} p_{i,\nu}^{k_\nu \alpha}\Big)^{\exp(G)} \in H.$$

If $a \in D$, then a has a unique decomposition of the form

$$a = \prod_{j \in I} a_j$$

with all $a_j \in D_j$ and $a_j = 1$ for all but finitely many $j \in I$. For all $i \in I$ and all $1 \leq \nu \leq s_i$ we set

$$v_{p_{i,\nu}}(a) = v_{p_{i,\nu}}(a_i).$$

LEMMA 5.5. *For every $i \in I$ let*

$$D_i' = \{c_i \in D_i : \min\{v_{p_{i,\nu}}(c_i) : 1 \leq \nu \leq s_i\} \leq \alpha + \beta\},$$

$D' = \coprod_{i\in I} D'_i$ and $H' = H \cap D'$. Then H' is divisor closed and $c(H') \le \mathcal{D}(G_1)(\alpha+\beta)$.

Proof. By Corollary 3.6, H' is divisor closed and we have

$$\begin{aligned} c(H') &\le w_H(H',\mathcal{U}(H)) && \text{by Proposition 3.7}\\ &\le \sup_{u\in\mathcal{U}(H)} \sup L_D(u)\cdot w_D(D',\mathcal{U}(D)) && \text{by Corollary 3.6}\\ &\le \mathcal{D}(G_1)\cdot w_D(D',\mathcal{U}(D)) && \text{by Lemma 4.1}\\ &\le \mathcal{D}(G_1)\sup_{i\in I} w_{D_i}(D'_i,\mathcal{U}(D_i)) && \text{by Proposition 3.4.} \end{aligned}$$

Let $i \in I$; since

$$D'_i \subseteq D''_i = \{c_i \in D_i : \sup L_{D_i}(c_i) \le \alpha+\beta\}$$

and $w_{D_i}(D''_i,\mathcal{U}(D_i)) \le \alpha+\beta$ (cf. Remark 3 after Definition 3.3), the assertion follows. ∎

LEMMA 5.6. *Let $J \subseteq I$ be finite and $a \in H \cap \coprod_{i\in J} D_i$. Then there exists a factorization $z \in \mathcal{Z}_H(a)$ such that*

$$\sigma(z) < (4\beta+\alpha)\#J + \sum_{\substack{i\in J\\ s_i=1}} v_{p_{i,1}}(a).$$

Proof. Let $a = \prod_{i\in J} a_i$ with $a_i \in D_i$. We set

$$J_1 = \{i \in J : v_{p_{i,\nu}}(a_i) \ge 2\beta+\alpha \text{ for all } 1 \le \nu \le s_i\}.$$

If $i \in J_1$, then

$$a_i = \varepsilon_i p_{i,1}^{k_1}\dots p_{i,s_i}^{k_{s_i}}$$

with $\varepsilon_i \in \widehat{D}_i^{\times}$ and $k_j = 2\beta l_j + \alpha + r_j$ with $l_j \in \mathbb{N}_+$ and $0 \le r_j < 2\beta$. If

$$b_i = (p_{i,1}^{2l_1}\dots p_{i,s_i}^{2l_{s_i}})^{\beta} \quad \text{and} \quad c_i = (\varepsilon p_{i,1}^{\alpha+r_1}\dots p_{i,s_i}^{\alpha+r_{s_i}}),$$

then $b_i \in H$, $c_i \in D_i$ and $a_i = b_i c_i$. For $i \in J\setminus J_1$ we set $c_i = a_i$, $b = \prod_{i\in J_1} b_i$, and $c = \prod_{i\in J} c_i$. Then $a = bc$ with $b \in H$, $c \in D$, and hence $c \in H$ since $H \subseteq D$ is saturated.

By construction we have, for all $i \in J$,

$$\min\{v_{p_{i,\nu}}(c_i) : 1 \le \nu \le s_i\} < 2\beta+\alpha$$

and hence

$$\max L_H(c) \le \max L_D(c) = \sum_{i\in J} \max L_{D_i}(c_i) < (2\beta+\alpha)\#J.$$

Next we consider the elements b_i. Let $i \in J_1$. If $s_i = 1$, then

$$\max L_H(b_i) \le v_{p_{i,1}}(b_i) \le v_{p_{i,1}}(a).$$

If $s_i \geq 2$, then set

$$b_{i,1} = (p_{i,1} p_{i,2}^{2l_2 - 1} \dots p_{i,s_i}^{2l_{s_i} - 1})^\beta \quad \text{and} \quad b_{i,2} = (p_{i,1}^{2l_1 - 1} p_{i,2} \dots p_{i,s_i})^\beta.$$

We infer that $b_{i,1}, b_{i,2} \in H$, $b_i = b_{i,1} b_{i,2}$ and

$$\min L_H(b_i) \leq \max L_H(b_{i,1}) + \max L_H(b_{i,2}) \leq 2\beta.$$

By combining these results the assertion follows. ∎

Proof of Theorem 5.4. Let $i \in I$. Since $(p_{i,1} \dots p_{i,s_i})^\beta \in H$ there exists an element $u_i \in \mathcal{U}(H) \cap D_i$ with

$$v_{p_{i,\nu}}(u_i) \leq \beta \quad \text{for } 1 \leq \nu \leq s_i.$$

From Lemma 4.1 we infer that $\sup L_D(u_i) \leq \mathcal{D}(G_1)$.

