COLLOQUIUM MATHEMATICUM

VOL. 72 1997 NO. 1

A REMARK ON THE NORM OF A RANDOM WALK ON SURFACE GROUPS

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We show that the norm of the random walk operator on the Cayley graph of the surface group in the standard presentation is bounded by $1/\sqrt{g}$ where g is the genus of the surface.

1. Introduction. Surface groups are classified by the genus g of the surface. They have the following presentations:

$$\Gamma_g = \{a_1, b_1, \ldots, a_g, b_g \mid a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = e\},$$

with the generating subset

$$S_g = \{a_1, a_1^{-1}, b_1, b_1^{-1}, \ldots, a_g, b_g, a_g^{-1}, b_g^{-1}\}.$$

With the pair (Γ, S) one can associate the Cayley graph (vertices $= \{\gamma : \gamma \in \Gamma\}$, edges $= \{\{\gamma, \gamma s\} : \gamma \in \Gamma,\ s \in S\}$) and consider the following random walk operator $M : l^2(\Gamma) \to l^2(\Gamma)$:

$$Mf(\gamma) = \frac{1}{\# S} \sum_{s \in S} f(\gamma s) \quad \text{ for } f \in l^2(\Gamma).$$

Our aim is to prove

THEOREM 1.

$$\|M\|_{l^2(\Gamma_g) \to l^2(\Gamma_g)} \le 1/\sqrt{g}.$$

Remarks. 1. It is interesting to compare this result with Kesten's theorem [5]: Consider the pair (Γ, S), where Γ is a discrete, finitely generated group and S is a symmetric ($S = S^{-1}$), finite generating subset. Assume that S contains no element of order 2 (which is the case for surface groups) and so one can write $S = S_+ \cup S_+^{-1}$ (disjoint union). Kesten's theorem then states that

$$\|M\|_{l^2(\Gamma_g) \to l^2(\Gamma_g)} \ge \frac{\sqrt{2n-1}}{n},$$

1991 *Mathematics Subject Classification*: Primary 60J15.

Key words and phrases: random walk, norm of operator, surface group.

where $n = \#S_+$ and equality holds if and only if Γ is a free group and S_+ is a free generating subset.

In the case of surface groups it follows that

$$\|M\|_{l^2(\Gamma_g)\to l^2(\Gamma_g)} > \frac{\sqrt{4g-1}}{2g}.$$

As

$$\lim_{g\to\infty} \frac{\frac{\sqrt{4g-1}}{2g}}{\frac{1}{\sqrt{g}}} = 1,$$

one can see that the norm $\|M\|$ and the corresponding norm for the free group are asymptotically the same.

2. In Proposition 2 it will be shown that using the method presented here the bound $1/\sqrt{g}$ can be improved slightly. However, the bound $1/\sqrt{g}$ is better than the other bounds established for this problem [1, 2] and has been chosen for aesthetic reasons.

The article is organized as follows. In Section 2 we present the method we use. In Section 3 we prove the upper bound on the norm of the random walk on surface groups.

Acknowledgments. The author is grateful to P. de la Harpe, A. Hulanicki, L. Saloff-Coste and A. Valette for some helpful discussions.

2. Presentation of the method. The upper bounds that we establish in this article are obtained using the following lemma, which is due to Gabber [3]:

LEMMA 1. *Suppose that on the set of oriented edges of the Cayley graph associated with the pair* (Γ, S), *there is a positive function* F *such that*

$$F((\gamma,\gamma s)) = \frac{1}{F((\gamma s,\gamma))}.$$

If for every vertex $\gamma \in \Gamma$ *one has*

$$\frac{1}{\#S}\sum_{s\in S} F((\gamma s,\gamma)) \le k$$

then

$$\|M\|_{l^2(\Gamma)\to l^2(\Gamma)} \le k.$$

For the proof, cf. [1], Proposition 1.

