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ON SOME SINGULAR INTEGRAL OPERATORS CLOSE TO THE HILBERT TRANSFORM

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Let $m:\mathbb{R}\to\mathbb{R}$ be a function of bounded variation. We prove the $L^p(\mathbb{R})$-boundedness, $1<p<\infty$, of the one-dimensional integral operator defined by

$$T_m f(x)=\text{p.v.}\int k(x-y)m(x+y)f(y)\,dy$$

where $k(x)=\sum_{j\in\mathbb{Z}}2^j\varphi_j(2^jx)$ for a family of functions $\{\varphi_j\}_{j\in\mathbb{Z}}$ satisfying conditions (1.1)–(1.3) given below.

1. Introduction. We denote by $\mathcal{M}$ the space of real functions of bounded variation on $\mathbb{R}$ with the norm $\|\ \|$ given by $\|m\|=\|m\|_\infty+V(m)$, where $V(m)$ is the variation of m on $\mathbb{R}$.

Let $\{\varphi_j\}_{j\in\mathbb{Z}}$ be a family of functions in $L^1(\mathbb{R})$ satisfying, for all $j\in\mathbb{Z}$,

$$\int\varphi_j(x)\,dx=0, \tag{1.1}$$

$$\operatorname{supp}\varphi_j\subseteq\{x\in\mathbb{R}:1/2\le|x|\le 2\}, \tag{1.2}$$

and for some $c>0, 0<\varepsilon<1$ and for all $j\in\mathbb{Z}$,

$$\int|\varphi_j(x+y)-\varphi_j(x)|\,dx\le c|y|^\varepsilon. \tag{1.3}$$

We define $\varphi_j^{(j)}(x)=2^j\varphi_j(2^jx)$. Let $m\in\mathcal{M}$, and let $T_{m,j}$ be defined by

$$T_{m,j}f(x)=\int\varphi_j^{(j)}(x-y)m(x+y)f(y)\,dy.$$

Our aim is to prove the $L^p(\mathbb{R})$-boundedness, $1<p<\infty$, of the one-dimensional integral operator defined by

$$T_mf(x)=\lim_{(N,M)\to(-\infty,\infty)}\sum_{j=N}^{M}T_{m,j}f(x).$$

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In [R-S] the authors prove the boundedness on $L^2(\mathbb{R})$ of T_m in the case where $m \in L^\infty(\mathbb{R})$ satisfies $|m'(x)| \le c/|x|$ and the family $\{\varphi_j\}_{j\in\mathbb{Z}}$ gives rise to the Hilbert kernel, i.e. $\sum_{j\in\mathbb{Z}} 2^j \varphi_j(2^j x) = x^{-1}$.

The boundedness of T_m on $L^p(\mathbb{R}), 1 < p < \infty$, for $m \in L^\infty(\mathbb{R})$ satisfying the local Lipschitz condition $|m(x+h) - m(x)| \le c(|h|/|x|)^\delta$ for $|h| < |x|/2$ is obtained in [G-S-U].

We first prove some auxiliary results. Next we begin proving the boundedness of T_m on $L^p(\mathbb{R})$, $1 < p < \infty$, for $m = \chi_{[a,b]}$, the characteristic function of $[a,b]$. Moreover, we find that $\|T_m\|$ is independent of a and b. From these facts we derive the general case.

2. The main result. As usual, we denote by $\mathcal{S}(\mathbb{R})$ the Schwartz class of functions rapidly decreasing at infinity. We recall that the convolution operator K with kernel $k = \sum_{j\in\mathbb{Z}} \varphi_j^{(j)}$ is bounded on $L^p(\mathbb{R})$, $1 < p < \infty$. The same result holds for the maximal operator given by

$$K^* f(x) = \sup_M \Big| \sum_{j=-\infty}^{M} \varphi_j^{(j)} * f(x) \Big|$$

(see [D-R]).

