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ASYMPTOTIC PROPERTIES OF STOCHASTIC SEMILINEAR EQUATIONS BY THE METHOD OF LOWER MEASURES

BY

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Introduction. The aim of the paper is to establish the convergence of probability laws of solutions of certain infinite-dimensional stochastic differential equations in the strong (variational) norm. This type of convergence has been previously studied in connection with investigation of ergodic and mixing properties of autonomous stochastic evolution equations. In the simplest case of a reaction-diffusion equation perturbed by a space-time white noise the strong law of large numbers and the strong mixing have been established by Maslowski [28] and Manthey and Maslowski [25] by a method going back essentially to Khas'minskiĭ [19], which consists in proving topological irreducibility and the strong Feller property for the induced Markov process. These results have been extended by Da Prato, Elworthy and Zabczyk [4], Maslowski [26] and, recently, by Chojnowska-Michalik and Gołdys [2] by means of suitable technical tools like the Elworthy formula and the mild backward Kolmogorov equation. Analogous results have been obtained by Da Prato and Gątarek [5] for the stochastic Burgers equation, by Da Prato and Debussche [3] for the stochastic Cahn–Hilliard equation and by Flandoli and Maslowski [9] for the two-dimensional stochastic Navier–Stokes equation.

The ergodicity for stochastic semilinear equations with a multiplicative noise term was established by Peszat and Zabczyk [30] and further extended by Gątarek and Gołdys [14]. The case of σ-finite invariant measures and related recurrence properties have been studied in Maslowski and Seidler [29] and, in a more general setting, in Seidler [32]. An alternative method based on a more direct verification of the geometric ergodicity has been

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developed by Jacquot and Royer [16], [17], and applied, for instance, to a two-dimensional stochastic parabolic equation of the 4th order (stochastic plate equation).

Most of the ergodic results mentioned above are based on the verification of the topological irreducibility and the strong Feller property. The method used in the present paper is different. Its idea comes basically from the theory of deterministic discrete-time dynamical systems (cf. Lasota [21], Lasota and Yorke [23], and Lasota and Mackey [22]) and it has been previously applied by Maslowski [27] to finite-dimensional stochastic differential equations. It allows us to establish the convergence in the variational norm of (in general, time-inhomogeneous) Markov evolution systems, and, in the time-homogeneous case, to prove the existence of an invariant measure. In comparison with the above quoted ergodicity and mixing results, there are, in some sense, more restrictive requirements on the nonlinear term of the equation in our case. On the other hand, the equation is allowed be nonautonomous (i.e., the induced Markov process need not be homogeneous). Also, in some cases, the speed of convergence can be estimated.

The paper is divided into three sections. Section 1 includes definitions and general results on lower measures for evolution systems of Markov operators. A general statement on convergence of the evolution system of Markov operators under the assumption of existence of a system of lower measures (the so-called l-condition, Theorem 1.4) is quoted in the form proved in [27]. In Theorem 1.5 a general estimate on the speed of convergence is established. Proposition 1.7 is in fact a corollary of Theorem 1.4 covering the case of some more particular Markov operators that are studied in Section 2.

In Section 2, the general theory is applied to the Markov process induced by an infinite-dimensional stochastic equation. The main result is contained in Theorem 2.6 where the variational convergence of the adjoint Markov evolution system and, in the autonomous case, the existence and uniqueness of the invariant measure are established. Corollaries 2.7 and 2.8 are further specializations simplifying the assumptions of Theorem 2.6. Propositions 2.2 and 2.4 provide verification of general assumptions of Proposition 1.7 in the concrete case of the the equation (2.1). At the end of Section 2 three examples are given: An equation of the form (2.1) with a more particular nonlinear term in the drift (Example 2.10), a stochastic nonautonomous semilinear parabolic equation (Example 2.11) and a stochastic integrodifferential equation (Example 2.12).

Section 3 contains the proof of the crucial Proposition 2.2. It provides a lower bound on the density of the transition probability of the Markov process induced by the equation (2.1) with respect to the corresponding Gaussian transition probability and could be of independent interest.

For Banach spaces Y, Z we denote by $\mathcal{L}(X,Z)$ the space of bounded linear operators $Y \to Z$, $\mathcal{L}(Y) := \mathcal{L}(Y,Y)$, and by $\mathbf{C}(Y,Z)$ the space of continuous functions $Y \to Z$. The symbol $\mathcal{D}(A)$ stands for the domain of the operator A. More notation is introduced at the beginning of Sections 1 and 2.

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1. The general method of lower measures. Let $(\mathbf{X},\mathcal{F})$ be a measurable space and denote by $\mathcal{M}$, $\mathcal{M}_+$ and $\mathcal{P}$, respectively, the sets of finite real, finite nonnegative and probability measures on $\mathcal{F}$. We consider the usual linear structure on $\mathcal{M}$.

1.1. Definition. A linear mapping $\mathcal{S} : \mathcal{M} \to \mathcal{M}$ is called a *Markov operator* if $\mathcal{S}(\mathcal{P}) \subset \mathcal{P}$.

1.2. Definition. A system $\Sigma = \{\mathcal{S}_{s,t} : 0 \le s \le t < \infty\}$ is called an *evolution system of Markov operators* if each $\mathcal{S}_{s,t}$ is a Markov operator, and $\mathcal{S}_{u,t}\mathcal{S}_{s,u} = \mathcal{S}_{s,t}$ for every $0 \le s \le u \le t < \infty$.

For $\nu \in \mathcal{M}$ denote by $\|\nu\|$, ν^+ and ν^-, respectively, the total, positive and negative variations of ν. We put simply $\|\nu\| = \|\nu\|(\mathbf{X})$. Note that $\|\nu^+\| = \nu^+(\mathbf{X})$ and $\|\nu^-\| = \nu^-(\mathbf{X})$.

1.3. Definition. A system $\{\mu_s : s \in \mathbb{R}_+\} \subset \mathcal{M}_+$ is called a *system of lower measures* (with respect to Σ) if

$$\inf_{s\in\mathbb{R}_+} \mu_s(\mathbf{X}) > 0 \tag{1.1}$$

and

$$\|(\mathcal{S}_{s,t}\nu - \mu_s)^-\| \to 0, \qquad t \to \infty, \tag{1.2}$$

for every $s \ge 0$ and $\nu \in \mathcal{P}$. If $\mu_s \equiv \mu$ does not depend on s, then μ is called a *lower measure* (with respect to Σ).

The condition (1.2) is sometimes called the *l-condition* (cf. [21], [27]).

1.4. Theorem. (a) *If there exists a system of lower measures with respect to Σ then*

$$\|\mathcal{S}_{s,t}\nu_1 - \mathcal{S}_{s,t}\nu_2\| \to 0, \qquad t \to \infty, \tag{1.3}$$

for every $s \ge 0$ and $\nu_1, \nu_2 \in \mathcal{P}$.

(b) *Assume that the system Σ is homogeneous, i.e., $\mathcal{S}_{s,t} = \mathcal{S}_{s+h,t+h}$ does not depend on $h \ge 0$ for $0 \le s \le t < \infty$, and set $\mathcal{S}_t := \mathcal{S}_{0,t}$. Then there exists a lower measure with respect to Σ iff there exists an invariant measure*

$\mu^* \in \mathcal{P}$ with respect to Σ (i.e., $\mathcal{S}_t\mu^* = \mu^*$ for $t \geq 0$) and

$$\|\mathcal{S}_t\nu - \mu^*\| \to 0, \quad t \to \infty, \tag{1.4}$$

for every $\nu \in \mathcal{P}$.

The proof of Theorem 1.4 is based on an idea due to Lasota and Yorke [23], [21], and in the present setting is contained in [27].

1.5. THEOREM. *Let $\Sigma = \{\mathcal{S}_{s,t} : 0 \leq s \leq t < \infty\}$ be an evolution system of Markov operators and let there exist a system $\{\mu_s : s \in \mathbb{R}_+\} \subseteq \mathcal{M}_+$ such that*

$$\lambda = \inf_{s\geq 0} \mu_s(\mathbf{X}) > 0.$$

Let $\nu_1, \nu_2 \in \mathcal{P}$, $\varepsilon \in [0, 1/2]$ and $\alpha \in (0, 1/2)$ and assume that there exists a set $B \in \mathcal{F}$ such that

(i) *$\mathcal{S}_{\sigma,t}\nu_1(\mathbf{X} \setminus B) \leq \varepsilon$ and $\mathcal{S}_{\sigma,t}\nu_2(\mathbf{X} \setminus B) \leq \varepsilon$ for every $0 \leq \sigma \leq t < \infty$, and*

(ii) *there exists $\tau \geq 0$ such that for every $\mu \in \mathcal{P}$ with $\mu(B) = 1$, and $\sigma \geq 0$, we have*

$$\|(\mathcal{S}_{\sigma,\sigma+\tau}\mu - \mu_\sigma)^-\| < \lambda\alpha.$$

Then

$$\|\mathcal{S}_{s,t}\nu_1 - \mathcal{S}_{s,t}\nu_2\| \leq q_\alpha^n\|\nu_1 - \nu_2\| + 12\varepsilon\sum_{i=0}^{n} q_\alpha^i \leq q_\alpha^n\|\nu_1 - \nu_2\| + \frac{12\varepsilon}{1 - q_\alpha} \tag{1.5}$$

for every $s \geq 0$, $t \geq n\tau + s$ and $n \in \mathbb{N}$, where $q_\alpha := 1 - (1 - 2\alpha)\lambda \in (0, 1)$.