We will use the above lemma in the following way. On the set of nonoriented edges we construct an orientation and a partition into two subsets such that there are only a small number of possible configurations of oriented edges around a vertex. This partition of edges will enable us to define the function F on the edges needed in Lemma 1. The value of F on an edge will depend only on the type of the edge and its orientation.

Let us be more precise. Suppose that we are able to divide the set of non-oriented edges of the Cayley graph into two subsets—edges of type a and those of type b. Suppose further that for each non-oriented edge $\{\gamma, \gamma s\}$ we have chosen orientation, i.e. we decided whether the edge $(\gamma, \gamma s)$ starts at γ and ends at γs or the other way. For a given vertex $\gamma \in \Gamma$ consider all the edges containing γ. We will say that an edge is

- *negatively oriented* at γ if it ends at γ,
- *positively oriented* at γ if it starts at γ.

Now we define the function F on the set of oriented edges of the graph Γ in the following way. Let us fix positive numbers α and β. Then

- For the edge $(\gamma, \gamma s)$ of type a we define

$$F((\gamma, \gamma s)) = \begin{cases} \alpha & \text{if } (\gamma, \gamma s) \text{ has positive orientation at } \gamma, \\ 1/\alpha & \text{if } (\gamma, \gamma s) \text{ has negative orientation at } \gamma. \end{cases}$$

- For the edge $(\gamma, \gamma s)$ of type b we define

$$F((\gamma, \gamma s)) = \begin{cases} \beta & \text{if } (\gamma, \gamma s) \text{ has positive orientation at } \gamma, \\ 1/\beta & \text{if } (\gamma, \gamma s) \text{ has negative orientation at } \gamma. \end{cases}$$

As at each vertex $\gamma \in \Gamma$ there are only a few possible configurations of edges of type a and b and their orientations, it will be possible to find positive numbers α and β so that

$$\frac{1}{\#S} \sum_{s \in S} F((\gamma s, \gamma))$$

is small for all $\gamma \in \Gamma$.

We will apply this method to the surface groups Γ_g. It will be shown that one can divide the set of non-oriented edges of the Cayley graph of the group Γ_g into two disjoint subsets, say type a and type b, and choose an orientation of the edges so that:

At each vertex γ there are only 3 *possible configurations of edges* (see Figures 1.1–1.3 where the edges of type a are represented as thick lines and edges which are positively oriented are drawn above γ):

- *There are two edges of type a. These edges have a positive orientation and the other edges are of type b and have a negative orientation at γ* (see Fig. 1.1).
- *All edges are of type b and all except one have a negative orientation at γ* (see Fig. 1.2).

- *There is one edge of type a. This edge has a negative orientation. The other edges are of type b and all except one have a negative orientation at γ* (see Fig. 1.3).

Now we define F on the set of oriented edges of the Cayley graph of Γ_g in the following way:

- For the edge $(\gamma,\gamma s)$ of type a we define

$$F((\gamma,\gamma s))=\begin{cases}\sqrt{g} & \text{if } (\gamma,\gamma s) \text{ has positive orientation at } \gamma,\\ 1/\sqrt{g} & \text{if } (\gamma,\gamma s) \text{ has negative orientation at } \gamma.\end{cases}$$

- For the edge $(\gamma,\gamma s)$ of type b we define

$$F((\gamma,\gamma s))=\begin{cases}\dfrac{2g-1}{\sqrt{g}} & \text{if } (\gamma,\gamma s) \text{ has positive orientation at } \gamma,\\ \dfrac{\sqrt{g}}{2g-1} & \text{if } (\gamma,\gamma s) \text{ has negative orientation at } \gamma.\end{cases}$$

For all the situations described above we obtain

$$\frac{1}{\#S}\sum_{s\in S}F((\gamma s,\gamma))\le\frac{1}{\sqrt{g}}.$$

This implies that

$$\|M\|_{l^2(\Gamma_g)\to l^2(\Gamma_g)}\le\frac{1}{\sqrt{g}},$$

which is the bound given in Theorem 1.

The required partition of the edges will be obtained in the next section through a geometric approach.

