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THE UNIVERSAL SKEW FIELD OF FRACTIONS OF A TENSOR PRODUCT OF FREE RINGS

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1. Introduction. Let D be a skew field containing a subfield K and consider the *free D-ring* over K on a set X:

$$D_K\langle X\rangle, \tag{1}$$

defined as the ring generated by X over D, with defining relations $\alpha x = x\alpha$ for all $x \in X$, $\alpha \in K$. In the special case $D = K$ we write $K\langle X\rangle$ for $K_K\langle X\rangle$; further when K is commutative, $K\langle X\rangle$ is called the *free K-algebra* on X.

It is known that $D_K\langle X\rangle$ is always a *fir* (= free ideal ring) and hence has a universal field of fractions (see Th. 2.4.1, p. 105f. and Cor. 7.5.11, p. 417 of [1]). This leaves open the question whether a tensor product $D_K\langle X_1\rangle \otimes_D D_K\langle X_2\rangle$ has a universal field of fractions. When $D = K$ is commutative, we shall answer this question affirmatively in Theorem 3.1 below. This question is of some interest because the multiplication algebra of (1), that is, the subring of $\mathrm{End}(K\langle X\rangle)$ generated by all left and right multiplications, has the form of such a tensor product. Our indirect approach is needed, for as we shall see, the tensor product is not even a Sylvester domain as soon as the sets X_i each have more than one element, or when the tensor product has more than two factors. Some limitation on D is also necessary because in general $D \otimes_K D$ need not be embeddable in a field; indeed, it may not even be an integral domain.

2. The multiplication algebra of a free ring. All rings are assumed to be associative, with a unit element denoted by 1, which is inherited by subrings, preserved by homomorphisms and which acts unitally on modules.

Let R be a ring. Generally we shall write maps on the right, so that the right multiplication $\varrho_a : x \mapsto xa$ $(a, x \in R)$ gives rise to a homomorphism from R to $\mathrm{End}(R)$, while the left multiplication $\lambda_a : x \mapsto ax$ defines an anti-homomorphism. The maps $a \mapsto \varrho_a$ and $a \mapsto \lambda_a$ are injective (thanks to the presence of 1), so the right multiplications form a ring isomorphic to R

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while the ring of left multiplications is isomorphic to the opposite ring R^{o}. When R is a k-algebra (where k is a commutative ring), the multiplication is k-bilinear, by definition, and we have a homomorphism

$$\phi : R^{\mathrm{o}} \otimes_k R \to \mathrm{End}(R), \quad \text{where } f = \sum a_i \otimes b_i \text{ maps to } \phi_f : u \mapsto \sum a_i u b_i. \tag{2}$$

The image of this mapping is often called the *multiplication algebra* $M(R)$ of R. Our aim in this section is to prove the elementary result that for a free ring of rank greater than 1 the map (2) is injective. This is of course well known, but we include a proof since no convenient reference seems available. By the *rank* of a free D-ring $D_K\langle X\rangle$ we understand the cardinal of X: this can be shown to depend only on the ring, not on X (see p. 60 of [1] for the case $D = k$; the same proof works in general).

THEOREM 2.1. *Let D be a skew field, k a central subfield and $R = D_k\langle X\rangle$ the free D-ring over k on a set X. The map* (2) *defines an isomorphism of $R^{\mathrm{o}} \otimes_k R$ with the multiplication ring of R, provided that either* (i) *the rank of R is greater than* 1, *or* (ii) *$X \neq \emptyset$ and $D \neq k$.*

Proof. Let X be indexed as $X = \{x_\lambda\}$ and take a k-basis $\{u_\alpha\}$ of D including $1 = u_0$. Then the finite products of terms $x_\lambda u_\alpha$ form a left D-basis of R; we have to show that ϕ given by (2) is injective.

Consider $f = \sum a_i \otimes b_i$ in the kernel of ϕ. We can take each b_i to be a product of terms $x_\lambda u_\alpha$ (and possibly a factor u_β on the left). Write $b_0 = 1$; if no term in b_0 occurs, this just means that $a_0 = 0$, and our task is to show that $a_i = 0$ for all i. Suppose first that $|X| > 1$ and let x, y be distinct members of X. Choose n larger than the degree of any $a_i b_i$ and consider the result of applying f to x^n and y^n:

$$a_0 x^n + \sum_i a_i x^n b_i = 0, \tag{3}$$

$$a_0 y^n + \sum_i a_i y^n b_i = 0. \tag{4}$$

Since the b_i are distinct, they are linearly independent over k and from (3) we see, by the choice of n, that there can be no b_i that is not a power of x. Similarly (4) shows that each b_i is a power of y^n; this means that there can be no b_i apart from b_0. Hence each a_i must vanish and $f = 0$, as we wished to show.

