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ON A THEOREM OF P. S. ALEKSANDROV

BY

ZBIGNIEW KARNO (BIAŁYSTOK)

Dedicated to the memory of my father, Zygmunt Karno

1. Background and statement of results. In [1] P. S. Aleksandrov has proved that if X is an n-dimensional compactum (where n is finite ≥ 1), then there exist a closed subset A of X and an essential mapping v from the quotient space X/A to the n-dimensional sphere S^n. Actually, he has proved the following theorem (see also [3]): If f is an essential mapping from an n-dimensional compactum X to an n-dimensional disk D^n, then the pinch mapping induced by f on the quotient spaces $X/f^{-1}(\partial D^n)$ and $D^n/\partial D^n \approx S^n$ is also essential. The proof is strictly algebraic. In this paper, using geometric methods, we extend the Aleksandrov result in the following way.

1.1. THEOREM. *If f is an essential mapping from a compactum X to an n-dimensional disk D^n and if $n \leq 2$ or $\dim X < 2n-2$, then the pinch mapping $f_\bullet : X/f^{-1}(\partial D^n) \to D^n/\partial D^n$ induced by f is also essential.*

Recall that a mapping f of a space X into a sphere S is said to be *essential* if it is not homotopic to a constant mapping; equivalently, if every mapping $g : X \to S$ homotopic to f is surjective. A mapping f of a space X into a disk D is said to be *essential* (in the sense of Aleksandrov–Hopf) if there is no mapping $g : X \to \partial D$ such that $g(x) = f(x)$ for each $x \in f^{-1}(\partial D)$; equivalently, if every mapping $g : X \to D$ homotopic to f rel $f^{-1}(\partial D)$ is surjective (see [1]). Note that (see [12]) f is essential if and only if every mapping $g : X \to D$ which is homotopic to f as a map of pairs $(X, f^{-1}(\partial D)) \to (D, \partial D)$ is surjective (essential).

Let f be a mapping of a compactum X into a disk D. Consider the natural quotient mapping $q : X \to X/f^{-1}(\partial D)$ and similarly $p : D \to$

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$D/\partial D$. The mapping f induces a (continuous) mapping $f_\bullet : X/f^{-1}(\partial D) \to D/\partial D$ such that $p \circ f = f_\bullet \circ q$ (see [4]). It may be defined by $f_\bullet(a) = (p \circ f)(q^{-1}(a))$ for $a \in X/f^{-1}(\partial D)$. We call $f_\bullet$ the *pinch* mapping. Since $D/\partial D$ is homeomorphic to S^n, where $n = \dim D$, we will usually identify $f_\bullet$ with the corresponding mapping into S^n.

Actually we prove that the essentiality of $f \times \mathrm{id}_I$, where I is the unit interval, implies the essentiality of the pinch mapping $f_\bullet$. Therefore Theorem 1.1 is a consequence of the following theorem.

1.2. THEOREM. *If f is an essential mapping from a compactum X to an n-dimensional disk D^n and if $n \le 2$ or* $\dim X < 2n - 2$, *then* $f \times \mathrm{id}_I : X \times I \to D^n \times I$ *is essential* (*moreover*, $f \times \mathrm{id}_{I^k} : X \times I^k \to D^n \times I^k$ *is also essential for all* k).

This result is related to two theorems proved by W. Holsztyński [9], [10] and by K. Morita [16], [17] (see also [11]); these claim in the case of compacta that given an essential mapping $f : X \to I^n$ the product mapping $f \times \mathrm{id}_I : X \times I \to I^{n+1}$ is essential if $n \le 2$ or $\dim X = n$ (for the case $n = 1$ see also [8], [7], [2], [12], [13]). Both proofs use algebraic topology. Our proof is strictly geometric and it consists in proving the assertion first for compact polyhedra and afterwards in using the Freudenthal theorem to generalize it to arbitrary compacta.

Note that an essential mapping $f : I^4 \to I^3$ with $f \times \mathrm{id}_I : I^5 \to I^4$ inessential, constructed by W. Holsztyński in [10], has inessential pinch $f_\bullet$. So, Theorem 1.1 (and also Theorem 1.2) cannot be strengthened for compacta with $\dim X \ge 2n - 2$ and $n > 2$.

The paper is organized as follows. In Section 2 we establish some connections between mappings into disks and spheres, and their cones. Furthermore, we deduce 1.1 from 1.2. In the same section we show that an essential mapping $f : I^4 \to I^3$ with $f \times \mathrm{id}_I$ inessential has inessential pinch $f_\bullet$. In the next two sections we prove 1.2 for mappings of compact polyhedra into n-dimensional cubes, with $\dim X < 2n - 2$ in Section 3, and with $n = 2$ in Section 4. In Section 5 we complete the proof of 1.2 in the general case.

Throughout the paper, all spaces are metric and all mappings are continuous. Compactum means a compact metric space. By "dimension" we understand the covering dimension. By an n-dimensional sphere S we mean here any space homeomorphic to the unit n-dimensional sphere S^n, and by an n-dimensional disk D we mean any space homeomorphic to the unit n-dimensional cube I^n ($I = I^1$ is the unit interval $[0, 1]$), by ∂D we denote its geometric boundary, and by D° its geometric interior.