If $a \in H \backslash H^\times$ is given, then there exists a finite subset $J \subseteq I$ such that

$$a = \prod_{i \in J} a_i \in \coprod_{i \in J} D_i$$

with $1 \neq a_i \in D_i$. For every $i \in J$ we write a_i in the form

$$a_i = u_i^{\kappa_i} b_i$$

with $\kappa_i \in \mathbb{N}$ maximal such that $b_i \in D_i$. Then

$$\min \left\{ v_{p_{i,\nu}}(b_i) : 1 \leq \nu \leq s_i \right\} < \alpha + \max \left\{ v_{p_{i,\nu}}(u_i) : 1 \leq \nu \leq s_i \right\} \leq \alpha + \beta.$$

Hence we obtain

$$a = \prod_{i \in J} u_i^{\kappa_i} \cdot b$$

with $b = \prod_{i \in J} b_i$. Since $a \in H$, $\prod_{i \in J} u_i^{\kappa_i} \in H$ and $b \in D$, it follows that $b \in H$ because $H \subseteq D$ is saturated.

Define $Z \subseteq \mathcal{Z}(a)$ as

$$Z = \left\{ \prod_{i \in J} u_i^{\kappa_i} \cdot y : y \in \mathcal{Z}(b) \right\} \subseteq \mathcal{Z}(a).$$

For any two factorizations $z = \prod_{i \in J} u_i^{\kappa_i} \cdot y \in Z$ and $z' = \prod_{i \in J} u_i^{\kappa_i} \cdot y' \in Z$, Lemma 5.5 guarantees the existence of a $(\mathcal{D}(G_1)(\alpha + \beta))$-chain of factorizations from z to z'.

Hence it remains to verify that for every $z \in \mathcal{Z}(a)$ there exists an $(\alpha + \beta)\mathcal{D}(G_1)\big[4\beta + \alpha + (\alpha + \beta)\mathcal{D}(G_1)(2\alpha - 1)\big]$-chain of factorizations from z to some $z' \in Z$. Let

$$z = \prod_{i \in J} u_i^{\varrho_i} \prod_{k=1}^{\lambda} v_k \in \mathcal{Z}(a)$$

be given with $\varrho_i \in \mathbb{N}$ and $v_k \in \mathcal{U}(H)$. The maximality of the κ_i's implies that $\varrho_i \leq \kappa_i$. We set $\varrho = \sum_{i \in J} \varrho_i$ and complete the proof by induction on ϱ from $\varrho = \sum_{i \in J} \kappa_i$ to $\varrho = 0$.

If $\varrho = \sum_{i \in J} \kappa_i$, then $z \in Z$ and we are done. So suppose $\varrho < \sum_{i \in J} \kappa_i$. Set

$$\prod_{k=1}^{\lambda} v_k = c = \prod_{i \in J} c_i \in H$$

with $c_i \in D_i$. We distinguish two cases:

Case 1: For all $i \in J$ we have $\min\{v_{p_{i,\nu}}(c_i) : 1 \le \nu \le s_i\} \le \alpha + \beta$. If $y \in \mathcal{Z}(b)$, then $\prod_{i \in J} u_i^{\kappa_i - \varrho_i} \cdot y \in \mathcal{Z}(c)$ and by Lemma 5.5 there is a $\mathcal{D}(G_1)(\alpha + \beta)$-chain of factorizations from

$$z = \prod_{i \in J} u_i^{\varrho_i} v_1 \ldots v_\lambda \quad \text{to} \quad z' = \prod_{i \in J} u_i^{\varrho_i} \Big(\prod_{i \in J} u_i^{\kappa_i - \varrho_i} \cdot y \Big) \in Z.$$

Case 2: There exists some $i \in J$ such that $v_{p_{i,\nu}}(c_i) \ge \alpha + \beta$ for all $1 \le \nu \le s_i$. Each v_k, $1 \le k \le \lambda$, is a product of irreducibles in D. Taking these irreducibles, we obtain a factorization (in D) of each c_j, say

$$c_j = c_{j,1} \ldots c_{j,\lambda_j}$$

for all $j \in J$. There exists a $\mu' \le \min\{\lambda_i, \alpha + \beta\} \le \alpha + \beta$ such that

$$v_{p_{i,\nu}} \Big(\prod_{l=1}^{\mu'} c_{i,l} \Big) \ge \alpha + \beta \quad \text{for all } 1 \le \nu \le s_i.$$

Now each $c_{i,l}$ comes from a factorization of some v_k. Thus (after some renumbering), there is a $\mu \le \mu'$ such that for

$$d = \prod_{k=1}^{\mu} v_k$$

we have

$$v_{p_{i,\nu}}(d) \ge \alpha + \beta \quad \text{for all } 1 \le \nu \le s_i.$$

We write d in the form

$$d = \prod_{j \in J'} d_j$$

with $1 \ne d_j \in D_j$. Since each v_k is a product of at most $\mathcal{D}(G_1)$ irreducibles in D, it follows that $\# J' \le \mu \mathcal{D}(G_1) \le (\alpha + \beta)\mathcal{D}(G_1)$.