3. Partition of the edges for surface groups. It is well known that surface groups are isomorphic to subgroups of the group $PSL(2,\mathbb{R})=\{A\in M(2\times 2):\det A=1\}/\{\pm\mathrm{Id}\}$. This is the group of isometries of the hyperbolic upper half-plane, i.e. $H=\{x+iy\in\mathbb{C}:x\in\mathbb{R},\ y\in\mathbb{R}_+\}$ with the riemannian metric $\sqrt{dx^2+dy^2}/y$. It acts on H by homographies, i.e. for $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)$ and $z\in H$,

$$\gamma(z)=\frac{az+b}{cz+d}.$$

Now we can embed the Cayley graph of Γ into the hyperbolic plane H. Namely, for an embedding $\varphi:\Gamma_g\hookrightarrow PSL(2,\mathbb{R})$ the set of vertices in H will be $\{\gamma(i):\gamma\in\Gamma_g\}$. To do this we recall some classical properties of Γ_g.

For any φ the action of Γ_g on H is proper, discontinuous and co-compact and one can find φ with the following properties:

- the fundamental domain $\mathcal{F}_g$ of Γ_g,
$$\mathcal{F}_g = \{z \in H : d(z,i) < \inf_{e\neq\gamma\in\Gamma_g} d(\gamma(z),i)\},$$
is a regular hyperbolic $4g$-gon,
- for $\gamma \neq \gamma'$ the points $\gamma(i)$ and $\gamma'(i)$ do not lie on the same horizontal line (horocycle, i.e. the set $\{z \in H : \Im(z) = \mathrm{const}\}$).

The first property is well known and the second can be obtained by conjugating φ. We fix φ with the above properties and will be referring to elements of Γ_g as isometries of H.

LEMMA 2. *The hyperbolic distance D between two neighbouring vertices in the Cayley graph of Γ_g in H is equal to*

$$D = 2\operatorname{arccosh}\left(\cot\frac{\pi}{4g}\right). \tag{1}$$

Proof. The fundamental domain $\mathcal{D}$ is a regular hyperbolic $4g$-gon with inner angles equal to $\pi/(2g)$. Let A and A' be two neighboring vertices of $\mathcal{D}$ and let B be the middle of AA'. Consider the triangle ABC where C is the center of $\mathcal{D}$. This triangle has angles $\alpha = \pi/(4g)$, $\beta = \pi/2$, $\gamma = \pi/(4g)$ and one side equal to $c = D/2$ (c is opposite to γ). To get (1) we apply the cosine rule [4]:

$$\cosh c = \frac{\cos\alpha\cos\beta + \cos\gamma}{\sin\alpha\sin\beta}.$$

to the triangle ABC. ∎

LEMMA 3. *For any vertex $\gamma(i)$ the number of its neighbors that lie above the horizontal line* (*horocycle*) *passing through $\gamma(i)$ is equal to* 1 *or* 2.

Proof. Without loss of generality we can suppose that $\gamma = e$ (i.e. $\gamma(i) = i$). The situation for which the number of neighbors above the horocycle is maximal or minimal is when one of the neighbors is on the horocycle. This is because by slightly rotating the neighbors of γ around γ we can alter the number of those that are above the horocycle. By symmetry we can suppose that the neighbor $s_1(i)$ ($s_j \in S_g$) lying on the horocycle is on the left hand side of i, i.e. it is the point $-a + i$ ($a > 0$). Using the formula (see [4])

$$\cosh\left(\frac{1}{2}\operatorname{dist}(z,w)\right) = \frac{|z-\overline{w}|}{2\sqrt{\Im(z)\Im(w)}}$$

for $z = i$ and $w = -a + i$ it follows that

$$a = 2\sqrt{\cot^2\frac{\pi}{4g} - 1}.$$

If we want to know the position of the next neighbor $s_2(i)$ of i clockwise to $s_1(i)$ we rotate the point $s_1(i)=-a+i$ around i by $\pi/(2g)$, i.e. we apply to $-a+i$ the isometry

$$\begin{pmatrix} \cos\frac{\pi}{4g} & -\sin\frac{\pi}{4g} \\ \sin\frac{\pi}{4g} & \cos\frac{\pi}{4g} \end{pmatrix}. \tag{2}$$