2. Cones of mappings into disks and spheres. Let X be a compactum. By the (unreduced) *cone* $C(X)$ we mean the quotient space

$(X\times I)/(X\times 1)$. Obviously, $C(X)$ is compact and metrizable (see [4]). We will denote by $[x,t]\in C(X)$ the image of $(x,t)\in X\times I$ under the natural quotient mapping $c_X : X\times I\to C(X)$. We will identify X with the closed subspace $\{[x,0] : x\in X\}$ of $C(X)$. Let $f : X\to Y$ be a mapping between compacta. By the *cone mapping* $C(f) : C(X)\to C(Y)$ induced by f we mean the (continuous) mapping defined by $C(f)([x,t]) = [f(x),t]$ for $[x,t]\in C(X)$. It satisfies the equality $c_Y\circ(f\times \mathrm{id}_I) = C(f)\circ c_X$. Under the identifications mentioned above, $C(f)^{-1}(Y) = X$ and $C(f)_{|X} = f$. Note that the cone on an n-dimensional sphere S, and similarly the cone on an n-dimensional disk D, are $(n+1)$-dimensional disks.

2.1. Proposition. *Let g be a mapping from a compactum X into a sphere S. Then g is essential if and only if $C(g)$ is essential.*

Proof. Follows from [6, Lemma 4.7] and [6, Theorem 4.11]. ∎

2.2. Theorem. *If f is a mapping from a compactum X into a disk D, then $f\times \mathrm{id}_I : X\times I\to D\times I$ is essential if and only if $C(f)$ is essential.*

Proof. Suppose $C(f)$ is not essential. Since $c_D\circ(f\times \mathrm{id}_I) = C(f)\circ c_X$, it follows that $C(f)\circ c_X$, and therefore $c_D\circ(f\times \mathrm{id}_I)$, is not essential either. Then there exists $g : X\times I\to \partial C(D)$ such that

$$c_D\circ(f\times \mathrm{id}_I)_{|Y} = g_{|Y} : Y\to \partial C(D), \tag{0}$$

where $Y = (c_D\circ(f\times \mathrm{id}_I))^{-1}(\partial C(D))$. Observe that $\partial(D\times I)$ and $\partial C(D)$ are spheres of the same dimension, $\partial(D\times I)/(D\times 1) = \partial C(D)$ and $D\times 1$ is a tame disk in $\partial(D\times I)$. It follows that there exists a homeomorphism $h : \partial C(D)\to \partial(D\times I)$ such that $(h\circ c_D)_{|\partial(D\times I)}\simeq \mathrm{id}_{\partial(D\times I)}$. Since $(f\times \mathrm{id}_I)^{-1}(\partial(D\times I)) = Y$, we conclude from (0) that $(f\times \mathrm{id}_I)_{|Y}\simeq h\circ g_{|Y}$ as mappings $Y\to\partial(D\times I)$. Applying the homotopy extension theorem, we get a mapping $f^* : X\times I\to\partial(D\times I)$ such that $f^*_{|Y} = (f\times \mathrm{id}_I)_{|Y}$, which contradicts the essentiality of $f\times \mathrm{id}_I$.

The reverse implication is quite obvious; $f\times \mathrm{id}_I$ is equivalent to $C(f)_{|c_X(X\times[0,t])} : c_X(X\times[0,t])\to c_D(D\times[0,t])$, where $0<t<1$. ∎

2.3. Theorem. *Let f be a mapping from a compactum X into a disk D. If $f\times \mathrm{id}_I : X\times I\to D\times I$ is essential, then the pinch mapping $f_\bullet : X/f^{-1}(\partial D)\to D/\partial D$ is also essential.*

Proof. According to 2.2, the cone mapping $C(f)$ is essential. By [6, Lemma 4.7], $g = C(f)_{|Y} : Y\to\partial C(D)$ is essential, where $Y = (C(f))^{-1}(\partial C(D)) = X\cup C(f^{-1}(\partial D))$.

Consider the natural quotient mappings $p : \partial C(D)\to\partial C(D)/C(\partial D)$ and $q : Y\to Y/C(f^{-1}(\partial D))$. We have the identifications $\partial C(D) = D\cup C(\partial D)$, $\partial C(D)/C(\partial D) = D/\partial D$ and $Y/C(f^{-1}(\partial D)) = X/f^{-1}(\partial D)$. It follows that $f_\bullet\circ q = p\circ g$. Since $\partial C(D)$ is a sphere and $C(\partial D)$ is a tame disk in $\partial C(D)$ such that $\dim \partial C(D) = \dim C(\partial D)$, we infer that there is a homeomorphism $h : \partial C(D)/C(\partial D) \to \partial C(D)$ such that $h \circ p \simeq \mathrm{id}_{\partial C(D)}$. Thus $h \circ f_\bullet \circ q \simeq g$. Since g is essential, so is $h \circ f_\bullet \circ q$. Hence $h \circ f_\bullet$ is essential. Since h is a homeomorphism, we conclude that $f_\bullet$ is essential. ■

Proof of Theorem 1.1. Combine 2.3 and 1.2. ■

From 1.1, 1.2, 2.1 and 2.2, we get the following observation.

2.4. COROLLARY. *If f is an essential mapping from a compactum X to I^n and if $n \le 2$ or $\dim X < 2n-2$, then*

(a) $C(f) : C(X) \to C(I^n) \approx I^{n+1}$ *is essential*,

(b) $C(f_\bullet) : C(X/f^{-1}(\partial I^n)) \to C(I^n/\partial I^n) \approx I^{n+1}$ *is essential.* ■

2.5. EXAMPLE. In [10, Example 3.8] W. Holsztyński gave an example of an essential mapping $f : I^4 \to I^3$ with $f^{-1}(\partial I^3) = \partial I^4$ such that $f \times \mathrm{id}_I : I^5 \to I^4$ is not essential (see also [9, Proposition 1.1]). According to Theorem 2.2, the cone mapping $C(f) : C(I^4) \to C(I^3)$ is not essential either.