P r o o f. Assume that

$$\|\mathcal{S}_{s,s+n\tau}(\nu_1 - \nu_2)\| \leq q_\alpha^n\|\nu_1 - \nu_2\| + 12\varepsilon\sum_{i=1}^{n} q_\alpha^i + 6\varepsilon \tag{1.6}$$

for some $n \in \mathbb{N}$. Our aim is to show (1.6) with n replaced by $n + 1$. Set

$$\mu_1(A) = c_1\mathcal{S}_{s,s+n\tau}\nu_1(A \cap B), \quad \mu_2(A) = c_2\mathcal{S}_{s,s+n\tau}\nu_2(A \cap B), \quad A \in \mathcal{F},$$

where $c_1 = (\mathcal{S}_{s,s+n\tau}\nu_1(B))^{-1}$ and $c_2 = (\mathcal{S}_{s,s+n\tau}\nu_2(B))^{-1}$ are normalizing constants. Since μ_1 and μ_2 are probability measures we have

$$\eta := \|\mu^+\| = \|\mu^-\| = \tfrac{1}{2}\|\mu\|$$

for $\mu = \mu_1 - \mu_2$. Without loss of generality we can assume $\eta > 0$. Note that

$$\|\mathcal{S}_{s,s+n\tau}\nu_1 - \mu_1\| + \|\mathcal{S}_{s,s+n\tau}\nu_2 - \mu_2\| \leq 2\left(\varepsilon + \frac{\varepsilon}{1 - \varepsilon}\right) \leq 6\varepsilon \tag{1.7}$$

by (i). Furthermore, we have

$$\|\mathcal{S}_{s+n\tau,s+(n+1)\tau}\mu\| = \eta\|(\mathcal{S}_{s+n\tau,s+(n+1)\tau}\eta^{-1}\mu^{+} - \mu_{s+n\tau}) - (\mathcal{S}_{s+n\tau,s+(n+1)\tau}\eta^{-1}\mu^{-} - \mu_{s+n\tau})\|. \tag{1.8}$$

Since $\eta^{-1}\mu^{+} \in \mathcal{P}$, $\eta^{-1}\mu^{-} \in \mathcal{P}$ and $\eta^{-1}\mu^{+}(\mathbf{X} \setminus B) = \eta^{-1}\mu^{-}(\mathbf{X} \setminus B) = 0$, we obtain

$$\|(\mathcal{S}_{s+n\tau,s+(n+1)\tau}\eta^{-1}\mu^{+} - \mu_{n\tau+s})^{-}\| < \lambda\alpha$$

and

$$\|(\mathcal{S}_{s+n\tau,s+(n+1)\tau}\eta^{-1}\mu^{-} - \mu_{n\tau+s})^{-}\| < \lambda\alpha.$$

Therefore,

$$\begin{aligned}\|\mathcal{S}_{s+n\tau,s+(n+1)\tau}\eta^{-1}\mu^{+} - \mu_{n\tau+s}\| &\le \mathcal{S}_{s+n\tau,s+(n+1)\tau}\eta^{-1}\mu^{+}(\mathbf{X}) - \mu_{s+n\tau}(\mathbf{X}) \\ &\quad + 2\|(\mathcal{S}_{s+n\tau,s+(n+1)\tau}\eta^{-1}\mu^{+} - \mu_{s+n\tau})^{-}\| \\ &\le 1 - \lambda + 2\alpha\lambda = q_\alpha,\end{aligned}$$

and similarly, $\|\mathcal{S}_{s+n\tau,s+(n+1)\tau}\eta^{-1}\mu^{-} - \mu_{s+n\tau}\| \le q_\alpha$. By (1.8) this yields

$$\|\mathcal{S}_{s+n\tau,s+(n+1)\tau}\mu\| \le 2\eta q_\alpha = \|\mu\| q_\alpha,$$

and by (1.7) it follows that

$$\begin{aligned}\|\mathcal{S}_{s,s+(n+1)\tau}(\nu_1 - \nu_2)\| &\le \|\mathcal{S}_{s+n\tau,s+(n+1)\tau}(\mathcal{S}_{s,s+n\tau}(\nu_1 - \nu_2))\| \\ &\le \|\mathcal{S}_{s+n\tau,s+(n+1)\tau}\mu\| + 6\varepsilon \le q_\alpha\|\mu\| + 6\varepsilon \\ &\le q_\alpha(\|\mathcal{S}_{s,s+n\tau}(\nu_1 - \nu_2)\| + 6\varepsilon) + 6\varepsilon \\ &\le q_\alpha\Big(q_\alpha^n\|\nu_1 - \nu_2\| + 12\varepsilon\sum_{i=1}^{n} q_\alpha^i + 6\varepsilon + 6\varepsilon\Big) + 6\varepsilon \\ &\le q_\alpha^{n+1}\|\nu_1 - \nu_2\| + 12\varepsilon\sum_{i=1}^{n+1} q_\alpha^i + 6\varepsilon.\end{aligned}$$

Thus we have obtained (1.6) with n replaced by $n+1$ and since for $n = 0$, (1.6) holds evidently, it is satisfied for all $n \in \mathbb{N}$ by induction. From the definition of a Markov operator it easily follows that $\mathcal{S}_{s,r} : (\mathcal{M}, \|\cdot\|) \to (\mathcal{M}, \|\cdot\|)$ is bounded for any $0 \le s \le r < \infty$, with the operator norm less than or equal to one. Therefore,

$$\|\mathcal{S}_{s,t}(\nu_1 - \nu_2)\| = \|\mathcal{S}_{s+n\tau,t}\mathcal{S}_{s,s+n\tau}(\nu_1 - \nu_2)\| \le \|\mathcal{S}_{s,s+n\tau}(\nu_1 - \nu_2)\|$$

for any $t \ge s + n\tau$, which concludes the proof. ■

1.6. Remark. The preceding theorem gives a possibility to estimate the speed of convergence in (1.3), (1.4) in the case when, given any $\varepsilon \ge 0$ and $\nu_1, \nu_2 \in \mathcal{P}$, we are able to find the set B and to estimate the value of τ from above. Usually, B is a "large" ball in a suitable state space, in which case the conditions (i) and (ii) of Theorem 1.5 are a kind of boundedness in probability and a "locally uniform" l-condition, respectively (see Remark 2.9 below).

If Σ is homogeneous and $\varepsilon = 0$ then Theorem 1.5 tells us that Σ is geometrically ergodic.

In the rest of the section we consider the case when the family of operators $\Sigma = \{\mathcal{S}_{s,t}\}$ is a two-parameter adjoint Markov evolution system corresponding to a Markov process in $\mathbf{X}$. More precisely, let $P = P(s,x,t,A)$, $0 \le s \le t < \infty$, $x \in \mathbf{X}$, $A \in \mathcal{F}$, be the transition probability function of a nonhomogeneous $\mathbf{X}$-valued Markov process $X = (X_t)$, i.e.,

$$P(s,x,t,A) = \mathbb{E}_{s,x}\chi_A(X_t), \quad 0 \le s \le t < \infty,\ x \in \mathbf{X},\ A \in \mathcal{F}, \tag{1.9}$$

and let $P^*_{s,t} : \mathcal{M} \to \mathcal{M}$ be defined as

$$P^*_{s,t}\nu(A) = \int_{\mathbf{X}} P(s,x,t,A)\,\nu(dx), \quad 0 \le s \le t < \infty,\ \nu \in \mathcal{M}. \tag{1.10}$$

It is easy to see that

$$\Sigma := \{P^*_{s,t} : 0 \le s \le t < \infty\} \tag{1.11}$$

is an evolution system of Markov operators in the sense of Definition 1.2. Below we present a useful sufficient condition for existence of a system of lower measures with respect to Σ defined by (1.11).

1.7. Proposition. *Assume that for every $s \ge 0$ there exist $\beta(s) > 0$, $\widehat{\mu}_s \in \mathcal{M}_+$, and $B(s) \in \mathcal{F}$ such that*

(i) *for any $x \in \mathbf{X}$ there exists $t_0 = t_0(s,x)$ such that $P(s,x,t,B(s)) \ge \beta(s)$ for all $t \ge t_0$,*

(ii) $\inf\{P(t,x,t+1,A) : x \in B(s),\ t \ge s\} \ge \widehat{\mu}_s(A)$ *for all $A \in \mathcal{F}$, and*

(iii) $\inf_{s\ge 0} \beta(s)\widehat{\mu}_s(\mathbf{X}) > 0$.