LEMMA 2.1. *Let* $\{\varphi_j\}_{j\in\mathbb{Z}}$ *be a family of functions satisfying* (1.1)–(1.3). *Let* $f \in \mathcal{S}(\mathbb{R})$ *and* $m \in \mathcal{M}$. *Then*

$$\lim_{(N,M)\to(-\infty,\infty)} \sum_{j=N}^{M} T_{m,j} f(x)$$

exists and is finite for a.e. $x \in \mathbb{R}$.

Proof. Since $\{\varphi_j\}_{j\in\mathbb{Z}}$ satisfies (1.3) there exist $q_0 > 1$ and $c > 0$ such that $\|\varphi_j\|_{q_0} \le c$ for all $j \in \mathbb{Z}$ (see [S]). Then

$$\sum_{j=N}^{0} \int |\varphi_j^{(j)}(x-y) m(x+y) f(y)|\, dy$$

$$\le \|m\|_\infty \sum_{j=N}^{0} |\varphi_j^{(j)}| * |f|(x) \le \|m\|_\infty \sum_{j=N}^{0} \|\varphi_j^{(j)}\|_q \|f\|_{q'}$$

$$= \|m\|_\infty \sum_{j=N}^{0} 2^{j(1-1/q)} \|\varphi_j\|_q \|f\|_{q'}$$

$$\le c\|m\|_\infty \|f\|_{q'} \sum_{j=N}^{0} 2^{j(1-1/q)},$$

and this geometric sum converges as $N \to -\infty$. Now

$$\sum_{j=0}^{M} \int \varphi_j^{(j)}(x-y)m(x+y)f(y)\,dy$$
$$= \sum_{j=0}^{M} \int \varphi_j^{(j)}(x-y)m(x+y)(f(y)-f(x))\,dy$$
$$+ f(x)\sum_{j=0}^{M} \int \varphi_j^{(j)}(x-y)m(x+y)\,dy.$$

To estimate the first term we observe that

$$\sum_{j=0}^{M} \int |\varphi_j^{(j)}(x-y)| \cdot |m(x+y)| \cdot |f(y)-f(x)|\,dy$$
$$\le \|m\|_\infty \|\nabla f\|_\infty \sum_{j=0}^{M} \int |\varphi_j^{(j)}(x-y)| \cdot |x-y|\,dy.$$

But (1.2) implies that $|x-y| \le 2^{-j+1}$ for $2^j(x-y) \in \operatorname{supp}\varphi_j$, thus the last expression can be bounded by $c\|m\|_\infty \|\nabla f\|_\infty \sum_{j=0}^{M} 2^{-j}$ and this geometric sum converges.

To estimate the second term we note that for $l \in \mathbb{N} \cup \{0\}$, $l \le |x| \le l+1$ and $2^j(x-y) \in \operatorname{supp}\varphi_j$, we have $|x+y| \le 2l+4$, and thus

$$f(x)\sum_{j=0}^{M} \int \varphi_j^{(j)}(x-y)m(x+y)\,dy$$
$$= f(x)\sum_{j=0}^{M} \int \varphi_j^{(j)}(x-y)m(x+y)\chi_{[-2l-4,2l+4]}(x+y)\,dy$$
$$= f(x)\sum_{j=0}^{M} \varphi_j^{(j)} * (m\chi_{[-2l-4,2l+4]})(2x),$$

and this sum converges as $M \to \infty$ almost everywhere for $|x| \in [l, l+1]$ since it is a partial sum of the series defining $K(m\chi_{[-2l-4,2l+4]})(2x)$. ■

For $f \in L^p(0,\infty)$ we denote also by f its extension to $\mathbb{R}$ by zero on the negative real axis.