We now prove that the pinch mapping $f_\bullet : I^4/\partial I^4 \to I^3/\partial I^3$ is not essential. Consider the natural quotient mappings $p : C(I^3) \to C(I^3)/C(\partial I^3)$ and $q : C(I^4) \to C(I^4)/C(\partial I^4)$. There exists a mapping $f_* : C(I^4)/C(\partial I^4) \to C(I^3)/C(\partial I^3)$ such that $f_* \circ q = p \circ C(f)$. Since $I^4 \approx C(I^3) \approx C(I^3)/C(\partial I^3)$ and $I^5 \approx C(I^4) \approx C(I^4)/C(\partial I^4)$, it follows that there exist homeomorphisms $h : C(I^3)/C(\partial I^3) \to C(I^3)$ and $g : C(I^4)/C(\partial I^4) \to C(I^4)$ such that $h \circ p$ and $\mathrm{id}_{C(I^3)}$ are homotopic as mappings of the pair $(C(I^3), \partial C(I^3))$, and $q \circ g$ and $\mathrm{id}_{C(I^4)/C(\partial I^4)}$ are homotopic as mappings of the pair $(C(I^4)/C(\partial I^4), \partial(C(I^4)/C(\partial I^4)))$. Since $C(f)$ is not essential, from the homotopy extension theorem applied to appropriate mappings, we infer that $h \circ p \circ C(f)$ is not essential. Since h is a homeomorphism, $p \circ C(f)$ is not essential. Thus $f_* \circ q$ is not essential. Since g is a homeomorphism, $f_* \circ q \circ g$ is not essential. Applying the homotopy extension theorem again, we conclude that f_* is not essential. Observe that $I^3/\partial I^3 = \partial(C(I^3)/C(\partial I^3))$, $I^4/\partial I^4 = \partial(C(I^4)/C(\partial I^4))$, and furthermore $f_\bullet = (f_*)_{|\partial(C(I^3)/C(\partial I^3))}$. Since $C(I^3)/C(\partial I^3)$ is contractible, from [6, Theorem 4.11] we conclude that $f_\bullet$ is not essential. Moreover, since $f_\bullet$ is not essential, from 2.1 we infer that $C(f_\bullet) : C(I^4/\partial I^4) \to C(I^3/\partial I^3)$ is not essential. ■

3. Mappings of polyhedra into high dimensional cubes. In this section and subsequently we will denote by q_X the projection of $X \times I$ to X and by q_I the projection of $X \times I$ to I. By p_{I^n} and p_I we will denote the analogous projections from $I^n \times I$ to I^n and I, respectively. Note that if $f : X \to I^n$, then $f \times \mathrm{id}_I = (p_{I^n} \circ (f \times \mathrm{id}_I), p_I \circ (f \times \mathrm{id}_I)) = (f \circ q_X, q_I)$.

We prove here a polyhedral version of Theorem 1.2 under the assumption $\dim X < 2n-2$ (see also [11, Lemma 2.2] for the analogous fact but with a complicated proof). In this case we have $n > 2$ and $\dim X \geq n$; the last inequality follows from the essentiality assumption. The case $n \leq 2$ will be discussed in the remaining sections.

3.1. THEOREM. *If f is an essential mapping from a compact polyhedron X to I^n and if $\dim X < 2n-2$, then $f \times \mathrm{id}_I$ is essential.*

As a preliminary to the proof of this theorem, we state an auxiliary lemma. It will also be needed in the next section.

3.2. LEMMA. *Let f be a mapping from a compact polyhedron X into I^n. Suppose $f \times \mathrm{id}_I$ is not essential. Then there exists a PL-mapping $G : X \times I \to \partial I^{n+1}$ such that*

(a) *$X \times i \subset G^{-1}(I^n \times i)$ for $i = 0, 1$, and*
(b) *$f^{-1}(\partial I^n) \times I \subset G^{-1}((\partial I^n) \times I)$.*

Moreover, for $i = 0, 1$ the mapping $g_i : X \to I^n$ defined by the formula $g_i(x) = (p_{I^n} \circ G)(x, i)$, $x \in X$, is PL and has the following properties:

(c) *$f^{-1}(\partial I^n) \subset g_i^{-1}(\partial I^n)$, and g_i and f are homotopic as mappings $(X, f^{-1}(\partial I^n)) \to (I^n, \partial I^n)$,*
(d) *if f is essential, then g_i is also essential.*

Proof. Since $f \times \mathrm{id}_I$ is not essential, there exists $F_0 : X \times I \to \partial I^{n+1}$ such that

(1) $$(f \times \mathrm{id}_I)(x,t) = F_0(x,t) \quad \text{for } (x,t) \in (f \times \mathrm{id}_I)^{-1}(\partial I^{n+1}).$$

Let $u : I^n \to I^n$ be any mapping such that

(2) $$u_{|\partial I^n} = (\mathrm{id}_{I^n})_{|\partial I^n} \quad \text{and} \quad \partial I^n \subset \operatorname{Int} u^{-1}(\partial I^n).$$

Define $F : X \times I \to \partial I^{n+1}$ by $F(x,t) = (u \circ p_{I^n} \circ F_0(x,t), t)$ for $(x,t) \in X \times I$. It is easy to check, using (1) and (2), that

(3) $$X \times i \subset F^{-1}(I^n \times i) \quad \text{for } i = 0, 1,$$

(4) $$f^{-1}(\partial I^n) \times I \subset \operatorname{Int} F^{-1}((\partial I^n) \times I),$$

and for $i = 0, 1$, the mapping $f_i : X \to I^n$ defined by $f_i(x) = (p_{I^n} \circ F)(x, i)$, $x \in X$, satisfies