Then there exists a system of lower measures with respect to $\Sigma = \{P^*_{s,t}\}$. If β, B and $\widehat{\mu}$ can be found independent of $s \in \mathbb{R}_+$ then there exists a lower measure with respect to Σ.

Proof. For $t \ge s \ge 0$, $\nu \in \mathcal{P}$ and $A \in \mathcal{F}$, we have

$$\begin{aligned} P^*_{s,t+1}\nu(A) &= \int_{\mathbf{X}}\int_{\mathbf{X}} P(t,y,t+1,A)\,P(s,x,t,dy)\,\nu(dx) \\ &\ge \int_{\mathbf{X}}\int_{B(s)} P(t,y,t+1,A)\,P(s,x,t,dy)\,\nu(dx) \\ &\ge \widehat{\mu}_s(A)\int_{\mathbf{X}} P(s,x,t,B(s))\,\nu(dx) \end{aligned}$$

by (ii). Since $\liminf_{t\to\infty}\int_{\mathbf{X}} P(s,x,t,B(s))\,\nu(dx) \ge \beta(s)$ by (i), we obtain

$$\|(P^*_{s,t}\nu - \beta(s)\widehat{\mu}_s)^-\| \to 0, \quad t \to \infty.$$

Hence $\{\mu_s := \beta(s)\widehat{\mu}_s : s \in \mathbb{R}_+\}$ is a system of lower measures with respect to $\Sigma = \{P^*_{s,t}\}$. ∎

2. L-condition for semilinear stochastic equations. In this section the general method developed in the previous part is applied to the case when the Markov process is induced by a semilinear stochastic equation of the general form

$$dX_t = [AX_t + f(t, X_t)]dt + dW_t, \quad t \ge s \ge 0, \quad X_s = x \in H, \tag{2.1}$$

where H is a separable Hilbert space. Throughout the section we assume that W_t is a cylindrical Wiener process on H with identity covariance, A is a self-adjoint and negative unbounded linear operator on H with a nuclear inverse. Hence there exists an orthonormal basis $\{e_n\}$ in H consisting of eigenvectors of A and the corresponding eigenvalues satisfy

$$Ae_n = -\lambda_n e_n, \quad \lambda_n \ge \lambda_0 > 0,\ n \in \mathbb{N}, \tag{2.2}$$

and

$$\sum_{n=0}^{\infty} \lambda_n^{-1} < \infty. \tag{2.3}$$

It is well known that under the assumptions (2.2) and (2.3) there exists a unique mild solution to the linear counterpart of the equation (2.1),

$$dZ_t = AZ_t dt + dW_t, \quad Z_s = x \in H, \quad 0 \le s \le t < \infty. \tag{2.4}$$

This solution is an H-valued Ornstein–Uhlenbeck process defined by the formula

$$Z_t = S(t-s)x + \int_s^t S(t-r)\, dW_r, \quad t \ge s, \tag{2.5}$$

where $S(\cdot)$ is the (analytic) semigroup generated on H by the operator A (see, e.g., [7] for details on the semigroup theory of stochastic equations). The function $f : \mathbb{R} \times H \to H$ is assumed to be at least measurable and such that (2.1) has a unique weak solution and induces in a natural way a Markov process in the space H (cf. [1], [7] for explicit sufficient conditions on f). However, in this section, more restrictive assumptions on f have to be selectively used and they are specified below.

Denote by $\mathcal{B}(H)$ and $\mathcal{P}$ the σ-algebra of Borel sets of H and the set of probability mesures defined on $\mathcal{B}(H)$, respectively, and let

$$P = P(s, x, t, \Gamma) = \mathbb{E}\chi_\Gamma(X_t^{s,x}), \quad x \in H,\ \Gamma \in \mathcal{B}(H),\ t \ge s \ge 0,$$

and

$$Q = Q(t-s, x, \Gamma) = \mathbb{E}\chi_\Gamma(Z_t^{s,x}), \quad x \in H,\ \Gamma \in \mathcal{B}(H),\ t \ge s \ge 0,$$

be the transition probability kernels corresponding to the processes X_t and Z_t, respectively, where $X_t^{s,x}$ and $Z_t^{s,x}$ stand for the solutions of the respective equations (2.1) and (2.4). (This notation is used to emphasize the initial conditions $X_s^{s,x} = x$ and $Z_s^{s,x} = x$.) We shall apply the results of the previous section to the adjoint Markov evolution system P^* defined by

$$P^*_{s,t}\nu(\Gamma) = \int_H P(s,x,t,\Gamma)\,\nu(dx), \qquad \nu \in \mathcal{P},\ \Gamma \in \mathcal{B}(H), \tag{2.6}$$

with $\mathbf{X} = H$ and $\mathcal{F} = \mathcal{B}(H)$. In order to formulate some assumptions we recall the concept of the conditioned (or "pinned") Ornstein–Uhlenbeck process defined by the equation (2.4), which was studied in the infinite-dimensional context by Simão [33], [34]. For $t \geq 0$ and $x, y \in H$, set $x_n = \langle x, e_n\rangle$, $y_n = \langle y, e_n\rangle$, $w_n(t) = \langle W_t, e_n\rangle$, and

$$Y_n(s) = \frac{1 - e^{-2\lambda_n(t+1-s)}}{e^{-\lambda_n(t+1-s)}} \int_t^s \frac{e^{-\lambda_n(t+1-u)}}{1 - e^{-2\lambda_n(t+1-u)}}\, dw_n(u) \tag{2.7}$$

for $t \leq s \leq t+1$ and $n \in \mathbb{N}$. The Ornstein–Uhlenbeck process given by (2.4), conditioned to go from x at $s = t$ to y at $s = t+1$, $\widehat{Z}(s) = Z^{t,x}_{t+1,y}(s)$, is given by the expansion

$$\widehat{Z}(s) = Z^{t,x}_{t+1,y}(s) = \sum_{n=1}^{\infty} \widehat{Z}_n(s) e_n, \tag{2.8}$$

where

$$\begin{aligned}\widehat{Z}_n(s) &= e^{-\lambda_n(s-t)} \frac{1 - e^{-2\lambda_n(t+1-s)}}{1 - e^{-2\lambda_n}} x_n \\ &\quad + e^{\lambda_n(t+1-s)} \frac{e^{-2\lambda_n(t+1-s)} - e^{-2\lambda_n}}{1 - e^{-2\lambda_n}} y_n + Y_n(s)\end{aligned} \tag{2.9}$$

for $s \in [t, t+1]$ (cf. [34]). Set

$$\delta_n = \sup\{|\langle f(s,x), e_n\rangle| : s \geq 0,\ x \in H\}, \qquad n \in \mathbb{N}. \tag{2.10}$$

Below, the assumptions (A1)–(A3) are formulated.

(A1) $\delta_n < \infty$ for $n \in \mathbb{N}$ and

$$\sum_{n=1}^{\infty} \delta_n \lambda_n^{-1/2} = \sum_{n=1}^{\infty} \sup\{|\langle f(s,x), (-A)^{-1/2} e_n\rangle| : s \geq 0,\ x \in H\} < \infty.$$

(A2) There exists $\alpha > 0$ such that

$$\mathbb{E} \exp\{\alpha |f(s, Z^{t,x}(s))|^2\} < \infty$$

for all $t \in \mathbb{R}_+$, $s \in [t, t+1]$ and $x \in H$.

(A3) There exist $\beta > 1$ and functions $v_1, v_2 : H \to \mathbb{R}_+$, bounded on bounded sets in H, such that

$$\mathbb{E}\exp\Big\{\beta \int_t^{t+1} |f(s,\widehat{Z}(s))|^2\,ds\Big\} \le v_1(x)v_2(y)$$

for $t \in \mathbb{R}_+$ and $x, y \in H$, where $\widehat{Z}(s) = Z^{t,x}_{t+1,y}(s)$ is defined by (2.7)–(2.9).