Our next purpose is to construct an analogue to the Hilbert integral. We will need the result proved in [G-U] that we now state:

LEMMA 2.2. *Let q and q' be conjugate exponents*, $1 < q < \infty$, $q^{-1} + q'^{-1} = 1$. *For $g : \mathbb{R}^n \to \mathbb{C}$ define $g^{(j,q)}(x) = 2^{jn/q} g(2^j x)$. Let* $\{\varphi_j\}_{j\in\mathbb{Z}}$

and $\{\psi_j\}_{j\in\mathbb{Z}}$ *be two families of measurable functions on* $\mathbb{R}^n$ *with support contained in* $\{t : 2^{-1} \leq |t| \leq 2\}$ *such that*

$$\|\varphi_j\|_{q_0} \leq c_1, \quad \|\psi_j\|_{q_1} \leq c_2$$

for some $q_0 > q$, $q_1 > q'$, $c_1 > 0$, $c_2 > 0$, *and for all* $j \in \mathbb{Z}$. *Then the integral operator defined by*

$$Uf(\xi) = \int_{\mathbb{R}^n} K_1(\xi - y)K_2(\xi + y)f(y)\,dy,$$

where $K_1(x) = \sum_{j\in\mathbb{Z}} \varphi_j^{(j,q)}(x)$ *and* $K_2(x) = \sum_{j\in\mathbb{Z}} \psi_j^{(j,q')}(x)$, *is bounded on* $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

LEMMA 2.3. *The operator defined by*

$$I(f)(x) = \sum_{j\in\mathbb{Z}} \int |\varphi_j^{(j)}(y)| \cdot |f(y - x)|\,dy$$

is bounded from $L^p(0,\infty)$ *into* $L^p(0,\infty)$.

Proof. Since $\operatorname{supp} f \subseteq (0,\infty)$, we have, for $x > 0$,

$$I(f)(x) = \Big(\sum_{j\in\mathbb{Z}} \int_{2x<y} + \sum_{j\in\mathbb{Z}} \int_{x<y<2x}\Big)|\varphi_j^{(j)}(y)| \cdot |f(y - x)|\,dy.$$

We note that the first term equals

$$\sum_{j\in\mathbb{Z}} \int |\varphi_j^{(j)}(y)|\chi_{(-\infty,0)}(2x - y)|f(y - x)|\,dy$$
$$= \sum_{j,k\in\mathbb{Z}} \int |\varphi_j^{(j)}(y)|\chi_{(-1,-1/2)}(2^k(2x - y))|f(y - x)|\,dy.$$

Now, for $x > 0$ and $j \geq k + 2$, the j, k term of the last sum vanishes. So we only consider $j < k + 2$. In this case, for $1 < q < q_0$,

$$\sum_{j<k+2} \int |\varphi_j^{(j)}(y)|\chi_{(-1,-1/2)}(2^k(2x - y))|f(y - x)|\,dy$$
$$= \sum_{j<k+2} \int 2^{j/q}|\varphi_j(2^j y)|2^{j/q'}\chi_{(-1,-1/2)}(2^k(2x - y))|f(y - x)|\,dy$$
$$\leq \sum_{j<k+2} \int 2^{j/q}|\varphi_j(2^j y)|2^{(k+2)/q'}\chi_{(-1,-1/2)}(2^k(2x - y))|f(y - x)|\,dy$$
$$\leq 2^{2/q'} \sum_{j,k} \int |\varphi_j^{(j,q)}(y)|\chi^{(k,q')}_{(-1,-1/2)}(2x - y)|f^{\vee}(x - y)|\,dy,$$

where $f^{\vee}(t) = f(-t)$.

On the other hand, we observe that for each fixed x the second term of $I(f)(x)$ is bounded by $5M(|f|)(x)$ where $M(f)(x) = \sup(|\varphi_j^{(j)}|{*}f)(x)$.