(15) $$f^{-1}(\partial I^n) \subset \operatorname{Int} f_i^{-1}(\partial I^n) \quad \text{and} \quad f_i(x) = f(x) \quad \text{for } x \in f^{-1}(\partial I^n).$$

From (4) and (5) it follows that there is a polyhedron $P \subset X$ such that

(6) $$P \times I \subset F^{-1}((\partial I^n) \times I),$$

(7) $$f^{-1}(\partial I^n) \subset P \subset f_i^{-1}(\partial I^n) \quad \text{for } i = 0, 1.$$

Let K be any triangulation of $X \times I$ such that $P \times I$ and $X \times i$ are the underlying polyhedra of some subcomplexes of K. Similarly, let L be any triangulation of ∂I^{n+1} with an analogous property with respect to $(\partial I^n) \times I$

and $I^n \times i$. Consider any simplicial approximation $G : X \times I \to \partial I^{n+1}$ to $F : K \to L$. We show that G, and g_i defined as in the assertion of Lemma 3.2, meet the requirements.

Clearly, G is PL and so is g_i for $i = 0, 1$. From (3) we have $F(X \times i) \subset I^n \times i$. Since G is a simplicial approximation to F, we conclude that $G(X \times i) \subset I^n \times i$ and

(8) $$g_i \text{ is a simplicial approximation to } f_i,$$

by [18, p. 127]. Thus, in particular, (a) is satisfied. From (6) we have $F(P \times I) \subset (\partial I^n) \times I$. In the same manner we can see that $G(P \times I) \subset (\partial I^n) \times I$. Therefore $G(f^{-1}(\partial I^n) \times I) \subset (\partial I^n) \times I$, by (7). Thus (b) is satisfied. From what has already been shown, we can also see that $G(P \times i) \subset (\partial I^n) \times i$. It follows that $P \subset g_i^{-1}(\partial I^n)$. Therefore $f^{-1}(\partial I^n) \subset g_i^{-1}(\partial I^n)$, by (7). Moreover, from (7) and (8) it follows that g_i and f_i are homotopic as mappings $(X, P) \to (I^n, \partial I^n)$ (see [18, p. 128]). But f_i and f are homotopic rel $f^{-1}(\partial I^n)$, by (5). From (7), we conclude that g_i and f are homotopic as mappings $(X, f^{-1}(\partial I^n)) \to (I^n, \partial I^n)$. Thus (c), and hence (d), is satisfied. This completes the proof. ■

Proof of Theorem 3.1. Set $k = \dim X$. Suppose that f is essential and, on the contrary, $f \times \mathrm{id}_I$ is not. By Lemma 3.2, there exists a PL-mapping $G : X \times I \to \partial I^{n+1}$ such that

(1) $$X \times i \subset G^{-1}(I^n \times i) \quad \text{for } i = 0, 1,$$

(2) $$f^{-1}(\partial I^n) \times I \subset G^{-1}((\partial I^n) \times I),$$

and for the PL-mapping $g : X \to I^n$ defined by $g(x) = (p_{I^n} \circ G)(x, 0)$, $x \in X$, we have

(3) $f^{-1}(\partial I^n) \subset g^{-1}(\partial I^n)$, and g and f are homotopic as mappings $(X, f^{-1}(\partial I^n)) \to (I^n, \partial I^n)$.

Consider a triangulation K of $X \times I$ and a triangulation L of ∂I^{n+1} such that $G : K \to L$ is simplicial. For $i = 0, 1$, choose an n-dimensional simplex A_i in L contained in $I^n \times i$. Now pick a point a_0 in the geometric interior of A_0. It follows that $G^{-1}(a_0)$ is a subpolyhedron of $X \times I$ and $\dim G^{-1}(a_0) \le k + 1 - n$. Since $a_0 \in I^n \times 0$ and $a_0 \notin (\partial I^n) \times I$, from (1) and (2) we have $G^{-1}(a_0) \cap ((X \times 1) \cup (f^{-1}(\partial I^n) \times I)) = \emptyset$.

Set $P_0 = q_X(G^{-1}(a_0))$. Since q_X is PL, we infer that P_0 is a subpolyhedron of X such that

(4) $$\dim P_0 \le k + 1 - n,$$

(5) $$P_0 \cap f^{-1}(\partial I^n) = \emptyset,$$

(6) $$G^{-1}(a_0) \subset P_0 \times [0, 1).$$

Observe that $A_1 \cap G(P_0 \times I) = G(G^{-1}(A_1) \cap (P_0 \times I))$, hence by (4) we have $\dim(A_1 \cap G(P_0 \times I)) \le k + 2 - n < n$, because according to the assumption $k < 2n - 2$. Therefore there exists a point a_1 in the geometric interior of A_1 such that $a_1 \notin G(P_0 \times I)$. Set $P_1 = q_X(G^{-1}(a_1))$. Obviously,

$$P_0 \cap P_1 = \emptyset. \tag{7}$$

Since $a_1 \in I^n \times 1$, from (1) we have $G^{-1}(a_1) \cap (X \times 0) = \emptyset$. Thus

$$G^{-1}(a_1) \subset P_1 \times (0, 1]. \tag{8}$$

By (5) and (7) there exists a mapping $u : X \to I$ such that

$$u(P_0) \subset \{1\} \text{ and } u(P_1 \cup f^{-1}(\partial I^n)) \subset \{0\}. \tag{9}$$

From (6) and (8) it follows that

$$\{(x, u(x)) : x \in X\} \cap (G^{-1}(a_0) \cup G^{-1}(a_1)) = \emptyset. \tag{10}$$