Therefore the imaginary part of $s_2(i)$ equals

$$\Im(s_2(i)) = \frac{1}{1+4\big(\cot^2\frac{\pi}{4g}-1\big)\sin^2\frac{\pi}{4g} - 4\sqrt{\cot^2\frac{\pi}{4g}-1}\,\sin\frac{\pi}{4g}\cos\frac{\pi}{4g}}. \tag{3}$$

We now want to show that

$$\Im(s_2(i)) > 1.$$

Since $0<\frac{\pi}{4g}<\frac{\pi}{4}$ we know that $\cos\frac{\pi}{4g}>\sin\frac{\pi}{4g}>0$ and the following holds:

$$\begin{aligned}
1+4\bigg(\cot^2\frac{\pi}{4g}-1\bigg)\sin^2\frac{\pi}{4g} &- 4\sqrt{\cot^2\frac{\pi}{4g}-1}\,\sin\frac{\pi}{4g}\cos\frac{\pi}{4g} \\
&= 1+4\sqrt{\cot^2\frac{\pi}{4g}-1}\,\sin\frac{\pi}{4g}\bigg(\sqrt{\cos^2\frac{\pi}{4g}-\sin^2\frac{\pi}{4g}}-\cos\frac{\pi}{4g}\bigg) \\
&< 1+4\sqrt{\cot^2\frac{\pi}{4g}-1}\,\sin\frac{\pi}{4g}\bigg(\sqrt{\cos^2\frac{\pi}{4g}}-\cos\frac{\pi}{4g}\bigg) = 1.
\end{aligned}$$

This shows that the quantity (3) is greater than 1 and therefore that $s_2(i)$ lies above the horocycle.

Now we want to find the position of the next neighbor $s_3(i)$. To do this we rotate the point $s_1(i)=-a+i$ around i by π/g, i.e.

$$s_3(i) = \begin{pmatrix} \cos\frac{\pi}{2g} & -\sin\frac{\pi}{2g} \\ \sin\frac{\pi}{2g} & \cos\frac{\pi}{2g} \end{pmatrix}(-a+i).$$

Therefore the imaginary part of $s_3(i)$ is

$$\Im(s_3(i)) = \frac{1}{1+4\big(\cot^2\frac{\pi}{4g}-1\big)\sin^2\frac{\pi}{2g} - 4\sqrt{\cot^2\frac{\pi}{4g}-1}\,\sin\frac{\pi}{2g}\cos\frac{\pi}{2g}}.$$

We aim to show that

$$\Im(s_3(i)) < 1.$$

As $0<\frac{\pi}{2g}<\frac{\pi}{4}$ we have $\sin\frac{\pi}{2g}>\sin\frac{\pi}{4g}>0$ and therefore

$$1+4\bigg(\cot^2\frac{\pi}{4g}-1\bigg)\sin^2\frac{\pi}{2g} - 4\sqrt{\cot^2\frac{\pi}{4g}-1}\,\sin\frac{\pi}{2g}\cos\frac{\pi}{2g}$$

$$= 1 + 4\sqrt{\cot^2 \frac{\pi}{4g} - 1}\,\sin\frac{\pi}{2g}\left(\cos\left(\frac{\pi}{2g}\right)\frac{\sin\frac{\pi}{2g}}{\sin\frac{\pi}{4g}} - \cos\frac{\pi}{2g}\right)$$

$$> 1 + 4\sqrt{\cot^2 \frac{\pi}{4g} - 1}\,\sin\frac{\pi}{2g}\left(\cos\frac{\pi}{2g} - \cos\frac{\pi}{2g}\right) = 1.$$

Therefore $s_3(i)$ lies beneath the horocycle. Since all the other neighbors must lie below the horocycle as well, the lemma is proved. ■

Lemma 3 enables us to divide the set of edges into a disjoint union of edges of types a and b. Any non-oriented edge $\{x, y\}$ can be written uniquely as $\{\gamma s, \gamma\}$ with $\Im(\gamma s(i)) > \Im(\gamma(i))$ and we say that this edge is of

- *type* a if there exists only one $s' \in S_g$ $(s' \neq s)$ with $\Im(\gamma s'(i)) > \Im(\gamma(i))$,
- *type* b if there is no $s' \in S_g$ $(s' \neq s)$ with $\Im(\gamma s'(i)) > \Im(\gamma(i))$.