A straightforward application of Lemma 2.2 and the boundedness on $L^p(\mathbb{R})$ of the maximal operator M (see [D-R]) give us the desired result. ∎

LEMMA 2.4. *Let $\{m_j\}_{j\in\mathbb{N}}$ be a sequence of functions in $L^\infty(\mathbb{R})$ satisfying*

(i) *There exists $\alpha > 0$ such that $\|m_j\|_\infty \le \alpha$ for all $j \in \mathbb{N}$.*
(ii) *$m(x) = \lim_{j\to\infty} m_j(x)$ exists for a.e. $x \in \mathbb{R}$.*
(iii) *If $1 < p < \infty$, then there exists $c > 0$ such that for $j \in \mathbb{N}$, $N, M \in \mathbb{Z}$ and $f \in \mathcal{S}(\mathbb{R})$,*

$$\Big\|\sum_{k=N}^{M} T_{m_j,k} f\Big\|_p \le c\|f\|_p.$$

Then, for $f \in \mathcal{S}(\mathbb{R})$,

$$\Big\|\sum_{k=N}^{M} T_{m,k} f\Big\|_p \le c\|f\|_p \quad \text{and} \quad \|T_m f\|_p \le c\|f\|_p.$$

Proof. We have

$$\Big\|\sum_{k=N}^{M} T_{m_j,k} f\Big\|_p^p = \int\Big|\int \sum_{k=N}^{M} \varphi_k^{(k)}(t) m(2x-t) f(x-t)\,dt\Big|^p\,dx.$$

By the dominated convergence theorem and the Fatou lemma the last expression is bounded by

$$\liminf_{j\to\infty}\int\Big|\int \sum_{k=N}^{M} \varphi_k^{(k)}(t) m_j(2x-t) f(x-t)\,dt\Big|^p\,dx = \liminf_{j\to\infty}\Big\|\sum_{k=N}^{M} T_{m_j,k} f\Big\|_p^p \le c\|f\|_p^p.$$

A direct application of the Fatou lemma gives us the second assertion. ∎

We now study the operator T_m in the case where $m = \chi_{[a,b]}$, the characteristic function of the interval $[a,b]$. We obtain the following

LEMMA 2.5. *Let $m = \chi_{[a,b]}$ and $f \in L^p(\mathbb{R})$, $1 < p < \infty$. Then there exists c_p such that for all $N, M \in \mathbb{Z}$ with $N < M$,*

$$\Big\|\sum_{k=N}^{M} T_{m,k} f\Big\|_p \le c_p\|f\|_p.$$

Proof. We can assume $f > 0$.

Case 1: $x \in (-\infty, a/2)$. We will prove that for all $M, N \in \mathbb{Z}$,

$$\Big|\sum_{k=N}^{M} T_{m,k} f(x)\Big| \le I^\vee((\tau_{-a/2} f)_{|\mathbb{R}^+})(a/2 - x) \tag{2.6}$$

where $\tau_a f(x) = f(x-a)$ and $I^\vee$ is the operator provided by Lemma 2.3 associated with the family $\{\varphi_k^\vee\}_{k\in\mathbb{Z}}$ defined by $\varphi_k^\vee(t) = \varphi_k(-t)$. Indeed,

$$\begin{aligned}
\Big|\sum_{k=N}^{M} T_{m,k}f(x)\Big| &= \Big|\int \sum_{k=N}^{M} \varphi_k^{(k)}(t)\chi_{[a,b]}(2x-t)f(x-t)\,dt\Big| \\
&\le \int_{2x-b}^{2x-a} \sum_{k=N}^{M} |\varphi_k^{(k)}(t)| f(x-t)\,dt \\
&= \int_{a-2x}^{b-2x} \sum_{k=N}^{M} |\varphi_k^{(k)}(-y)| f(x+y)\,dy \\
&\le \int_{a/2-x}^{\infty} \sum_{k=N}^{M} |\varphi_k^{(k)}(-y)| f(x+y)\,dy \\
&= \int_{a/2-x}^{\infty} \sum_{k=N}^{M} |\varphi_k^{(k)}(-y)| (\tau_{-a/2}f)(y-(a/2-x))\,dy \\
&= \int \sum_{k=N}^{M} |\varphi_k^{(k)}(-y)| (\tau_{-a/2}f)_{|\mathbb{R}^+}(y-(a/2-x))\,dy,
\end{aligned}$$

and so we obtain (2.6).