Set $a = p_{I^n}(a_0)$. Let $p : \partial I^{n+1} \to I^n$ be any mapping such that

$$p^{-1}(a) = \{a_0, a_1\} \quad \text{and} \quad p(x, t) = x \quad \text{for } (x, t) \in (I^n \times 0) \cup (\partial I^n \times I) \tag{11}$$

The task is now to construct a homotopy $H : X \times I \to I^n$, which connects g rel $f^{-1}(\partial I^n)$ with a mapping that is not surjective. Such a homotopy can be defined by $H(x, t) = (p \circ G)(x, u(x)t)$ for $x \in X$ and $t \in I$. It has all the required properties. By (2), (9) and (11), H is a homotopy rel $f^{-1}(\partial I^n)$. By (11), $H(x, 0) = g(x)$ for $x \in X$. By (10) and (11), the mapping $h : X \to I^n$ defined by $h(x) = H(x, 1)$ for $x \in X$ is not surjective because $a \notin h(X)$.

Consequently, we conclude from (3) that f and h are homotopic as mappings $(X, f^{-1}(\partial I^n)) \to (I^n, \partial I^n)$. Since h is not surjective, f is not essential, a contradiction. ∎

3.3. Theorem. *If f is an essential mapping from a compact polyhedron X to I^n and if $\dim X < 2n - 2$, then $f \times \mathrm{id}_{I^k} : X \times I^k \to I^{n+k}$ is also essential for all k.*

Proof. By 3.1, $f \times \mathrm{id}_I : X \times I \to I^{n+1}$ is essential. Assuming the assertion to hold for $k \ge 1$, we will prove it for $k+1$. We have $\dim(X \times I^k) = k + \dim X < 2(n + k) - 2$. Since $f \times \mathrm{id}_{I^k}$ is essential, from 3.1 we conclude that $(f \times \mathrm{id}_{I^k}) \times \mathrm{id}_I = f \times \mathrm{id}_{I^{k+1}}$ is essential, which completes the proof. ∎

4. Mappings of polyhedra into the square. In this section we prove a polyhedral version of Theorem 1.2 under the assumption $n = 2$; first in the case $\dim X = 2$, and afterwards for arbitrary polyhedra.

4.1. Theorem. *If f is an essential mapping from a 2-dimensional compact polyhedron X to I^2, then $f \times \mathrm{id}_I$ is essential.*

For the proof we need the concept of T-modifications of mappings. Let X be a 2-dimensional compact polyhedron and let $f : X \to I^2$. By a *T-modification* of f we mean any composition $f_* = f \circ \varphi : X_* \to I^2$, where X_* and $\varphi : X_* \to X$ have the following properties: there are two collections of pairwise disjoint sets $\{D_1, \ldots, D_k\}$ and $\{T_1, \ldots, T_k\}$ such that

(a) D_i is a 2-dimensional disk in $X - f^{-1}(\partial I^2)$ with D_i° open in X,

(b) T_i is a once-punctured 2-dimensional torus in $X_* - (f \circ \varphi)^{-1}(\partial I^2)$ with T_i° open in X_*, and

(c) φ maps homeomorphically $X_* - \bigcup_i T_i^\circ$ onto $X - \bigcup_i D_i^\circ$ so that $\varphi(\partial T_i) = \partial D_i$ and $\varphi(T_i^\circ) \subset D_i^\circ$.

Here T_i° and ∂T_i denote the geometric interior and the geometric boundary of T_i, respectively.

4.2. Lemma. *If f is an essential mapping from a 2-dimensional compact polyhedron X to I^2, then every T-modification f_* of f is essential.*

Proof. On the contrary, suppose that $f_* = f \circ \varphi : X_* \to I^2$ is an inessential T-modification of f, where X_* and $\varphi : X_* \to X$ have the above mentioned properties (a)–(c). There is a mapping $g : X_* \to \partial I^2$ such that $f_{*|f_*^{-1}(\partial I^2)} = g_{|f_*^{-1}(\partial I^2)}$. In particular, we have $g(T_i) \subset \partial I^2$, for $i = 1, \ldots, k$. We wish to examine the mappings $g_{|\partial T_i} : \partial T_i \to \partial I^2$.

We first make the following:

Claim. *If g is a mapping of a once-punctured 2-dimensional torus T into a circle S^1, then the restriction mapping $g_{|\partial T} : \partial T \to S^1$ is not essential.*

Proof. Fix a base point $*$ in ∂T. Let $\iota : \partial T \to T$ denote the inclusion mapping. It induces the homomorphism $\iota_\# : \pi_1(\partial T, *) \to \pi_1(T, *)$ of the fundamental groups. Let c be a generator of $\pi_1(\partial T, *) = \mathbb{Z}$ and let a and b be generators of the free group $\pi_1(T, *) = \mathbb{Z} * \mathbb{Z}$. We may assume that $[a, b] = \iota_\#(c)$, where $[a, b] = a^{-1}b^{-1}ab$ is the commutator of a and b in $\pi_1(T, *)$. The mapping g induces a homomorphism $g_\# : \pi_1(T, *) \to \pi_1(S^1)$. Since $\pi_1(S^1) = \mathbb{Z}$ is abelian, we have $[a, b] \in \ker g_\#$. Thus $\iota_\#(\pi_1(\partial T, *)) \subset \ker g_\#$. Since $(g_{|\partial T})_\# = g_\# \circ \iota_\#$, it follows that $(g_{|\partial T})_\# : \pi_1(\partial T, *) \to \pi_1(S^1)$ is a trivial homomorphism, and, in consequence, $g_{|\partial T}$ is not essential. ∎