We now want to prove that there are only three possible configurations of edges of types a and b around a vertex.

The three figures below represent the only three possible configurations around the vertex $\gamma(i)$. Edges of type a and b are represented by thick and thin lines respectively.

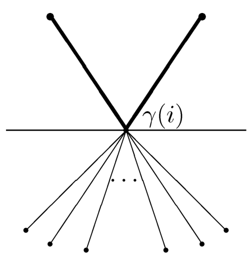

Fig. 1.1

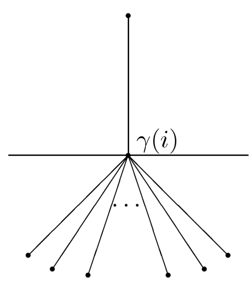

Fig. 1.2

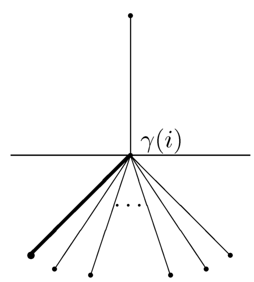

Fig. 1.3

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Proposition 1. *For a given vertex $\gamma(i)$ in the Cayley graph of Γ_g only one of the following can happen*:

1. *There are only two edges $\{\gamma s(i), \gamma(i)\}$,$\{\gamma s'(i), \gamma(i)\}$ such that $\Im(\gamma s(i)) > \Im(\gamma(i))$ and $\Im(\gamma s'(i)) > \Im(\gamma(i))$. These two edges are of type a and the other edges ending at $\gamma(i)$ are of type b* (see Fig. 1.1).
2. *There is only one edge $\{\gamma s(i), \gamma(i)\}$ such that $\Im(\gamma s(i)) > \Im(\gamma(i))$. This edge is of type b. The other edges ending at $\gamma(i)$ are either*

   a) *all of type b* (see Fig. 1.2), *or*

   b) *all of type b apart from one which is of type a* (see Fig. 1.3).

P r o o f o f P r o p o s i t i o n 1. We can assume without loss of generality that $\gamma = e$, i.e. $\gamma(i) = i$. Proposition 1 will be a consequence of the lemmas that follow.

LEMMA 4. *Suppose that* $\Im(e) > \Im(s)$. *If the edge* $\{e, s\}$ *is of type* a *then, for any* $s' \in S_g$ *such that*

$$\Im(e) > \Im(s') \quad \textit{and} \quad \Im(s') \geq \Im(s),$$

the edge $\{e, s'\}$ *is of type* a *as well.*

P r o o f. By symmetry we can suppose that $\Re(s) \geq 0$. If the edge $\{e, s\}$ is of type a then there must be an $s'' \in S_g$ such that $\Im(ss'') > \Im(s)$.

We will suppose that $\Re(s') \geq 0$; the proof in the case $\Re(s') \leq 0$ is the same.

Let R_α be the rotation by the angle α about $i = e$. Choose θ such that $R_\theta(s) = s'$. In order to prove that the edge $\{e, s'\}$ is of type b we must show that $\Im(R_\theta(ss'')) > \Im(s' = R_\theta(s))$.