Case 2: $x \in (b/2, \infty)$. Analogously to the first case we obtain

$$\Big|\sum_{k=N}^{M} T_{m,k}f(x)\Big| \le I((\tau_{b/2}f^\vee)_{|\mathbb{R}^+})(x-b/2). \tag{2.7}$$

Indeed,

$$\begin{aligned}
\Big|\sum_{k=N}^{M} T_{m,k}f(x)\Big| &\le \int_{2x-b}^{2x-a} \sum_{k=N}^{M} |\varphi_k^{(k)}(t)| f(x-t)\,dt \\
&\le \int_{x-b/2}^{\infty} \sum_{k=N}^{M} |\varphi_k^{(k)}(t)| f^\vee(t-x)\,dt \\
&= \int_{x-b/2}^{\infty} \sum_{k=N}^{M} |\varphi_k^{(k)}(t)| \tau_{b/2}f^\vee(t-(x-b/2))\,dt \\
&= \int \sum_{k=N}^{M} |\varphi_k^{(k)}(t)| (\tau_{b/2}f^\vee)_{|\mathbb{R}^+}(t-(x-b/2))\,dt,
\end{aligned}$$

and so we obtain (2.7).

Case 3: $x \in (a/2, b/2)$. We will prove that for all $M, N \in \mathbb{Z}$,

$$\Big|\sum_{k=N}^{M} T_{m,k} f(x)\Big| \le |2K^* f(x)| + I^{\vee}((\tau_{-b/2} f)_{|\mathbb{R}^+})(b/2 - x) + I((\tau_{a/2} f^{\vee})_{|\mathbb{R}^+})(x - a/2). \tag{2.8}$$

Indeed,

$$\begin{aligned}\Big|\sum_{k=N}^{M} T_{m,k} f(x)\Big| &= \Big|\Big(\int_{-\infty}^{\infty} - \int_{2x-a}^{\infty} - \int_{-\infty}^{2x-b}\Big)\sum_{k=N}^{M} \varphi_k^{(k)}(t) f(x-t)\,dt\Big| \\ &\le \Big|\int \sum_{k=N}^{M} \varphi_k^{(k)}(t) f(x-t)\,dt\Big| \\ &\quad + \Big|\int_{2x-a}^{\infty} \sum_{k=N}^{M} \varphi_k^{(k)}(t) f(x-t)\,dt\Big| \\ &\quad + \Big|\int_{-\infty}^{2x-b} \sum_{k=N}^{M} \varphi_k^{(k)}(t) f(x-t)\,dt\Big|,\end{aligned}$$

and as before we obtain (2.8).

Next, we estimate the L^p-norm of $\sum_{k=N}^{M} T_{m,k} f(x)$. (2.6)–(2.8) imply that

$$\begin{aligned}\Big\|\sum_{k=N}^{M} T_{m,k} f(x)\Big\|_p^p &= \Big(\int_{-\infty}^{a/2} + \int_{a/2}^{b/2} + \int_{b/2}^{\infty}\Big)\Big|\sum_{k=N}^{M} T_{m,k} f(x)\Big|^p\,dx \\ &\le \int_{-\infty}^{a/2} |I^{\vee}((\tau_{-a/2} f)_{|\mathbb{R}^+})(a/2 - x)|^p\,dx \\ &\quad + \int_{a/2}^{b/2} [2|K^* f(x)| + |I^{\vee}((\tau_{-b/2} f)_{|\mathbb{R}^+})(b/2 - x)| \\ &\quad + |I((\tau_{a/2} f^{\vee})_{|\mathbb{R}^+})(x - a/2)|^p]\,dx \\ &\quad + \int_{b/2}^{\infty} |I((\tau_{b/2} f^{\vee})_{|\mathbb{R}^+})(x - b/2)|^p\,dx.\end{aligned}$$