The assumptions (A2) and (A3) are trivially satisfied if $|f|$ is bounded on $\mathbb{R}_+ \times H$, in which case we also have $\delta_n < \infty$ for any $n \in \mathbb{N}$. It is not difficult to construct an example of an unbounded function f so that (A1)–(A3) are satisfied. However, we have the following statement, which was communicated to us by Jan Seidler:

2.1. Remark. Let $D \subset \mathbb{R}^d$ be a bounded domain, $H = L_2(D)$, $F \in \mathbf{C}(\mathbb{R})$ with a linear growth, and define

$$f(x)(\xi) = F(x(\xi)), \qquad x \in H,\ \xi \in D.$$

Assume that there exists a $g \in \mathcal{D}((-A)^{1/2})$ such that

$$0 < \Big|\int_D g(\xi)\,d\xi\Big| < \infty$$

and $|(-A)^{1/2}g| \le 1$. Then (A1) yields

$$\sup\{|F(y)| : y \in \mathbb{R}\} < \infty.$$

To see this, note that (A1) implies that

$$M := \sup\{|(-A)^{-1/2}f(x)| : x \in H\} < \infty, \tag{2.11}$$

that is,

$$\begin{aligned} M &= \sup\{|\langle f(x), h\rangle| : |(-A)^{1/2}h| \le 1,\ h \in \mathcal{D}((-A)^{1/2}),\ x \in H\} \\ &\ge \sup\{|\langle f(x), g\rangle| : x \in H\} = \sup\Big\{\Big|\int_D F(x(\xi))g(\xi)\,d\xi\Big| : x \in H\Big\}. \end{aligned}$$

Thus, if we assume that there exists a sequence $y_k \in \mathbb{R}$ such that $|F(y_k)| \to \infty$, then setting $x_k(\xi) = y_k$, we get

$$\begin{aligned} M &\ge \sup\{|(-A)^{-1/2}f(x_k)| : k \in \mathbb{N}\} \ge \sup\Big\{\Big|\int_D F(y_k)g(\xi)\,d\xi\Big| : k \in \mathbb{N}\Big\} \\ &\ge \Big|\int_D g(\xi)\,d\xi\Big| \sup\{|F(y_k)| : k \in \mathbb{N}\} = \infty, \end{aligned}$$

which contradicts (2.11).

The proof of the main result of the paper is based on the following proposition whose proof is given in Section 3.

2.2. Proposition. *Assume* (A1)–(A3), *set* $\gamma := Q(1,0,\cdot)$ *and*

$$D = \Big\{ y \in H : \sum_{n=1}^{\infty} \delta_n |\langle y, e_n\rangle| < \infty \Big\}, \tag{2.12}$$

and let $B \in \mathcal{B}(H)$ *be a bounded set in* H. *Then* $P(t,x,t+1,\cdot)$ *is absolutely continuous with respect to* $Q(1,x,\cdot)$ *for any* $t \geq 0$ *and* $x \in H$, *and*

$$\inf \Big\{ \frac{dP(t,x,t+1,\cdot)}{dQ(1,x,\cdot)}(y) : x \in B,\ t \in \mathbb{R}_+ \Big\} \geq h(y) \tag{2.13}$$

γ*-almost everywhere on* H, *where* $h : H \to \mathbb{R}_+$ *is a bounded measurable function depending on* B *such that* $h(y) > 0$ *for* $y \in D$.

2.3. Proposition. *For every bounded set* $B \subset H$ *there exist constants* $\kappa_1, \kappa_2 > 0$ *such that*

$$\frac{dQ(1,x,\cdot)}{d\gamma}(y) \geq \kappa_1 \exp\{-\kappa_2 |y|\} \tag{2.14}$$

for $x \in B$ *and* γ*-almost all* $y \in H$.

Proof. We have $Q(1,x,\cdot) = N(S(1)x, Q_1)$ and $\gamma = N(0, Q_1)$, where $Q_1 = \int_0^1 S(2t)\,dt$ satisfies

$$Q_1 e_n = \int_0^1 e^{-2\lambda_n t} e_n\,dt = \frac{1}{2\lambda_n}(1 - e^{-2\lambda_n}) e_n, \qquad n \in \mathbb{N}. \tag{2.15}$$

It is clear that $\mathrm{Range}(S(1)) \subset \mathrm{Range}(Q_1)$ and by the closed graph theorem it follows that $Q_1^{-1} S(1) \in \mathcal{L}(H)$. By the Cameron–Martin formula (see, e.g., [7], Proposition 2.24) we obtain

$$\begin{aligned}
\frac{dQ(1,x,\cdot)}{d\gamma}(y) &= \exp\big\{\langle Q_1^{-1/2} S(1)x, Q_1^{-1/2} y\rangle - \tfrac{1}{2}|Q_1^{-1/2} S(1)x|^2\big\} \\
&\geq \exp\big\{-|Q_1^{-1} S(1)x| \cdot |y| - \tfrac{1}{2}|Q_1^{-1/2} S(1)x|^2\big\} \\
&\geq \exp\big\{-\tfrac{1}{2}|Q_1^{1/2}|^2_{\mathcal{L}(H)} \cdot |Q_1^{-1} S(1)|^2_{\mathcal{L}(H)} |x|^2\big\} \\
&\quad \times \exp\{-|Q_1^{-1} S(1)|_{\mathcal{L}(H)} |x| \cdot |y|\} \\
&\geq \kappa_1 \exp\{-\kappa_2 |y|\}. \ \blacksquare
\end{aligned}$$

In the following Proposition 2.4 specific conditions on f are given for the general Proposition 1.7 and Theorem 1.5 to be applicable to the adjoint Markov semigroup generated by (2.1).

For $r > 0$ denote by $B_r = \{x \in H : |x| < r\}$ the open ball in H with center 0 and radius r. The following assumption (A4) is introduced:

(A4) For every $x \in H$ the function $f(\cdot, x) : \mathbb{R}_+ \to H$ is continuous and for every $r > 0$ there exists a constant $K_r > 0$ such that

$$|f(t,x) - f(t,y)| \le K_r |x-y|$$

for $t \in [0,r]$ and $x, y \in B_r$.

Set

$$m = \sup\Big\{\mathbb{E}\Big|\int_s^t S(t-r)\,dW_r\Big| : t \ge s \ge 0\Big\}.$$

From (2.2) and (2.3) it easily follows that $m < \infty$.

Note that under the assumptions (A4) and (2.16)–(2.18) below there exists a unique mild solution to (2.1), and that (2.1) induces a Markov process on H (cf. [8], [7]).

2.4. Proposition. *Assume that f satisfies* (A4) *and*

$$\langle Ax + f(t, x+y), x\rangle \le -\omega(t)|x|^2 + a(t, |y|)|x| \tag{2.16}$$

for $t \in \mathbb{R}_+$, $x \in \mathcal{D}(A)$ and $y \in H$, where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ and $a : \mathbb{R}_+^2 \to \mathbb{R}_+$ are measurable and bounded on bounded sets, $a(t,\cdot)$ *is increasing for every* $t \ge 0$,

$$\inf\Big\{\int_\sigma^{\sigma+T} \omega(\lambda)\,d\lambda : \sigma \in \mathbb{R}_+\Big\} \to \infty \qquad \text{as } T \to \infty, \tag{2.17}$$

and

$$M := \sup\Big\{\int_s^t \exp\Big\{-\int_r^t \omega(\lambda)\,d\lambda\Big\}\mathbb{E}a(r, |\phi(r)|)\,dr : t \ge s \ge 0\Big\} < \infty, \tag{2.18}$$

where

$$\phi(r) = \int_s^r S(r-u)\,dW_u.$$

Then for every $0 \le s \le t < \infty$ we have

(i) $\mathbb{E}|X^{s,x}(t)| \le m + |x|e^{-\int_s^t \omega(\lambda)d\lambda} + M$.

Furthermore,

(ii) *for every $\varepsilon > 0$ and $R_1 > 0$ there exists $R_0 > 0$ such that $P(\sigma, x, t, H \setminus B_{R_0}) < \varepsilon$ for $x \in B_{R_1}$ and $0 \le \sigma \le t < \infty$, and*

(iii) *for any constants $\widetilde{\beta}, L > 0$, satisfying*

$$(1-\widetilde{\beta})L > m + M \tag{2.19}$$

and every $R_2 > 0$ there exists $T > 0$ such that $P(\sigma, x, \sigma + t, B_L) \ge \widetilde{\beta}$ for all $x \in B_{R_2}$, $\sigma \in \mathbb{R}_+$ and $t \ge T$.

P r o o f. By a straightforward modification of the proof of Theorem 7.10 in [7] it can be shown that under the present conditions there exists a unique solution $v = v(t, s, x, \psi)$ of the equation

$$v(t) = S(t-s)x + \int_s^t S(t-r)f(r, v(r) + \psi(r))\,dr, \quad s \le t \le T, \tag{2.20}$$

for every $T \ge s \ge 0$, $x \in H$ and $\psi \in \mathbf{C}_0([s,T], H) := \{\varphi \in \mathbf{C}([s,T], H) : \varphi(s) = 0\}$, such that $v(\cdot, s, x, \psi) \in \mathbf{C}([s,T], H)$, and the mild solution of (2.1) has the form $X(t) = v(t, s, x, \phi(\cdot)) + \phi(t)$, $t \ge s$. Set $R(n) = nR(n, A)$, where $R(n, A) = (nI - A)^{-1}$ is the resolvent of A, $n > 0$, and define a sequence $v_n \in \mathbf{C}([s,T], H)$ by

$$v_n(t) = R(n)S(t-s)x + \int_s^t R(n)S(t-r)f(r, v(r) + \psi(r))\,dr, \qquad s \le t \le T,$$

where v is defined by (2.20). Note that for $s \le T$ we have

$$\sup_{s \le r \le t \le T} |(R(n) - I)(S(t-r)f(r, v(r) + \psi(r)) + S(t-s)x)| \to 0$$

as $n \to \infty$, and hence it is easy to see that

$$\sup_{s \le t \le T} |v_n(t) - v(t)| \to 0, \quad n \to \infty, \tag{2.21}$$

and

$$\sup_{s \le t \le T} |\delta_n(t)| \to 0, \quad n \to \infty, \tag{2.22}$$

where

$$\delta_n(t) := \frac{dv_n}{dt}(t) - Av_n(t) - f(t, v_n(t) + \psi(t)).$$

Since

$$\begin{aligned}|v_n(t)| \cdot \frac{d^-}{dt}|v_n(t)| \\ \le \langle Av_n(t) + f(t, v_n(t) + \psi(t)), v_n(t)\rangle + \langle \delta_n(t), v_n(t)\rangle, \qquad t \ge s,\end{aligned}$$

we obtain

$$\frac{d^-}{dt}|v_n(t)| \le -\omega(t)|v_n(t)| + a(t, |\psi(t)|) + |\delta_n(t)|, \quad t \ge s,$$

by (2.16), and, consequently,

$$\begin{aligned}v_n(t) \le |x|e^{-\int_s^t \omega(\lambda)d\lambda} \\ + \int_s^t \exp\Big\{-\int_r^t \omega(\lambda)\,d\lambda\Big\}(a(r, |\psi(r)|) + |\delta_n(r)|)\,dr, \qquad t \ge s.\end{aligned}$$