Next assume that $X = \{x\}$ and the k-basis of D includes 1, $u \neq 1$. We now have (3) and

$$a_0 (xu)^n + \sum_i a_i (xu)^n b_i = 0. \tag{5}$$

As before, (3) shows that each b_i is a power of x while (5) shows that it is a power of xu. Hence there can be no b_i and again $f = 0$. This shows ϕ to be injective in all cases, and it is therefore an isomorphism between $R^{\circ} \otimes_k R$ and the multiplication algebra of R. ∎

In the excluded case we have either $R = k[x]$; then the conclusion is clearly false. Or we have $R = D$ and then the situation depends on the precise nature of D.

3. Universal fields of fractions. Throughout, the term *field* will mean a not necessarily commutative division ring; sometimes the prefix *skew* is added for emphasis. As is well known, every commutative integral domain has a (commutative) field of fractions, which is unique up to isomorphism. By contrast, in the general case the absence of zero-divisors is necessary but not sufficient for a field of fractions to exist, and when it exists it need not be unique.

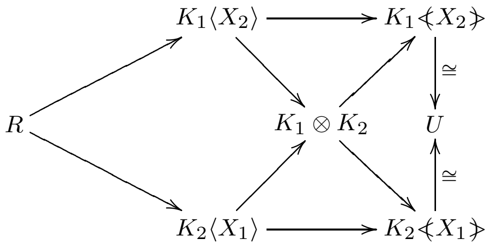
Let us recall the terminology. For any ring R an *R-field* is a field K with a homomorphism $R \to K$; if K is generated as a field by the image of R, it is called an *epic* R-field. An epic R-field for which the canonical map $R \to K$ is injective is called a *field of fractions*. In 7.2 of [1] (and 4.1 of [4]) it is explained that for a given ring R the epic R-fields may be regarded as the objects of a small category, and an initial object in this category, if it exists, is called a *universal R-field*, or if applicable, a *universal field of fractions* of R.

A matrix P over any ring R is said to be *full* if it is square, say $n \times n$, and cannot be written as $P = ST$, where S has fewer than n columns. Clearly any matrix P over R can be mapped to an invertible matrix over a given R-field only if P is full; thus the full matrices are the most that one can hope to invert. We recall that a ring R is called a *Sylvester domain* if in any matrix equation $AB = 0$ over R, one can write $A = A'A''$ and $B = B'B''$, where A'' is $r \times n$, B' is $n \times s$ and $r + s \leq n$. Sylvester domains have a field of fractions over which each full matrix can be inverted; clearly this must be the universal field of fractions, because any epic R-field is characterized up to isomorphism by the matrices over R that are inverted and only full matrices can be inverted. This property, of having "fully inverting" homomorphisms to a field, is actually characteristic of Sylvester domains (see Th. 7.5.10 of [1]). Since every fir is a Sylvester domain, any free ring $D_K\langle X\rangle$ has a universal field of fractions over which all full matrices are inverted. This field is denoted by $D_K \lt\!\!\!\lt X \gt\!\!\!\gt$.

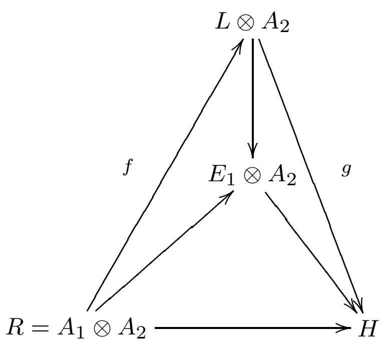
Of course there may well be rings that are not Sylvester domains and nevertheless have a universal field of fractions; this just means that some matrices that are full cannot be inverted over any R-field. Below we shall find examples of such a class.

Theorem 3.1. *Let k be a commutative field and $A_i = k\langle X_i\rangle$ $(X_i \neq \emptyset$, $i = 1, 2)$ be two free k-algebras. Then the tensor product $R = A_1 \otimes A_2$ has a universal field of fractions U containing the universal fields of fractions of each A_i. Moreover, R is a Sylvester domain if and only if one of X_1, X_2 has at most one element.*

Proof. Denote the universal field of fractions of A_i by K_i and let P be any square matrix over R. If P is full over $A_1 \otimes K_2 = K_2\langle X_1\rangle$, it will be invertible over $K_2\langle\!\langle X_1\rangle\!\rangle$, hence it will be full over $K_1 \otimes K_2$, therefore also over $K_1 \otimes A_2 = K_1\langle X_2\rangle$ and so invertible over $K_1\langle\!\langle X_2\rangle\!\rangle$. This and a symmetric argument interchanging 1 and 2 shows that $K_1\langle\!\langle X_2\rangle\!\rangle$ and $K_2\langle\!\langle X_1\rangle\!\rangle$ arise by inverting the same set of matrices over R, namely those that are full over $K_1 \otimes K_2$, and so these fields are isomorphic. We denote the corresponding localization by U; it now remains to show that U is the universal field of fractions of R.