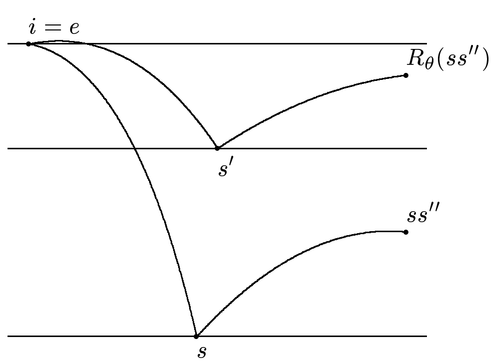
To see that, we imagine ourselves at the point $R_{t\theta}(s)$ ($t \in [0, 1]$). As we perform the rotations $R_{t\theta}$ the point $R_{t\theta}(ss'')$ is being rotated around $R_{t\theta}(s)$. The function

$$t \to \Im(R_{t\theta}(ss'')) - \Im(R_{t\theta}(s))$$

is then increasing.

As $\Im(ss'' = R_0(ss'')) > \Im(s = R_0(s))$ it follows that

$$\Im(R_\theta(ss'')) > \Im(s' = R_\theta(s)).$$



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LEMMA 5. *Suppose there is only one* $s \in S_g$ *such that* $\Im(s) > \Im(e)$. *Then there are no* $s', s'' \in S_g$ *which satisfy*

$$\Re(s') \geq 0, \qquad \Re(s'') \leq 0$$

such that both edges $\{e, s'\}, \{e, s''\}$ *are of type* a.

P r o o f. Suppose the contrary, i.e. that there are $s', s'' \in S_g$ such that $\Re(s') \geq 0$, $\Re(s'') \leq 0$ and that the edges $\{e, s'\}$, $\{e, s''\}$ are of type a.

By Lemma 4 we can suppose that the edges $\{e, s'\}$, $\{e, s''\}$ are respectively the right and left neighbors of the edge $\{e, s\}$. Let R_θ be the rotation around i such that $\Re(R_\theta(s)) = 0$ and $\Im(R_\theta(s)) > 1$. Again from Lemma 4 it follows that the edges $\{e, R_\theta(s')\}$ and $\{e, R_\theta(s'')\}$ are of type a.

This means that we only need to consider the situation where $\Re(s) = 0$ and the edges $\{e, s'\}$ and $\{e, s''\}$ are the right and left neighbors of $\{e, s\}$. Lemma 4 ensures that $\Im(s') = \Im(s'') < \Im(i = e)$. We have supposed that $\{e, s'\}$ is of type a, which means that there is an $s''' \in S_g$ such that $\Im(s's''') > \Im(s')$. The edge $\{s', s's'''\}$ is necessarily the right neighbor of $\{e, s'\}$.

Now let us rotate the points γ ($\gamma \in \Gamma$) about s' by the rotation R_ψ such that

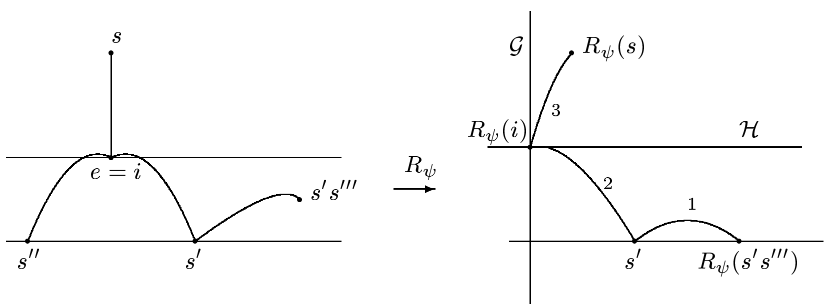
$$\Im(s') = \Im(R_\psi(s's''')) \quad \text{and} \quad \Re(s') < \Re(R_\psi(s's''')).$$