By (2.21) and (2.22) it follows that

$$v(t) \le |x| e^{-\int_s^t \omega(\lambda)\,d\lambda} + \int_s^t \exp\Big\{ -\int_r^t \omega(\lambda)\,d\lambda \Big\} a(r, |\psi(r)|)\,dr, \qquad t \ge s,$$

which yields

$$\mathbb{E}|X^{s,x}(t)| = \mathbb{E}|v(t,s,x,\phi(\cdot)) + \phi(t)| \le m + |x| e^{-\int_s^t \omega(\lambda)\,d\lambda} + M$$

for every $0 \le s \le t < \infty$ and $x \in H$, which is precisely (i). The assertions (ii) and (iii) follow immediately from (i) by the Chebyshev inequality and, in the case of (iii), by (2.17). Note that T is found by (2.17) so that

$$\inf\Big\{ \int_\sigma^{\sigma+T} \omega(\lambda)\,d\lambda : \sigma \in \mathbb{R}_+ \Big\} > -\log[1 - \widetilde{\beta} - R^{-1}(m+M)]. \ \blacksquare$$

2.5. Corollary. *Assume that* $f : \mathbb{R}_+ \times H \to H$ *satisfies* (A4) *and*

$$|f(t,x)| \le k_1 + k_2|x|, \qquad (t,x) \in \mathbb{R}_+ \times H, \tag{2.23}$$

for some $k_1, k_2 > 0$ *such that* $k_2 < \lambda_0$ (cf. (2.2)). *Then* (2.16)–(2.18) *are satisfied*, *and hence the assertions* (i)–(iii) *of Proposition* 2.4 *hold.*

Proof. Note that $\langle Ax, x\rangle \le -\lambda_0|x|^2$ for $x \in \mathcal{D}(A)$, and, therefore,

$$\begin{aligned}\langle Ax + f(t, x+y), x\rangle &\le -\lambda_0|x|^2 + k_1|x| + k_2|x|(|x| + |y|)\\ &\le (-\lambda_0 + k_2)|x|^2 + (k_1 + k_2|y|)|x|\end{aligned}$$

for $x \in \mathcal{D}(A)$, $t \in \mathbb{R}_+$, and $y \in H$. Thus we obtain (2.16) with a constant $\omega = \lambda_0 - k_2$ and with $a(t,\theta) = k_1 + k_2\theta$, $\theta \ge 0$. $\blacksquare$

Now we can state the main result of the present section.

2.6. Theorem. *Assume* (A1)–(A4), (2.16)–(2.18), *and*

$$\gamma(D) > 0, \tag{2.24}$$

where γ *and* D *are defined in Proposition* 2.2. *Then*

$$\|P^*_{s,t}\nu_1 - P^*_{s,t}\nu_2\| \to 0, \quad t \to \infty, \tag{2.25}$$

for all $s \ge 0$ *and* $\nu_1, \nu_2 \in \mathcal{P}$. *If*, *moreover*, $f(t,x) = f(x)$ *does not depend on* t, *then there exists a unique invariant measure* $\mu^* \in \mathcal{P}$ *for the equation* (2.1) *and*

$$\|P^*_t\nu - \mu^*\| \to 0, \quad t \to \infty,$$

for every $\nu \in \mathcal{P}$, *where* P^*_t *is the adjoint Markov semigroup of the* (*homogeneous*) *Markov process defined by* (2.1).

Proof. Take a fixed $L > m + M$, where m and M are the constants from Proposition 2.4, and $\beta > 0$ such that $(1-\beta)L > m + M$. We verify the assumptions of Proposition 1.7 with $B(s) = B_L$, $\beta(s) = \beta$ and a suitable lower measure $\widehat{\mu}$ (independent of s). The condition (i) of Proposition 1.7 is satisfied by Proposition 2.4(iii). Furthermore, for any $s \geq 0$, $\Gamma \in \mathcal{B}(H)$ and $x \in B_L$, we have

$$P(t,x,t+1,\Gamma) = \int_\Gamma \frac{dP(t,x,t+1,\cdot)}{dQ(1,x,\cdot)}(y)\,Q(1,x,dy) \geq \int_\Gamma h(y)\,Q(1,x,dy) \tag{2.26}$$

by Proposition 2.2, where $h \geq 0$ does not depend on $t \in \mathbb{R}_+$ and $x \in B_L$, and $h > 0$ on D. Proposition 2.3 yields

$$\int_\Gamma h(y)\,Q(1,x,dy) \geq \kappa_1 \int_\Gamma h(y)\exp\{-\kappa_2|y|\}\,\gamma(dy), \tag{2.27}$$

where κ_1 and κ_2 are independent of $x \in B_L$. Setting

$$\widehat{\mu}(\Gamma) = \kappa_1 \int_\Gamma h(y)\exp\{-\kappa_2|y|\}\,\gamma(dy)$$

we obtain

$$\inf\{P(t,x,t+1,\Gamma) : x \in B_L,\ t \in \mathbb{R}_+\} \geq \widehat{\mu}(\Gamma), \quad \Gamma \in \mathcal{B}(H),$$

hence the condition (ii) of Proposition 1.7 is satisfied. Since

$$\widehat{\mu}(H) \geq \kappa_1 \int_D h(y)\exp\{-\kappa_2|y|\}\,\gamma(dy) > 0$$

by (2.24), the remaining assumption (iii) of Proposition 1.7 is also satisfied and the proof is finished by applying Proposition 1.7 and Theorem 1.4. ∎

Note that if $\gamma(D) > 0$ then, in fact, $\gamma(D) = 1$ since D is a Borel linear subspace of H and so the Kallianpur 0-1 law applies (see, e.g., [18]).

If the mapping f is bounded in the norm of H, the assumptions of Theorem 2.6 are considerably simplified.

2.7. Corollary. *Assume that* $|f|$ *is bounded*, (A4) *is satisfied*, *and*

$$\sum_{n=1}^{\infty} \delta_n \lambda_n^{-1/2} < \infty, \tag{2.28}$$

$$\gamma\Big(\Big\{y \in H : \sum_{n=1}^{\infty} \delta_n |\langle y, e_n\rangle| < \infty\Big\}\Big) > 0, \tag{2.29}$$

where $\delta_n = \sup\{|\langle f(t,x), e_n\rangle| : t \geq 0,\ x \in H\}$. *Then the assumptions of Theorem* 2.6 *are satisfied.*

Proof. The conditions (2.28) and (2.29) are, in fact, the assumptions (A1) and (2.24), respectively. The assumptions (2.16)–(2.18) are satisfied by Corollary 2.5 and the remaining assumptions (A2), (A3) are satisfied trivially. ∎

Some more specific sufficient conditions for (2.28) and (2.29) to hold will be given. Denote by D^α_A, $\alpha > 0$, the domain of the fractional power $(-A)^\alpha$ equipped with the graph norm.

2.8. COROLLARY. *Assume that the function f satisfies* (A4) *and $|f|$ is bounded. Moreover, let one of the three conditions* (i)–(iii) *be satisfied*:

(i) $\sum_{n=1}^\infty \delta_n^2 = \sum_{n=1}^\infty \sup\{|\langle f(t,x), e_n\rangle|^2 : t \geq 0,\ x \in H\} < \infty$.

(ii) $f : \mathbb{R}_+ \times H \to D^\alpha_A$ *is bounded for some $\alpha > 0$ such that $(-A)^{-\alpha}$ is a Hilbert–Schmidt operator.*

(iii) $\sum_{n=1}^\infty \lambda_n^{-1/2} < \infty$.