Next we set

$$d = u_i e.$$

Since $v_{p_{i,\nu}}(e) = v_{p_{i,\nu}}(d) - v_{p_{i,\nu}}(u_i) \ge \alpha$ for all $1 \le \nu \le s_i$, we infer that $e \in \prod_{j \in J'} D_j$ and hence $e \in H$. By Lemma 5.6 there is a factorization $y \in \mathcal{Z}(e)$ with

$$\begin{aligned}
\sigma(y) &< (4\beta+\alpha)\#J' + \sum_{\substack{j\in J'\\ s_j=1}} v_{p_{j,1}}(e)\\
&\le (4\beta+\alpha)(\alpha+\beta)\mathcal{D}(G_1) + \sum_{\substack{j\in J'\\ s_j=1}} \sum_{k=1}^{\mu} v_{p_{j,1}}(v_k)\\
&\le (4\beta+\alpha)(\alpha+\beta)\mathcal{D}(G_1) + \#J'\cdot\mu(2\alpha-1)\mathcal{D}(G_1)\\
&\le (4\beta+\alpha)(\alpha+\beta)\mathcal{D}(G_1) + (\alpha+\beta)^2\mathcal{D}(G_1)^2(2\alpha-1).
\end{aligned}$$

Finally, if

$$z' = \prod_{i\in J} u_i^{\varrho_i} \prod_{k=\mu+1}^{\lambda} v_k\cdot u_i\cdot y,$$

then

$$\begin{aligned}
d(z,z') &= d\Big(\prod_{k=1}^{\mu} v_k, u_i y\Big) \le \max\{\mu, 1+\sigma(y)\}\\
&\le (\alpha+\beta)\mathcal{D}(G_1)[4\beta+\alpha+(\alpha+\beta)\mathcal{D}(G_1)(2\alpha-1)].
\end{aligned}$$

Now the assertion follows by induction hypothesis. ■

6. Monoids having infinite catenary degree. Let $H \subseteq D = \coprod_{i\in I} D_i$ be a saturated submonoid with class group G. Theorem 5.4 states that if G is finite and all D_i are finitely primary of some fixed exponent $\alpha\in\mathbb{N}_+$, then $c(H)<\infty$. Our first aim in this section is to show that $c(H)=\infty$ may happen if one of these two conditions fails (cf. Corollaries 6.2 and 6.4). We do even more. We prove that $\varrho(H)=\infty$ and that the set $\Delta(H)$ (defined below) is infinite.

For a subset $L\subseteq\mathbb{N}$ we set

$$\Delta(L)=\{l-k : k<l,\ k,l\in L \text{ and there is no } m\in L \text{ with } k<m<l\}\subseteq\mathbb{N}_+.$$

If $\#L\le 1$, then $\Delta(L)=\emptyset$. For an atomic monoid H we define

$$\Delta(H)=\bigcup_{a\in H}\Delta(L(a)).$$

For the relevance of $\Delta(H)$ cf. [Ge1] and [Ge2].

Proposition 6.1. *Let H, D be reduced atomic monoids, $H\subseteq D$ saturated and $U^*\subseteq\mathcal{U}(D)$ a subset having the following two properties*:

(a) *If $u_1,\dots,u_n\in U^*$ are pairwise distinct*, *then the product $u_1\dots u_n \in D$ has unique factorization in D (i.e., $\#\mathcal{Z}_D(u_1\dots u_n)=1$)*,

(b) *$G=\{[u]\in D/H : u\in U^*\}$ is an infinite group.*

Then H has infinite catenary degree, *infinite elasticity and $\Delta(H)$ is infinite.*

Proof. Let $g_1, \ldots, g_n \in G$ be pairwise distinct and $u_1, \ldots, u_n \in U^*$ with $[u_i] = g_i$ for $1 \le i \le n$, such that $a = \prod_{i=1}^n u_i \in H$ and $A = \prod_{i=1}^n g_i \in \mathcal{B}(G)$. By assumption, a has unique factorization in D. If $\emptyset \ne I \subseteq \{1, \ldots, n\}$ is a subset, then

$$\prod_{i \in I} u_i \in \mathcal{U}(H) \quad \text{and} \quad \prod_{i \in I} u_i \mid a \quad \text{in } H$$

if and only if

$$\prod_{i \in I} g_i \in \mathcal{U}(\mathcal{B}(G)) \quad \text{and} \quad \prod_{i \in I} g_i \mid A \quad \text{in } \mathcal{B}(G).$$