As we have supposed that

$$\Im(s') < \Im(s's''')$$

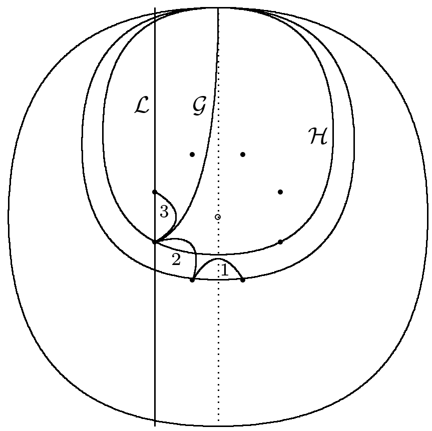
it follows that

$$\Re(R_\psi(s)) > \Re(R_\psi(e)). \tag{4}$$



We will show that (4) is impossible, which provides the required contradiction. To this end consider the disc model $\mathcal{D}$ of the hyperbolic space. Choose an isometry between the upper half-plane and the disc in such a way that $R_\psi(s)$, $R_\psi(i)$, s' and $R_\psi(s's''')$ are the vertices of a regular hyperbolic $4g$-gon $\mathcal{F}$ centered at the center of $\mathcal{D}$.

Let $\mathcal{G}$ be the geodesic orthogonal to the horocycle $\mathcal{H} = \{z \in H : \Im(z) = \Im(R_\psi(i))\}$ which passes through $R_\psi(i)$. The inequality (4) states that $R_\psi(s)$ is on the right hand side of $\mathcal{G}$. In the disc model $\mathcal{D}$ let us draw the Euclidean line $\mathcal{L}$ passing through $R_\psi(i)$ parallel (in the Euclidean sense) to the line which goes through the center of $\mathcal{D}$ and the point at which $\mathcal{H}$ is tangent to $\mathcal{D}$. As $R_\psi(s)$ is on the right hand side of $\mathcal{G}$ it must be on the right hand side of $\mathcal{L}$. However, as the number of vertices of $\mathcal{F}$ is at least 8, elementary Euclidean considerations show that $R_\psi(s)$ cannot be on the right hand side of $\mathcal{L}$.



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LEMMA 6. *Suppose there is only one $s \in S_g$ such that $\Im(s) > \Im(e)$. Then there are no $s', s'' \in S_g$ such that either*

$$\Re(s') \geq 0 \quad \textit{and} \quad \Re(s'') \geq 0$$

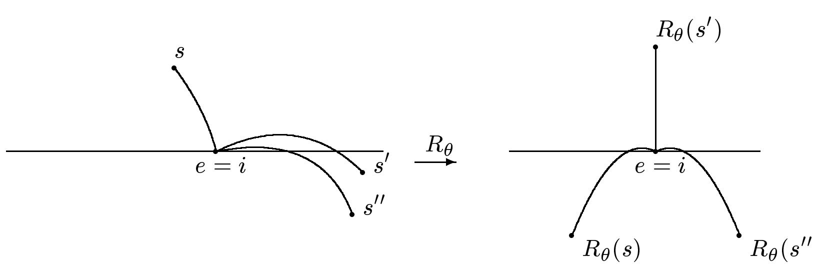
or

$$\Re(s') \leq 0 \quad \textit{and} \quad \Re(s'') \leq 0$$

and such that both edges $\{e, s'\}, \{e, s''\}$ are of type a.

Proof. Suppose the contrary, i.e. that there exist s', s'' such that the edges $\{e, s'\}$, $\{e, s''\}$ are of type b and $\Re(s') \geq 0$, $\Re(s'') \geq 0$ (the proof in the other case is the same). By Lemma 4 we can suppose that s' and s'' have the greatest imaginary parts of all the neighbors of e which lie below e and which have a positive real part.

Let R_θ be the rotation around i such that $\Re(R_\theta(s')) = 0$ and $\Im(R_\theta(s')) > 1$. As $\Im(R_\theta(s)) = \Im(R_\theta(s''))$, from Lemma 3, $\Im(R_\theta(s'')) < 1$. From Lemma 4 we know that the edge $\{e, R_\theta(s'')\}$ is of type a and by symmetry that the edge $\{e, R_\theta(s)\}$ is of type a as well. This contradicts Lemma 5.