Then the conclusions of Theorem 2.6 *hold true.*

Proof. By Corollary 2.7 it suffices to verify (2.28) and (2.29). If (i) holds then

$$\sum_{n=1}^\infty \lambda_n^{-1/2}\delta_n \leq \Big(\sum_{n=1}^\infty \lambda_n^{-1}\Big)^{1/2} \cdot \Big(\sum_{n=1}^\infty \delta_n^2\Big)^{1/2} < \infty$$

by (2.3), and

$$\sum_{n=1}^\infty \delta_n |\langle y, e_n\rangle| \leq |y| \Big(\sum_{n=1}^\infty \delta_n^2\Big)^{1/2}$$

for $y \in H$. It follows that $D = H$ and (2.28), (2.29) are obviously satisfied.

Part (ii) is a particular case of (i) since

$$\begin{aligned}
\sum_{n=1}^\infty \delta_n^2 &= \sum_{n=1}^\infty \sup\{|\langle f(t,x), e_n\rangle|^2 : t \geq 0,\ x \in H\} \\
&= \sum_{n=1}^\infty \sup\{|\langle (-A)^\alpha f(t,x), (-A)^{-\alpha} e_n\rangle|^2 : t \geq 0,\ x \in H\} \\
&\leq \sup\{|(-A)^\alpha f(t,x)|^2 : t \geq 0,\ x \in H\} \cdot \sum_{n=1}^\infty |(-A)^{-\alpha} e_n|^2 \\
&= \sup\{|f(t,x)|^2_{D^\alpha_A} : t \geq 0,\ x \in H\} \cdot |(-A)^{-\alpha}|^2_{\mathrm{HS}} < \infty,
\end{aligned}$$

where $|\cdot|_{\mathrm{HS}}$ stands for the Hilbert–Schmidt norm of operators on H.

If (iii) is satisfied then

$$\sum_{n=1}^\infty \lambda_n^{-1/2}\delta_n \leq \sup \delta_n \cdot \sum_{n=1}^\infty \lambda_n^{-1/2} < \infty$$

since $\sup \delta_n \leq \sup |f|$, and setting $y_n = \langle y, e_n\rangle$ for $y \in H$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}\sum_{n=1}^{\infty}|y_n|\delta_n &= \sum_{n=1}^{\infty}\lambda_n^{-1/4}|\lambda_n^{1/4}y_n|\delta_n\\ &\le \sup\delta_n\cdot\Big(\sum_{n=1}^{\infty}\lambda_n^{-1/2}\Big)^{1/2}\cdot\Big(\sum_{n=1}^{\infty}(\lambda_n^{1/4}y_n)^2\Big)^{1/2}\\ &\le \sup|f|\cdot\Big(\sum_{n=1}^{\infty}\lambda_n^{-1/2}\Big)^{1/2}\cdot|(-A)^{1/4}y|\end{aligned}$$

for $y \in D_A^{1/4}$, hence $D \supset D_A^{1/4}$. It is easy to check that (iii) implies that $\gamma(D_A^{1/4}) = 1$ and, therefore, (2.29) holds. ∎

2.9. Remark. Proposition 2.4(i) together with a lower estimate on $\widehat{\mu}(H)$, where $\widehat{\mu}$ is the measure defined in the proof of Theorem 2.6, make it possible to apply Theorem 1.5 to estimate the "speed" of convergence in (2.25) (cf. Remark 1.6). More precisely, given $\nu_1, \nu_2 \in \mathcal{P}$ and $\varepsilon > 0$ we can use the fact that

$$P(\sigma, x, t, H \setminus B_R) \le R^{-1}(m + |x| + M), \quad 0 \le \sigma \le t < \infty,\ x \in H,\ R > 0,$$

in order to find $R > m + M$ such that

$$P^*_{\sigma,t}\nu_1(H \setminus B_R) < \varepsilon, \quad P^*_{\sigma,t}\nu_2(H \setminus B_R) < \varepsilon$$

for every $0 \le \sigma \le t < \infty$. Furthermore, for any $\mu \in \mathcal{P}$ with $\mu(B_R) = 1$, $\sigma \ge 0$, $\tau \ge 1$ and $\Gamma \in \mathcal{B}(H)$, we have

$$\begin{aligned}P^*_{\sigma,\sigma+\tau}\mu(\Gamma) &\ge \int_{B_R} P(\sigma+\tau-1, x, \sigma+\tau, \Gamma)\, P^*_{\sigma,\sigma+\tau-1}\mu(dx)\\ &\ge \inf\{P(\sigma+\tau-1, x, \sigma+\tau, \Gamma) : x \in B_R\}\\ &\quad\times P^*_{\sigma,\sigma+\tau-1}\mu(B_R),\end{aligned}\tag{2.30}$$

and tracing the proof of Theorem 2.6 (with $B_L = B_R$) we obtain

$$\begin{aligned}\inf\{P(\sigma+\tau-1, x, \sigma+\tau, \Gamma) : x \in B_R\}\cdot P^*_{\sigma,\sigma+\tau-1}\mu(B_R)&\\ \ge \widehat{\mu}(\Gamma)P^*_{\sigma,\sigma+\tau-1}\mu(B_R)&.\end{aligned}\tag{2.31}$$

Choosing $\widetilde{\beta} > 0$ such that $\widetilde{\beta} < 1 - R^{-1}(m+M)$ we get, by Proposition 2.4(i),

$$P^*_{\sigma,\sigma+\tau-1}\mu(B_R) = \int_{B_R} P(\sigma, x, \sigma+\tau-1, B_R)\,\mu(dx) \ge \widetilde{\beta}\tag{2.32}$$

for $\tau = 1 + T$, where T can be found by (2.17) so that

$$\inf\Big\{\int_{\sigma}^{\sigma+T}\omega(\lambda)\,d\lambda : \sigma \in \mathbb{R}_+\Big\} > -\log[1 - \widetilde{\beta} - R^{-1}(m+M)].\tag{2.33}$$

From (2.30)–(2.32) it follows that

$$\|(P^*_{\sigma,\sigma+\tau}\mu(\cdot) - \widetilde{\beta}\widehat{\mu}(\cdot))^-\| = 0$$

for all $\sigma \in \mathbb{R}_+$ and for $\tau = 1 + T$ defined by (2.33). Thus (2.33) gives us the upper estimate on τ required in Remark 1.6.

2.10. Example. Consider the equation

$$dX_t = (AX_t + f(t, X_t))dt + dW_t, \quad t \geq s, \quad X_s = x, \tag{2.34}$$

on a Hilbert space H, where W_t is a standard cylindrical Wiener process on H and A satisfies (2.2)–(2.3). The mapping $f : \mathbb{R}_+ \times H \to H$ is assumed to have the form

$$f(t, x) = \sum_{j=1}^{N} a_j(t)\varphi_j(t, x), \qquad (t, x) \in \mathbb{R}_+ \times H, \tag{2.35}$$

where $a_j : \mathbb{R}_+ \to H$ and $\varphi_j : \mathbb{R}_+ \times H \to \mathbb{R}$ are bounded and continuous, and $\varphi_j(t, \cdot) : H \to \mathbb{R}$ are Lipschitz on bounded sets for all $t \geq 0$ and $j = 1, \ldots, N$. Moreover, assume that $a_j : \mathbb{R}_+ \to D^\alpha_A$ for some $\alpha > 0$ such that $(-A)^{-\alpha}$ is a Hilbert–Schmidt operator and $|(-A)^\alpha a_j(\cdot)|$ are bounded for $j = 1, \ldots, N$ (note that $(-A)^{-\alpha}$ is always Hilbert–Schmidt for $\alpha \geq 1/2$). Then by Corollary 2.8(ii) the conclusions of Theorem 2.6 hold true for the Markov evolution operator $\{P^*_{s,t} : 0 \leq s \leq t < \infty\}$ induced on $\mathcal{P}$ by the equation (2.34).

2.11. Example. Consider the nonautonomous stochastic parabolic equation

$$\begin{aligned} \frac{\partial u}{\partial t}(t, \xi) = -&\frac{\partial^4 u}{\partial \xi^4}(t, \xi) \\ &+ F(t, u(t, \xi)) + \dot{w}_{t\xi}, \qquad t \geq s \geq 0,\ \xi \in (0, 1), \end{aligned} \tag{2.36}$$

with the initial condition $u(s, \xi) = u_0(\xi)$, $\xi \in (0, 1)$, and the boundary conditions

$$u(t, 0) = u(t, 1) = \frac{\partial^2 u}{\partial \xi^2}(t, 0) = \frac{\partial^2 u}{\partial \xi^2}(t, 1) = 0, \quad t \geq s,$$

where $\dot{w}_{t\xi}$ stands symbolically for a space-time white noise and $F : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is bounded, continuous, and Lipschitz in the second variable. The formal equation (2.36) is rewritten in the usual way as an equation of the form

$$dX_t = (AX_t + f(t, X_t))dt + dW_t, \qquad t \geq s, \tag{2.1}$$

in the Hilbert space $H = L_2(0, 1)$, with the initial condition $X_s = u_0 \in H$, where

$$A = -\frac{\partial^4}{\partial \xi^4}, \qquad \mathcal{D}(A) = \left\{ u \in H_0^4(0,1) : \frac{\partial^2 u}{\partial \xi^2}(0) = \frac{\partial^2 u}{\partial \xi^2}(1) = 0 \right\}, \tag{2.37}$$
$$f(t,x)(\theta) = F(t, x(\theta)), \qquad x \in H,\ \theta \in (0,1),\ t \in \mathbb{R}_+,$$

and W_t is a standard cylindrical Wiener process on H. (See, e.g., [15], [6], [20] or [31] for related existence and regularity results.) It is obvious that $A = A^*$ is negative on H, A^{-1} is nuclear and the eigenvalues of $-A$ satisfy

$$\lambda_n \sim n^4, \tag{2.38}$$

so clearly $\sum_{n=1}^{\infty} \lambda_n^{-1/2} < \infty$. Thus Corollary 2.8(iii) may be applied. We conclude that the statements of Theorem 2.6 hold true.