With a change of variables and taking account of the boundedness of I and $I^{\vee}$ on $L^p(\mathbb{R}^+)$, we conclude that the sum of the first and the last integrals is bounded by $c\|f\|_p^p$. The boundedness of K^* implies that the central term is also bounded by $c\|f\|_p^p$. ∎

LEMMA 2.9. *For* $1 < p < \infty$, *there exists* $c_p > 0$ *such that for all* $f \in \mathcal{S}(\mathbb{R})$, *for all* $a, b \in \mathbb{R}$ *with* $a < b$, *and for all functions* $m : \mathbb{R} \to \mathbb{R}$ *such* *that* $m_{|[a,b]}$ *is increasing and continuous and* $\operatorname{supp} m \subseteq [a,b]$, *we have*

$$\Big\| \sum_{k=N}^{M} T_{m,k} f \Big\|_p \le c_p \|m\|_\infty \|f\|_p \quad \text{and} \quad \|T_m f\|_p \le c_p \|m\|_\infty \|f\|_p.$$

Proof. We can choose a sequence $\{m_n\}_{n\in\mathbb{N}}$ of step functions that converges pointwise to m and such that $\|m_n\|_\infty \le 2\|m\|_\infty$. Indeed, if $\{a = t_0, t_1, \dots, t_n = b\}$ is a partition of the interval $[a,b]$, we define $m_n(x) = \sum_{j=0}^{n-1} \lambda_j \chi_{(t_j,b)}(x)$ with $\lambda_0 = m(t_1)$ and $\lambda_j = m(t_{j+1}) - m(t_j)$, $1 \le j \le n-1$.

Now, we apply Lemma 2.5 to obtain

$$\begin{aligned}\Big\| \sum_{k=N}^{M} T_{m_n,k} f \Big\|_p &\le \sum_{j=0}^{n-1} |\lambda_j| \, \Big\| \sum_{k=N}^{M} T_{\chi_{(t_j,b)},k} f \Big\|_p \\ &\le c_p(|m(t_1)| + m(b) - m(t_1)) \|f\|_p \le 3c_p \|m\|_\infty \|f\|_p.\end{aligned}$$

So, the sequence $\{m_n\}$ satisfies the hypothesis of Lemma 2.4 and the assertion follows. ∎

THEOREM 2.10. *For* $1 < p < \infty$, *there exists* $c_p > 0$ *such that for all functions* m *of bounded variation on* $\mathbb{R}$ *and for all* $f \in \mathcal{S}(\mathbb{R})$ *we have* $\|T_m f\|_p \le c_p(\|m\|_\infty + V(m)) \|f\|_p$.

Proof. Lemma 2.4 implies that the theorem follows if we check that $\| \sum_{k=N}^{M} T_{m,k} f \|_p \le c_p(\|m\|_\infty + V(m)) \|f\|_p$ for m such that $\operatorname{supp} m \subset [a,b]$ for some $a, b \in \mathbb{R}$, with c_p depending only on p.

Since m is of bounded variation on $[a,b]$ we denote by $V_{[a,t]}$ the variation of m on $[a,t]$ and we can write

$$m(t) = V_{[a,t]} m - (V_{[a,t]} m - m(t)).$$

Both $V_{[a,t]}$ and $V_{[a,t]} m - m(t)$ are increasing functions on $[a,b]$; each of them can be approximated by a sequence of continuous and increasing functions, with $\| \, \|_\infty$ bounded by $V_{[a,b]} m$ and $V_{[a,b]} m + \|m\|_\infty$ respectively. So the theorem follows from Lemma 2.9. ∎

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