Consider any epic R-field H. The homomorphism $R \to H$ induces an epimorphism $A_1 \to E_1$, where E_1 is the subfield of H generated by the image of A_1. Since E_1 is an epic A_1-field, it arises as the residue-class field of a local ring L (Th. 7.2.2 of [1]). Now L is the universal localization of a set of matrices over A_1 and all these matrices are inverted over H, hence there is a natural homomorphism $g : L \otimes A_2 \to H$. Under this homomorphism the maximal ideal of L is mapped to 0, therefore g can be factored by the natural homomorphism $L \otimes A_2 \to E_1 \otimes A_2$ and we have the diagram



If P is any matrix over R which becomes invertible over H, then it must be full over $E_1 \otimes A_2$ and so it is full over $K_1 \otimes A_2$, because E_1 is an A_1-specialization of K_1. Thus P is full over $K_1\langle X_2\rangle$ and hence invertible over $K_1 \langle\!\langle X_2 \rangle\!\rangle \cong U$. This shows U to be the universal field of fractions of R.

If $X_1 = X_2 = \emptyset$, then $R = k$ and this is a fir; if one of X_1, X_2 has one element, say $X_1 = \{x\}$, then $R = k[x]\langle X_2\rangle$, and this is a Sylvester domain by Th. 5.5.12 of [1]. To complete the proof we have to show that R is not a Sylvester domain when $|X_i| > 1$ for $i = 1, 2$; clearly it will be enough to show this when $|X_1| = |X_2| = 2$. Let us write $X_1 = \{a, b\}$, $X_2 = \{x, y\}$, $R = k\langle a, b\rangle \otimes k\langle x, y\rangle$ and in R consider the equation

$$(6) \qquad (a\ b\ -x\ -y)\begin{pmatrix} x & 0 & y & 0 \\ 0 & x & 0 & y \\ a & b & 0 & 0 \\ 0 & 0 & a & b \end{pmatrix} = 0.$$

In a Sylvester domain every full matrix is a non-zerodivisor, as an easy consequence of the definition, so it will be enough to show that the 4×4 matrix in (6), C say, is full. If not, we would have an equation

$$(7) \qquad C = PQ, \quad \text{where } P \text{ is } 4 \times 3 \text{ and } Q \text{ is } 3 \times 4 \text{ over } R.$$

We shall show that this leads to a contradiction; in the proof we may assume that all the variables commute. Write P_4 for the 3×3 matrix consisting of the first three rows of P. We have

$$(8) \qquad P_4 Q = \begin{pmatrix} x & 0 & y & 0 \\ 0 & x & 0 & y \\ a & b & 0 & 0 \end{pmatrix}.$$

Leaving out one column at a time on the right we get four 3×3 matrices with determinants axy, bxy, ay^2, by^2 (up to sign). Each is $\det P_4$ times the determinant formed from three columns of Q, hence $\det P_4$ is either 1 or y. If it is 1, we can replace P, Q by PP_4^{-1}, P_4Q in (7) and find

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ d & e & f \end{pmatrix} \begin{pmatrix} x & 0 & y & 0 \\ 0 & x & 0 & y \\ a & b & 0 & 0 \end{pmatrix}.$$

By comparing the last row, we find: $dx + fa = 0$, $ex + fb = 0$, $dy = a$, $ey = b$. But this is impossible in R, or even in R made commutative, so $\det P_4 = y$. Let us take R commutative (i.e. take its quotient by the commutator ideal) and write

$$P_4^{-1} = y^{-1} \begin{pmatrix} p & p' & p'' \\ q & q' & q'' \\ r & r' & r'' \end{pmatrix}.$$

Then by (8),

$$Q = y^{-1}\begin{pmatrix} px + p''a & p'x + p''b & py & p'y \\ qx + q''a & q'x + q''b & qy & q'y \\ rx + r''a & r'x + r''b & ry & r'y \end{pmatrix}.$$