2.12. Example. Consider the stochastic integrodifferential equation of the form

$$\frac{\partial u}{\partial t}(t,\xi) = \frac{\partial^2 u}{\partial \xi^2}(t,\xi) + \int_0^1 K(t,\xi,r,u(t,r))\,dr + \dot{w}_{t\xi}, \qquad t \ge s \ge 0,\ \xi \in (0,1), \tag{2.39}$$

with the Neumann type boundary conditions

$$\frac{\partial u}{\partial \xi}(t,0) = \frac{\partial u}{\partial \xi}(t,1) = 0, \quad t \ge s, \tag{2.40}$$

and an initial condition $u(s,\xi) = u_s(\xi)$, $\xi \in (0,1)$, where $K : \mathbb{R}_+ \times (0,1)^2 \times \mathbb{R} \to \mathbb{R}$ is bounded and measurable and satisfies

$$|K(t,\xi,r,\theta_1) - K(t,\xi,r,\theta_2)| \le k_T |\theta_1 - \theta_2| \tag{2.41}$$

for all $t \in [0,T], T \ge 0$ and $(\xi, r, \theta_1, \theta_2) \in (0,1)^2 \times \mathbb{R}^2$, where the constant $k_T > 0$ depends just on T, $K(\cdot,\xi,r,\theta) : \mathbb{R}_+ \to \mathbb{R}$ is continuous for every $(\xi,r,\theta) \in (0,1)^2 \times \mathbb{R}$, $K(t,\cdot,r,\theta) \in \mathcal{C}^1(0,1)$ for every $(t,r,\theta) \in \mathbb{R}_+ \times (0,1) \times \mathbb{R}$, and $\partial K/\partial \xi$ is bounded on $\mathbb{R}_+ \times (0,1)^2 \times \mathbb{R}$. The system (2.39)–(2.41) can be rewritten in the usual way as an equation of the form

$$dX_t = (AX_t + f(t,X_t))dt + dW_t, \quad X_s = u_s \in H, \quad t \ge s \ge 0, \tag{2.1}$$

in the space $H = L_2(0,1)$, where W_t is a standard cylindrical Wiener process on H, $A = \partial^2/\partial \xi^2$ is defined on the subspace of $H^2(0,1)$ consisting of functions satisfying the boundary conditions (2.40), and $f : \mathbb{R}_+ \times H \to H$ is given by

$$f(t,x)(\xi) = \int_0^1 K(t,\xi,r,x(r))\,dr, \qquad x \in H,\ t \in \mathbb{R}_+,\ \xi \in (0,1). \tag{2.42}$$

It is well known that A satisfies the conditions (2.2)–(2.3) with a suitably defined orthonormal basis $\{e_n\}$ in H and with $\lambda_n \sim n^2$. From the above assumptions on the kernel K it easily follows that f defined by (2.42) is bounded in the norm of H and satisfies (A4). Furthermore, since $D_A^{1/2} = H^1(0,1)$ in the present case (cf. [12]) and

$$\int_0^1 \left| \frac{\partial}{\partial \xi} \int_0^1 K(t,\xi,r,x(r))\,dr \right|^2 d\xi$$

is bounded by a constant independent of $t \in \mathbb{R}_+$ and $x \in H$, we obtain

$$\sup\bigl\{|(-A)^{1/2} f(t,x)| : (t,x) \in \mathbb{R}_+ \times H\bigr\} < \infty.$$

Noting that $(-A)^{-1/2}$ is Hilbert–Schmidt we conclude that the statements of Theorem 2.6 hold in the present case by Corollary 2.8(ii).

3. Proof of Proposition 2.2. We keep the notation introduced at the beginning of the preceding section and the assumptions (2.2)–(2.3) and (A1)–(A3). In order to prove Proposition 2.2, two auxiliary results are established. For $n \in \mathbb{N}$ set $f_n(t,x) = \langle f(t,x), e_n\rangle$ for $(t,x) \in \mathbb{R}_+ \times H$.

3.1. Lemma. *Assume* (2.2)–(2.3) *and* (A1)–(A3). *Then*

(i) *there exist* $k > 1$ *and* $c_k > 0$ *such that*

$$\mathbb{E}\exp\biggl\{k\biggl(\sum_{n=1}^m \int_t^{t+1} f_n(s,\widehat{Z}(s))\,dw_n(s) - \frac{1}{2}\sum_{n=1}^m \int_t^{t+1} |f_n(s,\widehat{Z}(s))|^2\,ds\biggr)\biggr\} \le c_k v_1(x) v_2(y)$$

for every $m \in \mathbb{N}$ *and* $t \in \mathbb{R}_+$,

(ii) *for every* $x \in H$, $y \in D$, $k > 0$ *and* $m \in \mathbb{N}$ *we have*

$$\mathbb{E}\exp\biggl\{k\biggl|\sum_{n=1}^m \int_t^{t+1} f_n(s,\widehat{Z}(s)) a_n(s,x_n,y_n)\,ds\biggr|\biggr\} \le \exp\biggl\{2k\sum_{n=1}^\infty \delta_n \frac{|y_n - e^{-\lambda_n} x_n|}{1 - e^{-2\lambda_n}}\biggr\} < \infty,$$

where

$$a_n(s,x_n,y_n) = \frac{2\lambda_n e^{-\lambda_n(t+1-s)}}{1-e^{-2\lambda_n}}\,(y_n - e^{-\lambda_n} x_n), \tag{3.1}$$

(iii) *for every $k>0$ there exists a constant $c_k'>0$ depending only on k such that*

$$\mathbb{E}\exp\left\{k\sum_{n=1}^{m}\int_t^{t+1}\delta_n\frac{2\lambda_n e^{-2\lambda_n(t+1-s)}}{1-e^{-2\lambda_n(t+1-s)}}\,|Y_n(s)|\,ds\right\}\le c_k'$$

for all $m\in\mathbb{N}$ and $t\in\mathbb{R}_+$.

Proof. The above statements are straightforward modifications of known results: The estimate (i) follows from (A3) in the same way as, e.g., the inequality (1.10) in [10], Chapter 7, Section 1, while the proof of (iii) is completely analogous to the proof of Lemma 4.1 in [33]. Simple calculations yield

$$\Big|\sum_{n=1}^{m}\int_t^{t+1}f_n(s,\widehat{Z}(s))a_n(s,x_n,y_n)\,ds\Big|\le 2\sum_{n=1}^{m}\delta_n\frac{|y_n-e^{-\lambda_n}x_n|}{1-e^{-2\lambda_n}},$$

and (ii) follows. ∎

Set

$$\phi(t,x,y)=\sum_{n=1}^{\infty}\phi_n(t,x,y)=L_1\text{-}\lim_{m\to\infty}\sum_{n=1}^{m}\phi_n(t,x,y) \tag{3.2}$$

for $t\in\mathbb{R}_+$, $x\in H$ and $y\in D$, where L_1-lim denotes the limit in the mean,

$$\begin{aligned}\phi_n(t,x,y)={}&\int_t^{t+1}f_n(s,\widehat{Z}(s))\,dw_n(s)-\frac12\int_t^{t+1}|f_n(s,\widehat{Z}(s))|^2\,ds\\&+\int_t^{t+1}f_n(s,\widehat{Z}(s))a_n(s,x_n,y_n)\,ds\\&-2\int_t^{t+1}f_n(s,\widehat{Z}(s))\frac{\lambda_n e^{-2\lambda_n(t+1-s)}}{1-e^{-2\lambda_n(t+1-s)}}Y_n(s)\,ds,\end{aligned}$$

and a_n, Y_n and $\widehat{Z}(s)$ are defined by (3.1), (2.7) and (2.8), respectively. The limit passage in (3.2) is justified since the series in (3.2) converges in probability while the condition (A3) and Lemma 3.1(ii), (iii) imply that the partial sums in (3.2) are uniformly bounded in L_p for some $p>1$ and, therefore, uniformly integrable.