Hence, there is a bijection $\psi : \mathcal{Z}_H(a) \to \mathcal{Z}_{\mathcal{B}(G)}(A)$ with $\sigma(z) = \sigma(\psi(z))$ and $d(z, z') = d(\psi(z), \psi(z'))$ for all $z, z' \in \mathcal{Z}_H(a)$. In particular, $c_H(a) = c_{\mathcal{B}(G)}(A)$ and $L_H(a) = L_{\mathcal{B}(G)}(A)$.

Therefore it suffices to verify that there exists an element $A \in \mathcal{B}(G)$ which is squarefree in $\mathcal{F}(G)$, and with $c_{\mathcal{B}(G)}(A)$, $\varrho_{\mathcal{B}(G)}(A)$ and $\sup \Delta(L(A))$ arbitrarily large.

Let $N \ge 3$ be given. We consider two cases:

Case 1: G contains some element g of infinite order. We choose natural numbers $m_1, \ldots, m_{N-1} \in \mathbb{N}_+$ such that $m_j > \sum_{i=1}^{j-1} m_i$ for $1 < j \le N-1$ and we set $m_0 = -\sum_{i=1}^{N-1} m_i$. Then

$$A = \Big(\prod_{i=0}^{N-1} m_i g \Big) \Big(\prod_{i=0}^{N-1} -m_i g \Big) = \prod_{i=0}^{N-1} ((m_i g) \cdot (-m_i g)) \in \mathcal{B}(G)$$

has just the above two factorizations. Hence we infer that

$$c(A) = N, \quad \varrho(A) = \frac{N}{2} \quad \text{and} \quad \Delta(L(A)) = \{N-2\}.$$

Case 2: G contains $2N$ non-zero linearly independent elements $g_1, \ldots$ $\ldots, g_{2N}$. We set $g_0 = -\sum_{i=1}^{2N} g_i$ and define

$$\begin{aligned} A &= \Big(\prod_{i=0}^{2N} g_i \Big) \Big((-g_0 - g_{2N-1} - g_{2N}) \cdot \prod_{i=1}^{N-1} (-g_{2i-1} - g_{2i}) \Big) \\ &= \prod_{i=1}^{N-1} \big((-g_{2i-1} - g_i) \cdot g_{2i-1} \cdot g_{2i} \big) \\ &\quad \times \big((-g_0 - g_{2N-1} - g_{2N}) \cdot g_0 \cdot g_{2N-1} \cdot g_{2N} \big). \end{aligned}$$

Obviously, $A \in \mathcal{B}(G)$ and A is squarefree in $\mathcal{F}(G)$. The second factorization is the only one in which g_0 and $-g_0 - g_{2N-1} - g_{2N}$ are in the same irreducible block. If g_0 and $-g_0 - g_{2N-1} - g_{2N}$ are in distinct irreducible blocks appearing in a factorization z of A, then $\sigma(z) = 2$. Therefore,

$$c(A) = N, \quad \varrho(A) = \frac{N}{2} \quad \text{and} \quad \Delta(L(A)) = \{N-2\}. \ \blacksquare$$

COROLLARY 6.2. *Let H and $D = \coprod_{i\in I} D_i$ be reduced atomic monoids such that $H \subseteq D$ is saturated with infinite class group G.*

1. *Suppose there is a subset $U^* \subseteq \mathcal{U}(D)$ such that $g \cap U^* \neq \emptyset$ for every $g \in G$ and with $\#(U^* \cap D_i) \leq 1$ for all $i \in I$. Then $\varrho(H) = c(H) = \infty$ and $\Delta(H)$ is infinite.*

2. *Suppose that H is a Krull monoid, $H \hookrightarrow D$ a divisor theory, and that $g \cap \mathcal{U}(D) \neq \emptyset$ for every $g \in G$. Then $\varrho(H) = c(H) = \infty$ and $\Delta(H)$ is infinite.*

Proof. 1. Obviously, U^* has properties (a) and (b) of the previous proposition.