We now turn to the proof of Lemma 4.2. According to the above claim, no mapping $g_{|\partial T_i} : \partial T_i \to \partial I^2$ is essential. From (c) it follows that $(g \circ \psi)_{|\partial D_i} : \partial D_i \to \partial I^2$ is not essential, where $\psi = \varphi^{-1} : X - \bigcup_i D_i^\circ \to X_* - \bigcup_i T_i^\circ$. Hence there is an extension $g^* : X \to \partial I^2$ of $g \circ \psi : X - \bigcup_i D_i^\circ \to \partial I^2$. From (b), we have $T_i \cap f_*^{-1}(\partial I^2) = \emptyset$. Therefore ψ sends homeomorphically $f^{-1}(\partial I^2)$ onto $f_*^{-1}(\partial I^2)$, by (c) and (a). Hence $f_{|f^{-1}(\partial I^2)} = (g \circ \psi)_{|f^{-1}(\partial I^2)} = g^*_{|f^{-1}(\partial I^2)}$. This contradicts the essentiality of f. ∎

4.3. Lemma. *Let f be a mapping from a 2-dimensional compact polyhedron X to I^2. If every T-modification f_* of f is essential, then $f \times \mathrm{id}_I$ is essential.*

Proof. On the contrary, suppose that $f \times \mathrm{id}_I$ is not essential. By 3.2, there exists a PL-mapping $G : X \times I \to \partial I^3$ such that

(1) $$X \times i \subset G^{-1}(I^2 \times i) \quad \text{for } i = 0, 1,$$

(2) $$f^{-1}(\partial I^2) \times I \subset G^{-1}((\partial I^2) \times I),$$

and for the PL-mapping $g : X \to I^2$ defined by $g(x) = (p_{I^2} \circ G)(x, 0)$, $x \in X$, we have

(3) $f^{-1}(\partial I^2) \subset g^{-1}(\partial I^2)$, and g and f are homotopic as mappings $(X, f^{-1}(\partial I^2)) \to (I^2, \partial I^2)$.

Let K be a fixed triangulation of X such that $g^{-1}(\partial I^2)$ is the underlying polyhedron of some subcomplex of K. Set $P = (|K^{(1)}| \times I) \cup (X \times \{0, 1\})$ and $P_0 = (|K^{(0)}| \times I) \cup (|K^{(1)}| \times \{0, 1\})$, where $K^{(j)}$ denotes the j-dimensional skeleton of K. Consider a triangulation M of $X \times I$ and a triangulation L of ∂I^3 so that $G : M \to L$ is simplicial, $|K^{(1)}| \times I$ and $X \times \{0, 1\}$ are the underlying polyhedra of some subcomplexes of M, and similarly $(\partial I^2) \times I$ and $I^2 \times \{0, 1\}$ are the underlying polyhedra of some subcomplexes of L.

For $i = 0, 1$, choose a 2-dimensional simplex A_i in L contained in $I^2 \times i$. We may assume that A_i is disjoint from $\partial I^2 \times i$ and the geometric interiors of the disks $p_{I^2}(A_0)$ and $p_{I^2}(A_1)$ intersect. Now pick points a in I^2 and a_i in the geometric interior of A_i, $i = 0, 1$, such that $a = p_{I^2}(a_0) = p_{I^2}(a_1)$. Since G is simplicial, $Y = (p_{I^2} \circ G)^{-1}(a)$ is a subpolyhedron of $X \times I$ and $\dim Y \le 1$. From the construction, it follows that

(4) the set $P \cap Y$ is finite and disjoint from P_0, and

(5) for any $A \in K$, $A \subset X - g^{-1}(\partial I^2)$ if $(A \times I) \cap Y \neq \emptyset$.

Moreover, for every 2-dimensional simplex $A \in K$, $(A \times I) \cap Y$ is either empty or a 1-dimensional manifold properly embedded in $A \times I$. In particular,

(6) for every 2-dimensional simplex $A \in K$, every component of $(A \times I) \cap Y$ intersecting $\partial(A \times I)$ is a PL-arc properly embedded in $A \times I$.

Without loss of the properties (1)–(6) we may assume that

(7) the projection $q_X : X \times I \to X$ is an embedding on the set $P \cap Y$.

For otherwise one can replace G by $G \circ h^{-1}$, where h is a PL-homeomorphism of $X \times I$ to itself which is the identity on $P_0 \cup (g^{-1}(\partial I^2) \times I)$ and sending $A \times I$ to itself for any $A \in K$, so that q_X is an embedding on $h(P \cap Y)$.

Let Y_0 be the union of all components of Y intersecting $X \times 0$. From (1) we have

$$(X \times 1) \cap Y_0 = \emptyset. \tag{8}$$

Observe that the set $E = (|K^{(1)}| \times I) \cap Y_0 \subset P \cap Y$ is finite. By (4) and (6), $q_X(E)$ is disjoint from $|K^{(0)}|$, $q_X(Y - Y_0)$ and $q_X((X \times 0) \cap Y_0)$, because for any $e \in E$ every interval of the form $(q_X(e)) \times I$ intersects Y_0 exactly in e.

By (4) and (7), there exists a simplicial subdivision K_1 of K such that all $q_X(e)$, where $e \in E$, are vertices of K_1, and all closed stars $\mathrm{st}(q_X(e), K_1)$ are pairwise disjoint and do not intersect $|K^{(0)}|$, $q_X(Y - Y_0)$ and $q_X((X \times 0) \cap Y_0)$.

Let $u : X \to I$ be a linear extension on all simplices in K_1 of the vertex map $u : K_1^{(0)} \to I$ defined by $u(v) = 1$ if $v \in q_X(E)$, and $u(v) = 0$ if $v \notin q_X(E)$. For any $Z \subset X$ denote by $Z_u = \{(z, u(z)) \in X \times I : z \in Z\}$ the graph of $u_{|Z}$.