3.2. Lemma. *Assume* (2.2)–(2.3) *and* (A1)–(A3). *Then*

$$\frac{dP(t,x,t+1,\cdot)}{dQ(1,x,\cdot)}(y)=\mathbb{E}\exp\{\phi(t,x,y)\}\qquad \gamma\text{-a.e. } y\in D \tag{3.3}$$

for any $t\in\mathbb{R}_+$ and $x\in H$.

P r o o f. Let $x \in H$ and $t \in \mathbb{R}_+$ be fixed. By the Girsanov theorem we have

$$\frac{dP(t,x,t+1,\cdot)}{dQ(1,x,\cdot)}(y) = \mathbb{E}(\varrho_t \mid Z^{t,x}(t+1) = y)$$

$Q(1,x,\cdot)$-a.e., where $\varrho_t = L_1\text{-}\lim \varrho_t^m$ and

$$\varrho_t^m = \exp\Big\{\sum_{n=1}^m \int_t^{t+1} f_n(s, Z^{t,x}(s))\, dw_n(s) - \frac{1}{2}\sum_{n=1}^m \int_t^{t+1} |f_n(s,Z^{t,x}(s))|^2\, ds\Big\}$$

(cf. [26], [13], [33], Lemma 3.1). As $m \to \infty$, $\mathbb{E}(\varrho_t^m \mid Z^{t,x}(t+1) = y)$ converges in $L_1(H, \mathcal{B}(H), Q(1,x,\cdot))$ to $\mathbb{E}(\varrho_t \mid Z^{t,x}(t+1) = y)$ and, therefore, there exists a subsequence of $\mathbb{E}(\varrho_t^m \mid Z^{t,x}(t+1) = y)$ (denoted again by $\mathbb{E}(\varrho_t^m \mid Z^{t,x}(t+1) = y))$ which converges $Q(1,x,\cdot)$-almost everywhere. Since the Ornstein–Uhlenbeck process Z defined by (2.4) is strongly Feller in the present case (see, e.g., [26]), the measures $Q(1,x,\cdot)$ and $\gamma = Q(1,0,\cdot)$ are equivalent and we arrive at

$$\lim_{m\to\infty} \mathbb{E}(\varrho_t^m \mid Z^{t,x}(t+1) = y) = \mathbb{E}(\varrho_t \mid Z^{t,x}(t+1) = y) \quad \gamma\text{-a.e.} \tag{3.4}$$

Now, for each $m \in \mathbb{N}$, we have

$$\begin{aligned}
&\mathbb{E}(\varrho_t^m \mid Z^{t,x}(t+1) = y) \\
&\quad = \mathbb{E}\Big(\exp\Big\{\sum_{n=1}^m \int_t^{t+1} f_n(s, Z^{t,x}(s))\, dZ_n^{t,x}(s) \\
&\qquad + \sum_{n=1}^m f_n(s, Z^{t,x}(s))\lambda_n Z_n^{t,x}(s)\, ds \\
&\qquad - \frac{1}{2}\sum_{n=1}^m \int_t^{t+1} |f_n(s, Z^{t,x}(s))|^2\, ds\Big\} \,\Big|\, Z^{t,x}(t+1) = y\Big) \\
&\quad = \mathbb{E}\exp\Big\{\sum_{n=1}^m \int_t^{t+1} f_n(s, \widehat{Z}(s))\, d\widehat{Z}_n(s) \\
&\qquad + \sum_{n=1}^m \int_t^{t+1} f_n(s,\widehat{Z}(s))\lambda_n \widehat{Z}_n(s)\, ds - \frac{1}{2}\sum_{n=1}^m \int_t^{t+1} |f_n(s,\widehat{Z}(s))|^2\, ds\Big\},
\end{aligned} \tag{3.5}$$

where $\widehat{Z}(s)$ is the Ornstein–Uhlenbeck process conditioned to go from x at $s = t$ to y at $s = t+1$, and $\widehat{Z}_n(s) = \langle \widehat{Z}(s), e_n\rangle$. The processes $\widehat{Z}_n(s)$ satisfy the stochastic differential equation

$$
\begin{aligned}
d\widehat{Z}_n(s) &= dw_n(s) - \lambda_n \widehat{Z}_n(s)\,ds \\
&\qquad - 2\lambda_n \frac{e^{-2\lambda_n(t+1-s)}\widehat{Z}_n(s) - e^{-2\lambda_n(t+1-s)}y_n}{1-e^{-2\lambda_n(t+1-s)}}\,ds, \qquad (3.6)\\
\widehat{Z}_n(t) &= x_n
\end{aligned}
$$

(cf. [35] for the details from the theory of conditioned processes). Solving (3.6) we get the formula (2.9). It follows that

$$
\begin{aligned}
(3.7)\quad \mathbb{E}(\varrho_t^m \mid Z^{t,x}(t+1) = y) \\
= \mathbb{E}\exp\Big\{ &\sum_{n=1}^{m} \int_t^{t+1} f_n(s, \widehat{Z}(s))\,dw_n(s) \\
&- \frac{1}{2}\sum_{n=1}^{m} \int_t^{t+1} |f_n(s, \widehat{Z}(s))|^2\,ds \\
&- 2\sum_{n=1}^{m} \int_t^{t+1} f_n(s, \widehat{Z}(s))\lambda_n \\
&\times \frac{e^{-2\lambda_n(t+1-s)}\widehat{Z}_n(s) - e^{-\lambda_n(t+1-s)}y_n}{1-e^{-2\lambda_n(t+1-s)}}\,ds\Big\}.
\end{aligned}
$$

Using (2.9) we calculate that

$$
\begin{aligned}
2\lambda_n &\frac{e^{-2\lambda_n(t+1-s)}\widehat{Z}_n(s) - e^{-\lambda_n(t+1-s)}y_n}{1-e^{-2\lambda_n(t+1-s)}} \\
&= -a_n(s, x_n, y_n) + \frac{2\lambda_n e^{-2\lambda_n(t+1-s)}}{1-e^{-2\lambda_n(t+1-s)}} Y_n(s),
\end{aligned}
$$

therefore, (3.7) yields

$$
\mathbb{E}(\varrho_t^m \mid Z^{t,x}(t+1) = y) = \mathbb{E}\exp\Big\{\sum_{n=1}^{m} \phi_n(t,x,y)\Big\}. \tag{3.8}
$$

As for (3.2) it can be checked that

$$
\lim_{m\to\infty} \mathbb{E}\exp\Big\{\sum_{n=1}^{m} \phi_n(t,x,y)\Big\} = \mathbb{E}\exp\{\phi(t,x,y)\}
$$

for $t \in \mathbb{R}_+$, $x \in H$ and $y \in D$ (uniform integrability of the partial sums follows from Lemma 3.1(i)–(iii)). This, together with (3.8) and (3.4), implies that

$$
\mathbb{E}(\varrho_t \mid Z^{t,x}(t+1) = y) = \mathbb{E}\exp\{\phi(t,x,y)\}
$$

for γ-almost all $y \in D$. ■

Proof of Proposition 2.2. For all $m \in \mathbb{N}$ we have

$$\mathbb{E}\Big|\sum_{n=1}^{m}\int_{t}^{t+1} f_n(s,\widehat{Z}(s))\,dw_n(s) - \frac{1}{2}\sum_{n=1}^{m}\int_{t}^{t+1} |f_n(s,\widehat{Z}(s))|^2\,ds\Big|$$
$$\le 2\mathbb{E}\exp\Big\{\int_{t}^{t+1} |f(s,\widehat{Z}(s))|^2\,ds\Big\} \le 2v_1(x)v_2(y)$$

by (A3). This, together with (ii) of Lemma 3.1, implies that

$$\mathbb{E}|\phi(t,x,y)| \le c_1 + c_2^2 v_1(x)v_2(y) + 2\sum_{n=1}^{\infty}\delta_n\frac{|y_n - e^{-\lambda_n}x_n|}{1-e^{-2\lambda_n}}$$

for $(t,x,y) \in \mathbb{R}_+ \times H \times D$, where c_1 is a constant. Since v_1 is bounded on bounded sets in H, it follows that for every bounded set $B \subset H$ there exists a constant c_B such that

$$\mathbb{E}|\phi(t,x,y)| \le c_B\Big[1 + v_2(y) + \sum_{n=1}^{\infty}(\delta_n|y_n| + \delta_n e^{-\lambda_n})\Big].$$

The sum on the right-hand side converges for $y \in D$, so setting

$$g(y) = c_B\Big[1 + v_2(y) + \sum_{n=1}^{\infty}(\delta_n|y_n| + \delta_n e^{-\lambda_n})\Big]$$

for $y \in D$, we obtain

$$\mathbb{E}\exp\{\phi(t,x,y)\} \ge \exp\{\mathbb{E}\phi(t,x,y)\} \ge \exp\{-\mathbb{E}|\phi(t,x,y)|\} \ge \exp\{-g(y)\}$$

for all $t \in \mathbb{R}_+$, $x \in B$ and $y \in D$. Setting $h(y) = \exp\{-g(y)\}$ for $y \in D$ and $h(y) = 0$ otherwise we complete the proof of Proposition 2.2. ■

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