2 is a special case of 1. $\blacksquare$

PROPOSITION 6.3. *For every $s \in \mathbb{N}_+$ and every $\alpha \geq 2$ there exists a finitely primary monoid D of rank s and exponent α^2 with $c(D) \geq \alpha$.*

Proof. Let $s \in \mathbb{N}_+$ and $\alpha \geq 2$ be given. We set $u = (\alpha, \ldots, \alpha) \in \mathbb{N}^s$, $v = (\alpha+1, \ldots, \alpha+1) \in \mathbb{N}^s$ and define

$$\begin{aligned} D_\alpha = D = \{&(n_1, \ldots, n_s) \in \mathbb{N}^s : n_i \geq \alpha^2 \text{ for } 1 \leq i \leq s\} \\ &\cup \{ku + lv \in \mathbb{N}^s : 0 \leq k, l \leq \alpha\} \subseteq (\mathbb{N}^s, +). \end{aligned}$$

Then D is a finitely primary monoid of rank s and exponent α^2 (cf. Condition 2 in Definition 5.1). Next we consider factorizations of $a = \big(\alpha(\alpha+1), \ldots, \alpha(\alpha+1)\big) \in D$. Clearly, $u, v \in \mathcal{U}(D)$ and

$$a = \alpha v = (\alpha+1)u.$$

Let $z = \sum_{i=1}^r w_i \in \mathcal{Z}(a)$ be an arbitrary factorization with $w_i \in \mathcal{U}(D)$ and suppose $z \neq \alpha v$. Then $v \neq w_i$ for all $1 \leq i \leq r$, which implies that $d(\alpha v, z) \geq \alpha$. Therefore we have

$$c(D) \geq c(a) \geq \alpha. \ \blacksquare$$

COROLLARY 6.4. *There exists a weakly Krull monoid H with trivial class group but infinite catenary degree.*

Proof. Obviously, $H = \coprod_{\alpha\geq 2} D_\alpha$ is weakly Krull with trivial class group and

$$c(H) = \sup_{\alpha\geq 2} c(D_\alpha) = \infty. \ \blacksquare$$

Let H, D be atomic monoids and $\varphi : H \to D$ a divisor homomorphism with finite class group. In [G-L; Theorem 2] it was proved that $c(D) < \infty$

and $\varrho(D) < \infty$ imply that $c(H) < \infty$. (In fact, the terminology in [G-L] is different; cf. [HK4; §1] for the relationship between $\varrho(D)$ and $\mu_m(D)$.)

Now if $D = \coprod_{i\in I} D_i$, then $\varrho(D) = \sup\{\varrho(D_i) : i \in I\}$ (by [HK4; Proposition 4]) and if D_i is finitely primary of rank $s_i \geq 2$ then $\varrho(D_i) = \infty$ (by [HK4; Theorem 4]).

Hence Theorem 5.4 is not a consequence of [G-L; Theorem 2]. Even more, let $\varphi : H \to D$ be as above and suppose $c(D) < \infty$ and $\varrho(D) = \infty$. Then $c(H) = \infty$ might occur, as can be seen from the next result.

PROPOSITION 6.5. 1. *For every $n \in \mathbb{N}_+$ there exists a finitely generated monoid D_n and a saturated submonoid $H_n \subseteq D_n$ with $D_n/H_n \simeq \mathbb{Z}/2\mathbb{Z}$, $c(D_n) = 3$ and $c(H_n) \geq n + 1$.*

2. *There exists an atomic monoid D having a saturated submonoid $H \subseteq D$ with $D/H \simeq \mathbb{Z}/2\mathbb{Z}$ such that $c(D) = 3$ but $c(H) = \infty$.*

Proof. 1. Let $n \in \mathbb{N}_+$ and $U_n = \{u_0, u_0', u_1, \ldots, u_n, v_1, v_1', \ldots, v_n, v_n'\}$ an arbitrary set. We define D_n as the free abelian monoid with basis U_n modulo the following relations:

$$u_0u_0' = v_1v_1'u_1 \quad \text{and} \quad v_iv_i' = v_{i+1}v_{i+1}'u_{i+1} \quad \text{for } 1 \leq i < n.$$

Then D_n is finitely generated with $\mathcal{U}(D_n) = U_n$ and $c(D_n) = 3$. The element $a_n = u_0u_0' \in D_n$ has exactly $n+1$ factorizations in D_n, which are the following:

$$a_n = u_0u_0' = v_iv_i' \prod_{j=1}^{i} u_j \quad \text{for } 1 \leq i \leq n.$$

We define a monoid epimorphism

$$\varphi_n : D_n \to \mathbb{Z}/2\mathbb{Z}$$

by $\varphi_n(u_0') = \varphi_n(u_i) = 0 + 2\mathbb{Z}$ for $0 \leq i \leq n$ and $\varphi_n(v_j) = \varphi_n(v_j') = 1 + 2\mathbb{Z}$ for $1 \leq j \leq n$. Then $H_n = \mathrm{Ker}(\varphi_n) \subseteq D_n$ is saturated and $\mathrm{Cl}(\varphi_n) \simeq \mathbb{Z}/2\mathbb{Z}$. For every $1 \leq i < n$ the elements $v_iv_i' \in H_n$ are not irreducible in H_n and hence