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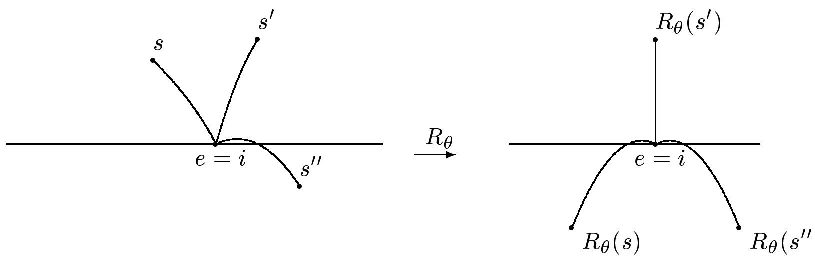
LEMMA 7. *Suppose there are $s, s' \in S_g$ such that $\Im(s) > \Im(e)$ and $\Im(s') > \Im(e)$, which means that the edges $\{e, s\}$ and $\{e, s'\}$ are of type a. Then all other edges $\{e, s''\}$ for $s'' \in S_g$ are of type b.*

Proof. Suppose the contrary, i.e. there exist s, s' such that $\Im(s) > \Im(i)$ and $\Im(s') > \Im(i)$ and there exists another s'' such that the edge $\{e, s''\}$ is of type a. The real parts of s and s' must be of different signs because otherwise there would be another point s''' with real part of opposite sign and $\Im(s''') > \Im(i)$, which would contradict Lemma 3. So suppose that $\Re(s') \geq 0$ and $\Re(s) \leq 0$. By symmetry we can suppose that $\Re(s'') \geq 0$, and by Lemma 4 that the edge $\{e, s''\}$ is the right neighbor of $\{e, s'\}$.

Let R_θ be the rotation around $e = i$ such that $\Re(R_\theta(s')) = 0$ and $\Im(R_\theta(s')) > 1$. As $\Im(R_\theta(s'')) = \Im(R_\theta(s))$ it follows by Lemma 3 that $\Im(R_\theta(s'')) = \Im(R_\theta(s)) < 1$.

From Lemma 4 one can see that the edge $\{e, R_\theta(s'')\}$ is of type a and by symmetry so is $\{e, R_\theta(s)\}$. This contradicts Lemma 5.



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This ends the proof of Proposition 1. ■

3.1. *The upper bound.* With the properties we have established one can slightly improve Theorem 1, i.e. we can prove:

PROPOSITION 2.

$$\|M\|_{l^2(\Gamma_g)\to l^2(\Gamma_g)} \leq \frac{2c + (4g-2)c/(2c^2-1)}{4g}$$

where

$$c = \frac{\sqrt{2g+1+\sqrt{(2g+1)^2+8g-8}}}{2}.$$

Proof. The proof differs from the proof of Theorem 1 given in Section 2 only through the use of different values of F on the edges of type a and b defined by geometrical considerations. Namely we minimize the expression

$$\frac{1}{\#S}\sum_{s\in S} F((\gamma s, \gamma))$$

over the possible values of F on oriented edges of type a and b. We have to minimize it for the situations represented in Figures 1.1–1.3. When we do this we find that the minimum of the above expression is attained for the following values of F:

1. For edges of type a:
   - if $\Im(\gamma s) < \Im(\gamma)$ then $F((\gamma s, \gamma)) = c^{-1}$,
   - if $\Im(\gamma s) > \Im(\gamma)$ then $F((\gamma s, \gamma)) = c$.
2. For edges of type b:
   - if $\Im(\gamma s) < \Im(\gamma)$ then $F((\gamma s, \gamma)) = 1/(2c - c^{-1})$,
   - if $\Im(\gamma s) > \Im(\gamma)$ then $F((\gamma s, \gamma)) = 2c - c^{-1}$.

It follows then that for all possible situations around the vertex γ, represented in Figures 1.1–1.3, we have

$$\frac{1}{\#S}\sum_{s\in S} F((\gamma s, \gamma)) \le \frac{2c + (4g-2)c/(2c^2-1)}{4g},$$

which is the value given in Proposition 2. ∎

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Received 1 August 1996