We wish to define the desired X_* as X_u which will be modified in the following way: first in X_u a finite collection of pairs of disjoint disks will be removed, and in the place of every pair a pipe will be inserted, i.e., a set homeomorphic to $S^1 \times I$, so that $X_* \cap Y = \emptyset$.

Observe that $|K^{(1)}|_u$ is disjoint from Y and X_u is disjoint from $Y - Y_0$. Hence $X_u \cap Y = \bigcup \{A_u^\circ \cap Y_0 : A \in K - K^{(1)}\}$. Consider $A \in K - K^{(1)}$ and let $D_A = \{(x,t) \in A \times I : t \geq u(x)\}$ be the shadow over A_u. Clearly, D_A is a 3-dimensional PL-disk, and A_u is a 2-dimensional disk in ∂D_A. By (6) and (8), one can choose K_1 so fine that every component of $D_A \cap Y_0$ is a PL-arc properly embedded in D_A with end points in A_u°. Take a sufficiently small regular neighborhood of such an arc, which is disjoint from $Y - Y_0$ and homeomorphic to $D \times I$, where D is a 2-dimensional disk. Now we replace each pair of disks corresponding to $D \times \{0,1\}$ in X_u by the pipe $(\partial D) \times I$, and we get the desired X_* disjoint from Y. In a natural way, we can define $\varphi : X_* \to X$ which agrees with q_X on $|K^{(1)}|_u$ and $g^{-1}(\partial I^2)$, and has the property that $\varphi(z) \in A^\circ$ if $q_X(z) \in A^\circ$.

We claim that the T-modification g_* of g defined by

$$g_* = g \circ \varphi : X_* \to I^2 \text{ is not essential.} \tag{9}$$

Indeed, we have $g_*^{-1}(\partial I^2) = \varphi^{-1}(g^{-1}(\partial I^2)) = (q_{X|X_*})^{-1}(g^{-1}(\partial I^2)) = g^{-1}(\partial I^2) \times 0$ and both g_* and $g \circ (q_{X|X_*})$ are homotopic rel $g_*^{-1}(\partial I^2)$, because there is a homotopy along vertical intervals connecting φ and $q_{X|X_*}$. The mapping $g \circ (q_{X|X_*})$ is also homotopic to $p_{I^2} \circ G \circ \iota$ rel $g_*^{-1}(\partial I^2)$, where $\iota : X_* \to X \times I$ is given by $\iota(x,t) = (x,0)$. On the other hand, $p_{I^2} \circ G \circ \iota$ is homotopic to $(p_{I^2} \circ G)_{|X_*}$ rel $g_*^{-1}(\partial I^2)$, because ι is homotopic to an embedding of X_* into $X \times I$ under a homotopy leaving $g_*^{-1}(\partial I^2)$ inside $(p_{I^2} \circ G)^{-1}(\partial I^2) = G^{-1}((\partial I^2) \times I)$. By the construction, the last mapping is not surjective, and (9) is proved.

Define a T-modification of f by $f_* = f \circ \varphi$. From (3) and from the construction of φ, it follows that g_* and f_* are homotopic as mappings $(X, f_*^{-1}(\partial I^2)) \to (I^2, \partial I^2)$. By (9), we conclude that f_* is not essential. ∎

P r o o f o f T h e o r e m 4.1. Combine 4.2 and 4.3. ∎

4.4. THEOREM. *If f is an essential mapping from a compact polyhedron X to I^2, then $f \times \mathrm{id}_{I^k} : X \times I^k \to I^{k+2}$ is essential for all k.*

P r o o f. Let Y be the 2-dimensional skeleton in any triangulation of X. We first prove that $f_{|Y} : Y \to I^2$ is essential.

On the contrary, suppose that $f_{|Y}$ is not essential. Then there exists $g : Y \cup f^{-1}(\partial I^2) \to \partial I^2$ such that $g_{|f^{-1}(\partial I^2)} = f_{|f^{-1}(\partial I^2)}$. Since $\partial I^2 \approx S^1$ is an Eilenberg–MacLane space $K(\mathbb{Z}, 1)$, we infer that there is a continuous extension $g^* : X \to \partial I^2$ of g. Hence $g^*_{|f^{-1}(\partial I^2)} = f_{|f^{-1}(\partial I^2)}$, which contradicts the essentiality of f.

Since $(f_{|Y}) \times \mathrm{id}_{I^k} = (f \times \mathrm{id}_{I^k})_{|(Y \times I^k)}$, it suffices to show that $(f_{|Y}) \times \mathrm{id}_{I^k} : Y \times I^k \to I^{k+2}$ is essential. From the essentiality of $f_{|Y} : Y \to I^2$ and from 4.1, it follows that $(f_{|Y}) \times \mathrm{id}_I : Y \times I \to I^3$ is essential. Since $Y \times I$ satisfies the dimensional condition in 3.3, we conclude that $((f_{|Y}) \times \mathrm{id}_I) \times \mathrm{id}_{I^{k-1}} = (f_{|Y}) \times \mathrm{id}_{I^k}$ is essential. ∎

5. Essential mappings of compacta into cubes. We prove here Theorem 1.2 for arbitrary compacta. In the case $n = 1$ we have the following well known fact [8], [7], [2], [12], [13].