$$a_n = u_0u_0' = (v_nv_n')u_n \ldots u_1$$

are the only two factorizations of a in H_n. Thus we infer that

$$c(H_n) \geq c(a_n) = n + 1.$$

2. We set $D = \coprod_{n\in\mathbb{N}_+} D_n$ and define a monoid epimorphism $\varphi : D \to \mathbb{Z}/2\mathbb{Z}$ by $\varphi|D_n = \varphi_n$ for all $n \in \mathbb{N}_+$. Then $H = \mathrm{Ker}(\varphi) \subseteq D$ is saturated with class group $\mathrm{Cl}(\varphi) \simeq \mathbb{Z}/2\mathbb{Z}$. Since $H_n \subseteq H$ is a divisor closed submonoid, we obtain

$$c_H(a_n) = c_{H_n}(a_n) = n + 1$$

for every $n \in \mathbb{N}_+$, and thus $c(H) = \infty$. ■

7. Weakly Krull domains. In this section we discuss weakly Krull domains with some additional properties. Our main result will be formulated in Theorem 7.3.

Let R be an integral domain with quotient field K. Then $R^\bullet = R\setminus\{0\}$ denotes its multiplicative monoid, $R^\times = R^{\bullet\times}$ its group of units and $R^\# = R^\bullet/R^\times$ the reduced multiplicative monoid. For a prime $\mathfrak{p} \in \operatorname{spec}(R)$, $h(\mathfrak{p})$ means the height of $\mathfrak{p}$ and $d(\mathfrak{p})$ the depth of $\mathfrak{p}$. We set $X^{(1)}(R) = \{\mathfrak{p} \in \operatorname{spec}(R) : h(\mathfrak{p}) = 1\}$. As usual, $\overline{R}$ denotes the integral closure of R in its quotient field K and $\widehat{R} = \widehat{R}^\bullet \cup \{0\}$ its complete integral closure. Clearly, $R \subseteq \overline{R} \subseteq \widehat{R} \subseteq K$ and if R is noetherian, then $\overline{R} = \widehat{R}$. If $\mathcal{I}(R)$ denotes the multiplicative monoid of integral invertible ideals of R and $\mathcal{H}(R) \subseteq \mathcal{I}(R)$ the submonoid of principal ideals, then $\operatorname{Pic}(R) = \mathcal{I}(R)/\mathcal{H}(R)$ is the Picard group of R. We denote by $\mathcal{I}_t(R)$ the monoid of integral t-invertible t-ideals (equipped with t-multiplication) and by $\operatorname{Cl}_t(R) = \mathcal{I}_t(R)/\mathcal{H}(R)$ the t-class group (cf. [An]). Clearly, $R^\# \simeq \mathcal{H}(R)$, $\mathcal{I}(R) \subseteq \mathcal{I}_t(R)$ and $\operatorname{Pic}(R) \subseteq \operatorname{Cl}_t(R)$. If $R = \widehat{R}$, then $\operatorname{Cl}(R)$ means the the divisor class group of R. We say that R is a *local domain* if it has just one maximal ideal.

If S/R is a ring extension, then the annihilator of the factor module S/R,

$$\operatorname{Ann}_R(S/R) = f_{S^\bullet/R^\bullet} \cup \{0\} = f_{S/R},$$

is just the conductor of the ring extension.

Suppose that $(V_i)_{i\in I}$ is a family of overrings of R (i.e., $R \subseteq V_i \subseteq K$ for all $i \in I$) such that $R = \bigcap_{i\in I} V_i$. We say the intersection is *of finite character* if for all $0 \neq x \in K$ (equivalently, all $x \in R^\bullet$) we have $x \in V_i^\times$ for all but finitely many $i \in I$.

DEFINITION 7.1. An integral domain R is said to be a *weakly Krull domain* if

$$R = \bigcap_{P\in X^{(1)}(R)} R_P$$

and the intersection is of finite character. If in addition all R_P are discrete valuation rings, then R is called a *Krull domain*.

In [A-M-Z; Theorem 1] various ideal-theoretic characterizations of weakly Krull domains are given. F. Halter-Koch showed that the notion of a weakly Krull domain is a purely multiplicative one.

LEMMA 7.2. *Let R be an integral domain. Then R is a weakly Krull domain if and only if $R^\bullet$ is a weakly Krull monoid. Furthermore, let $\varphi : R^\bullet \to D$ be a weak divisor theory. Then $D \simeq \mathcal{I}_t(R)$, $(R_P^\#)_{P\in X^{(1)}(R)}$ are (up to isomorphism) just the primary components of D and* $\operatorname{Cl}(R^\bullet) = \operatorname{Cl}_t(R)$. *If R is Krull, then* $\operatorname{Cl}_t(R) = \operatorname{Cl}(R)$.

Proof. See [HK3; Theorem 4.6]. ■

The above lemma allows us to apply our main result from monoid theory to weakly Krull domains.