Since all the entries of Q lie in R, we obtain from the first row $px + p''a \equiv 0$, $p'x + p''b \equiv 0 \pmod{y}$, hence $p = ua + vy$, $p' = hb + v'y$ for some $u, h, v, v' \in R$ and so there exist $v'', w'' \in R$ such that

$$p'' = -ux + v''y = -hx + w''y.$$

It follows that $(h - u)x + (v'' - w'')y = 0$, so $h = u + ty$, $w'' = v'' + tx$ for some $t \in R$ and we obtain

$$(p\ p'\ p'') = u(a\ b\ -x) + (v\ v' + tb\ v'')y.$$

Similarly for the second and third row, hence we have

$$P_4^{-1} = y^{-1}\begin{pmatrix} u \\ u' \\ u'' \end{pmatrix}(a\ b\ -x) + S,$$

for some matrix S. Writing $S = (s_{ij})$, we have

$$Q = P_4^{-1}\begin{pmatrix} x & 0 & y & 0 \\ 0 & x & 0 & y \\ a & b & 0 & 0 \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} & s_{13} & u \\ s_{21} & s_{22} & s_{23} & u' \\ s_{31} & s_{32} & s_{33} & u'' \end{pmatrix} C.$$

Denote the first factor on the right by T, so that $Q = TC = TPQ$. By (8) the first three columns of Q form a full matrix, so Q is left regular and we have $TP = I$. If we make a Binet–Cauchy expansion by 3×3 minors, we obtain

$$(t_1\ t_2\ t_3\ t_4)(p_1\ p_2\ p_3\ p_4)^T = 1, \tag{9}$$

where t_i is the 3×3 minor obtained by omitting row i from T and p_i is the 3×3 minor obtained by omitting column i from P, while the superscript T indicates transposition. We have seen that $p_4 = y$; by symmetry we have $p_3 = x$, $p_2 = b$, $p_1 = a$, so (9) has the form

$$t_1a + t_2b + t_3x + t_4y = 1,$$

where $t_i \in R$. This is clearly impossible, and it proves that C must be full. Therefore R is not a Sylvester domain and the proof is complete.

We note that even though the matrix C is full, it cannot be inverted over any R-field. This follows because it is not invertible over the universal field of fractions, but it can also be seen directly: if C becomes invertible, then $(a\ b\ -x\ -y)$ must become zero, by (6), but then $C = 0$ and we have a contradiction. In fact, the proof shows that C is not even full over $K_1 \otimes K_2$.

From the proof of Theorem 3.1 we see that U arises by inverting all full matrices over $K_1 \otimes K_2$; this shows the latter to be a Sylvester domain (by

Th. 7.5.10 of [1]), but in fact we can show that it must be a fir. To do so we need a definition. A set Σ of square matrices over a ring R is called *factor-complete* if whenever $AB \in \Sigma$, where A is $r \times n$ and B is $n \times r$, then $r \leq n$ and there is an $n \times (n-r)$ matrix B' such that $(B\ B')$ is invertible over the localization R_Σ. It can be shown that for a semifir R a set Σ is factor-complete if and only if R_Σ is again a semifir; moreover, if R is a fir (and Σ is factor-complete) then R_Σ is also a fir ([1], Th. 7.10.4 and 7.10.7). To apply these results to the present situation, consider the ring $K_1\langle X_2\rangle$; it is a fir and the ring $K_1 \otimes_k K_2$ is obtained from it by localization at the set Σ of all full matrices over $k\langle X_2\rangle$. Since K_2 is a fir, it follows that Σ is factor-complete in $k\langle X_2\rangle$ and it still has this property when considered as matrix set over $K_1\langle X_2\rangle$. Therefore, by the results quoted, $K_1 \otimes K_2$ is a fir and we obtain

Corollary 3.2. *Let $k\langle X_i\rangle$ be a free algebra with universal field of fractions K_i $(i = 1, 2)$. Then $K_1 \otimes_k K_2$ is a fir.*

The ring R of Theorem 3.1 has global dimension two, by Roganov's theorem (see Th. 3.6.10 of [3]); it would be of interest to know whether it is projective-free (i.e. every finitely generated projective module is free, of unique rank). It is known that any Sylvester domain is projective-free and of weak global dimension at most 2, and for commutative rings the converse holds, but not in general (see Cor. 5.5.5 of [1]).

We remark that the tensor product of a finite number of free algebras $k\langle X_i\rangle$ $(i = 1, \ldots, r)$, where each X_i is non-empty, is a fir for $r = 1$ and a Sylvester domain for $r \leq 2$ if at most one X_i has more than one element, but in no other cases. This follows because the polynomial ring $k[x_1, x_2, x_3]$ is not a Sylvester domain (see [1], p. 258, or for an elementary proof, [2]). However, it is not known whether such a tensor product has a universal field of fractions when there are more than two factors.

Finally, we may ask for an analogue for free D-rings, but this will depend on the relation of D to k. To find a universal field of fractions of $D_k\langle X_1\rangle \otimes_k D_k\langle X_2\rangle$ we need to examine $D \otimes_k D$ and this need not even be an integral domain, e.g. if D contains elements algebraic over k but not in k.

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