5.1. PROPOSITION. *If for $i = 1, \ldots, n$, f_i is an essential mapping from a compactum X_i to the unit interval I, then the product mapping*

$$f_1 \times \ldots \times f_n : X_1 \times \ldots \times X_n \to I^n$$

is essential; *in particular*, *$f_i \times \mathrm{id}_{I^k} : X_i \times I^k \to I^{k+1}$ is essential for all k.* ∎

P r o o f o f T h e o r e m 1.2. According to the Freudenthal theorem (see, e.g., [5, Theorem 1.13.2]), X is the inverse limit of an inverse sequence $\{X_i, \pi_{i,j}\}$ consisting of compact polyhedra X_i with

$$\dim X_i \le \dim X. \tag{1}$$

First we are going to use inverse limits to construct a compactum X^* which contains mutually disjoint copies of all X_i as closed subsets, and a copy of X as a closed subset approximated sufficiently closely by X_i (cf. [14], [11]).

Set $X_i^* = X_1 \sqcup \ldots \sqcup X_i$. Define the bonding mappings $\tau_{i,j} : X_j^* \to X_i^*$ by $\tau_{i,i} = \mathrm{id}_{X_i^*}$ and $\tau_{i,j} = \tau_{i,i+1} \circ \tau_{i+1,i+2} \circ \ldots \circ \tau_{j-1,j}$ for $j > i$, whereas $\tau_{i,i+1}(x) = x$ if $x \in X_i^*$, and $\tau_{i,i+1}(x) = \pi_{i,i+1}(x)$ if $x \in X_{i+1}$. Define $X^* = \varprojlim\{X_i^*, \tau_{i,j}\}$.

We may assume that $D^n = I^n$. Now set $A = f^{-1}(\partial I^n)$. Since A is a closed subset of X, $A_i = \pi_i(A)$ is also closed, where $\pi_i : X \to X_i$ is the projection. Therefore $A = \varprojlim\{A_i, \pi_{i,j|A_j}\}$ (see, e.g., [4, p. 138]). Let $A_i^* = A_1 \sqcup \ldots \sqcup A_i$. Clearly, A_i^* is a closed subset of X_i^*. Therefore $A^* = \varprojlim\{A_i^*, \tau_{i,j\,|A_j^*}\}$ is a closed subset of X^*.

Since ∂I^n is an absolute neighborhood retract, there exist an open neighborhood V of A in A^* and an extension $f_0 : V \to \partial I^n$ of $f_{|A} : A \to \partial I^n$. Then $A_i \subset V$ for almost all i. Without loss of generality we may assume that $A^* \subset V$. Using the same argument for the mapping $g_0 : X \cup A^* \to I^n$ defined by $g_0(x) = f(x)$ if $x \in X$ and $g_0(x) = f_0(x)$ if $x \in A^*$, we get an open neighborhood U of $X \cup A^*$ in X^* and an extension $g : U \to I^n$ of g_0. As before, we may assume that $X^* \subset U$. Observe that $g_{|X} = f$. Set $f_i = g_{|X_i} : X_i \to I^n$. We have $f_i(\pi_i(A)) = f_i(A_i) = f_0(A_i) \subset \partial I^n$. Hence

$$\pi_i(A) \subset f_i^{-1}(\partial I^n). \tag{2}$$

Now we prove that

$$f_i \text{ is essential for almost all } i. \tag{3}$$

According to [15, p. 71], there exists an $\varepsilon > 0$ such that for any mapping $f' : (X, f^{-1}(\partial I^n)) \to (I^n, \partial I^n)$ which is ε-near to f, f and f' are homotopic as mappings of pairs. Since for every $\delta > 0$ there exists i such that π_i is a δ-push, from the uniform continuity of g it follows that there is a j such that for every $i \geq j$, $\operatorname{dist}(g_{|X}, (g \circ \pi_i)_{|X}) < \varepsilon$, and therefore $\operatorname{dist}(f, f_i \circ \pi_i) < \varepsilon$. By (2), we have $f_i \circ \pi_i : (X, f^{-1}(\partial I^n)) \to (I^n, \partial I^n)$. Thus f and $f_i \circ \pi_i$ are homotopic as mappings of pairs. Since f is essential, we conclude that $f_i \circ \pi_i$, and in consequence f_i, is essential.

By the assumption and by (1), $n \leq 2$ or $\dim X_i < 2n - 2$. Therefore

$$f_i \times \mathrm{id}_{I^k} \text{ is essential for almost all } i, \tag{4}$$

by (3), 4.1 and by Theorems 3.3 and 4.4.

Suppose, on the contrary, that $f \times \mathrm{id}_{I^k} : X \times I^k \to I^{n+k}$ is not essential. We have $f \times \mathrm{id}_{I^k} = (g_{|X}) \times \mathrm{id}_{I^k} = (g \times \mathrm{id}_{I^k})_{|X \times I^k}$. By [12, Theorem I.1.10], there exists a neighborhood $U \subset X^* \times I^k$ of $X \times I^k$ such that $(g \times \mathrm{id}_{I^k})_{|U}$ is not essential. Since $X \times I^k$ is a compact subset of $X^* \times I^k$ approximated sufficiently closely by $X_i \times I^k$, it follows that there exists an index j such that for each $i \geq j$, $X_i \times I^k \subset U$. Hence for each $i \geq j$, $(g \times \mathrm{id}_{I^k})_{|X_i \times I^k} = (g_{|X_i}) \times \mathrm{id}_{I^k} = f_i \times \mathrm{id}_{I^k}$ is not essential, contrary to (4). ∎

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Institute of Mathematics
Warsaw University at Białystok
Akademicka 2
15-267 Białystok 1, Poland
E-mail: zkarno@math.uw.bialystok.pl

Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-950 Warszawa, Poland
E-mail: zkarno@impan.impan.gov.pl

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