THEOREM 7.3. *Let R be a weakly Krull domain such that for some $\alpha \in \mathbb{N}_+$ all $R_P^{\#}$, $P \in X^{(1)}(R)$, are finitely primary of exponent α.*

1. *The monoid $\mathcal{I}_t(R)$ of integral t-invertible t-ideals has finite catenary degree.*
2. *If $\mathrm{Cl}_t(R)$ is finite, then $R^\bullet$ has finite catenary degree.*

Proof. By Lemma 7.2 both assertions follow from Theorem 5.4. Note that $\mathcal{I}_t(R)$ is a weakly Krull monoid with trivial class group. ∎

Clearly, Krull domains (hence, in particular, integrally closed noetherian domains) with finite divisor class group satisfy the assumptions of the above theorem. The following elementary ring-theoretic lemmata provide further examples of (not necessarily integrally closed) noetherian domains satisfying the assumptions of Theorem 7.3. Indeed, these "examples" were our motivation for investigating weakly Krull monoids.

LEMMA 7.4. *Let R be an integral domain. Then the following conditions are equivalent*:

1. *$R^\bullet$ is finitely primary.*
2. *R is a one-dimensional local domain, $\widehat{R}$ is a semilocal principal ideal domain and $\mathfrak{f}_{\hat{R}/R} \neq (0)$.*

Proof. See [Ge4; Theorem 2]. ∎

LEMMA 7.5. *Let R be a noetherian domain.*

1. *For every $P \in X^{(1)}(R)$ the following conditions are equivalent*:
   (a) *$\overline{R_P}$ is a finitely generated R_P-module.*
   (b) *$R_P^{\#}$ is a finitely primary monoid.*
2. *If $\overline{R}$ is a finitely generated R-module, then all $R_P^{\#}$, $P \in X^{(1)}(R)$, are finitely primary of some fixed exponent $\alpha \in \mathbb{N}_+$.*

Proof. 1. Let $P \in X^{(1)}(R)$ be given.

(a)⇒(b). We check the conditions of Lemma 7.4. Clearly, R_P is a one-dimensional local noetherian domain; hence $\widehat{R_P} = \overline{R_P}$. By the Krull–Akizuki Theorem [Ma; Corollary to Theorem 11.7], $\overline{R_P}$ is a semilocal Dedekind domain and thus a principal ideal domain. Finally, (a) implies that $\mathfrak{f}_{\overline{R_P}/R_P} \neq (0)$.

(b)⇒(a). Since $R_P^{\#}$ is finitely primary, $\mathfrak{f}_{\widehat{R_P}/R_P} = \mathfrak{f}_{\overline{R_P}/R_P} \neq (0)$ by Lemma 7.4 and hence $\overline{R_P}$ is a finitely generated R_P-module (cf. [Ge4; Lemma 4] for the detailed argument).

2. Suppose that $\overline{R}$ is a finitely generated R-module and let $P \in X^{(1)}(R)$ be given. Then $\overline{R_P}$ is a finitely generated R_P-module and hence $R_P^{\#}$ is finitely primary by part 1. Furthermore,

$$\begin{aligned}\mathfrak{f}_{\overline{R_P}/R_P} &= \mathrm{Ann}_{R_P}(\overline{R_P}/R_P)\\ &= \mathrm{Ann}_{(R\backslash P)^{-1}\cdot R}\big((R\backslash P)^{-1}\cdot \overline{R}/R\big) = (R\backslash P)^{-1}\cdot \mathrm{Ann}_R(\overline{R}/R) \neq (0).\end{aligned}$$

Therefore we have $P \not\supset \mathfrak{f}_{\overline{R}/R}$ if and only if $\mathfrak{f}_{\overline{R}/R} \cap (R\backslash P) \neq \emptyset$ if and only if $(1) = \mathfrak{f}_{\overline{R_P}/R_P}$ if and only if $R_P = \overline{R_P}$. So, if $P \not\supset \mathfrak{f}_{\overline{R}/R}$, then $R_P^{\#}$ is finitely primary of exponent 1, and since there are only finitely many $P \in X^{(1)}(R)$ containing $\mathfrak{f}_{\overline{R}/R}$ the assertion follows. ∎

The next lemma shows that Cohen–Macaulay domains (cf. [B-H]) are weakly Krull.

LEMMA 7.6. *Let R be a noetherian domain, $\mathcal{P} \subseteq \mathrm{spec}(R)$ a set of non-zero prime ideals and $\mathcal{Q} = \bigcup_{a\in R^\bullet} \mathrm{Ass}(R/aR)$.*

1. *The following conditions are equivalent*:

(a) $R = \bigcap_{P\in\mathcal{P}} R_P$.

(b) *For every $Q \in \mathcal{Q}$ there is some $P \in \mathcal{P}$ with $Q \subseteq P$.*

2. *R is a weakly Krull domain if and only if every prime ideal of depth one has height one.*

Proof. 1. (a)⇒(b). Let $Q \in \mathcal{Q}$ be given; then $Q = \mathrm{Ann}_R(a + bR)$ for some $a, b \in R$ with $a \notin bR$. Since $a/b \notin R$, there exists some $P \in \mathcal{P}$ such that $a/b \notin R_P$, which implies $\mathrm{Ann}_R(a + bR) \subseteq P$.

(b)⇒(a). By [B-I-V; Lemma 7.15] we have

$$R = \bigcap_{Q\in\mathcal{Q}} R_Q$$

and thus

$$R \subseteq \bigcap_{P\in\mathcal{P}} R_P \subseteq \bigcap_{Q\in\mathcal{Q}} R_Q = R.$$

2. Suppose that R is weakly Krull and take some $Q' \in \mathrm{spec}(R)$ with $d(Q') = 1$. Then $Q' \subseteq Q$ for some $Q \in \mathcal{Q}$ (cf. [Ka; p. 67]) and by part 1 there is some $P \in X^{(1)}(R)$ with $Q \subseteq P$. Thus it follows that

$$1 \leq h(Q') \leq h(Q) \leq h(P) = 1.$$

Conversely, suppose that every prime ideal of depth one has height one. Then $\mathcal{Q} \subseteq X^{(1)}(R)$ and thus

$$R \subseteq \bigcap_{P\in X^{(1)}(R)} R_P \subseteq \bigcap_{Q\in\mathcal{Q}} R_Q = R.$$

Since R is noetherian, every $a \in R^\bullet$ is contained in only finitely many $P \in X^{(1)}(R)$ and therefore the intersection

$$R = \bigcap_{P \in X^{(1)}(R)} R_P$$

is of finite character. ■

Next we consider Z-rings. By definition, an integral domain R is said to be a *Z-ring* if it is noetherian and if every invertible ideal is a product of primary ideals. One-dimensional noetherian domains are Z-rings. If R is a Z-ring and $P \in X^{(1)}(R)$ let

$$\Omega(P) = \{Q \in \mathcal{U}(\mathcal{I}(R)) : \sqrt{Q} = P\}$$

denote the set of multiplicatively irreducible P-primary invertible ideals.

LEMMA 7.7. *Let R be a Z-ring and $P \in X^{(1)}(R)$.*

1. *R is a weakly Krull domain, $\mathcal{I}_t(R) = \mathcal{I}(R)$ and $\mathrm{Cl}_t(R) = \mathrm{Pic}(R)$.*
2. *$\Omega(P) = \{P\}$ if and only if R_P is a discrete valuation domain.*
3. *$\#\Omega(P) < \infty$ if and only if $R_P^\#$ is a finitely generated monoid if and only if $R_P^\#$ is finitely primary of rank 1.*
4. *Suppose that $\overline{R}$ is a finitely generated R-module. Then $\mathcal{I}(R)$ has finite catenary degree. If $\mathrm{Pic}(R)$ is finite, then $R^\bullet$ has finite catenary degree.*

Proof. Assertion 1 follows from [A-M-Z; Theorem 3.3] and 2 from [HK5; Lemma 2].

3. $\Omega(P)$ is a generating system of the submonoid $\mathcal{I}_P(R) \subseteq \mathcal{I}(R)$ consisting of P-primary invertible ideals. Since $\mathcal{I}_P(R)$ is isomorphic to $\mathcal{I}(R_P)$ and since invertible ideals in a local ring are principal, we infer that

$$\mathcal{I}_P(R) \simeq \mathcal{I}(R_P) = \mathcal{H}(R_P) \simeq R_P^\bullet / R_P^\times = R_P^\#.$$

Hence $\Omega(P)$ is finite if and only if $R_P^\#$ is finitely generated. The second assertion follows from [Ge4; Corollary 2].

Finally, 4 follows from Lemma 7.5 and Theorem 7.3. ■

Remarks. 1. Let R be a one-dimensional noetherian domain such that $\overline{R}$ is a finitely generated R-module and let $\mathfrak{f}$ denote the conductor (i.e., R is an order in the Dedekind domain $\overline{R}$). Then we have an exact sequence (cf. [Ne; I, §12])

$$1 \to \overline{R}^\times / R^\times \to (\overline{R}/\mathfrak{f})^\times / (R/\mathfrak{f})^\times \to \mathrm{Pic}(R) \to \mathrm{Cl}(\overline{R}) \to 1.$$

Hence if, for example, $\overline{R}$ has finite class group and the finite norm property, then $\mathrm{Pic}(R)$ is finite.

2. In [HK5; Satz 1], [Ge2; Theorem 4] and [Ge3; Theorem 3] finiteness theorems are derived for the arithmetic of a Z-ring R having finite Picard group, finite set $\Omega(P)$ for all $P \in X^{(1)}(R)$, and with $\Omega(P) = \{P\}$ for all but

finitely many $P \in X^{(1)}(R)$. All these results are valid for a weakly Krull domain R with finite t-class group, with $R_P^{\#}$ finitely primary of rank one for all $P \in X^{(1)}(R)$, and with R_P a discrete valuation domain for all but finitely many $P \in X^{(1)}(R)$.

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