COLLOQUIUM MATHEMATICUM

VOL. 71 1996 NO. 2

FOURIER TRANSFORM, OSCILLATORY MULTIPLIERS AND EVOLUTION EQUATIONS IN REARRANGEMENT INVARIANT FUNCTION SPACES

BY

LUCA BRANDOLINI AND LEONARDO COLZANI (MILANO)

Let

$$\mathcal{F}\phi(\xi)=\widehat{\phi}(\xi)=\int_{\mathbb{R}^N}\phi(x)\exp(-2\pi i\xi\cdot x)\,dx.$$

The Fourier transform operator $\mathcal{F}$ is naturally defined on $L^1(\mathbb{R}^N)$ with values in $L^\infty(\mathbb{R}^N)$, and can be extended to an isometry of $L^2(\mathbb{R}^N)$. In particular, the Fourier transform maps $L^1+L^2(\mathbb{R}^N)$ into $L^2+L^\infty(\mathbb{R}^N)$. However, P. Szeptycki proved that the largest "solid" space on which the Fourier transform can be defined as a function and not only as a distribution, is not the space $L^1+L^2(\mathbb{R}^N)$ as one could expect, but the amalgam of $\ell^2(\mathbb{Z}^N)$ and $L^1(Q)$, $Q=\{x\in\mathbb{R}^N:-1/2\le x_j<1/2\}$, and one has the inequality

$$\sup_{m\in\mathbb{Z}^N}\Big\{\int_Q|\mathcal{F}\phi(m+x)|^2\,dx\Big\}^{1/2}\le c\Big\{\sum_{n\in\mathbb{Z}^N}\Big(\int_Q|\phi(n+x)|\,dx\Big)^2\Big\}^{1/2}.$$

In particular, the Fourier transform is locally square integrable. See [11] and [3].

In the class of rearrangement invariant Banach function spaces the situation is different. A space of functions is rearrangement invariant if, roughly speaking, the norm of a function is determined by its distribution function. Beside the Lebesgue spaces $L^p(\mathbb{R}^N)$, the class of rearrangement invariant Banach function spaces includes, for example, the Lorentz and the Orlicz spaces, but does not include the amalgams.

Indeed, if we consider the Fourier transform in rearrangement invariant Banach function spaces we have the following result.

THEOREM 1. (i) *The largest rearrangement invariant Banach function space which is mapped by the Fourier transform into a space of locally integrable functions is the space* $L^1+L^2(\mathbb{R}^N)$.

1991 *Mathematics Subject Classification*: 35B30, 42B15.

Key words and phrases: Fourier transform, multipliers, Schrödinger and wave equations, rearrangement invariant Banach function spaces.

(ii) *The only rearrangement invariant Banach function space* $X(\mathbb{R}^N)$ *on which the Fourier transform is bounded*, $\|\mathcal{F}\phi\|_X \le c\|\phi\|_X$, *is the Hilbert space* $L^2(\mathbb{R}^N)$.

Although we have not found any explicit reference we suspect that this theorem is essentially known. We remark that C. Bennett has studied in [1] a generalization of the Hausdorff–Young inequality $\|\mathcal{F}\phi\|_{L^{p/(p-1)}} \le c\|\phi\|_{L^p}$ for rearrangement invariant Banach function spaces with Boyd indices $1/2 < \beta \le \alpha < 1$. We also point out that it is possible to prove an analogue of the above theorem for other operators, such as the Hankel or Fourier–Bessel transform for functions on $(0,\infty)$, which is defined by

$$\mathcal{H}\phi(\xi) = \int_0^\infty \phi(x)\sqrt{x\xi}J_\alpha(x\xi)\,dx.$$

Part (i) of the above theorem could be easily proved via the quoted results [11] and [3] on the extended domain of the Fourier transform, however we shall present an easy and independent proof. Part (ii) will be the main tool in our other results.

Let $P(\xi)$ be a real polynomial in N variables of degree strictly greater than one. For suitably smooth initial data, the Cauchy problem in $\mathbb{R}\times\mathbb{R}^N$

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = iP\left(\dfrac{1}{2\pi i}\dfrac{\partial}{\partial x}\right)u(t,x), \\ \quad u(0,x) = \phi(x), \end{cases}$$

has a solution given, via the Fourier transform in the x variable, by

$$\widehat{u}(t,\xi) = \widehat{\phi}(\xi)\ \exp(itP(\xi)).$$

It is an immediate consequence of the Plancherel formula that

$$\int_{\mathbb{R}^N} |u(t,x)|^2\,dx = \int_{\mathbb{R}^N} |\phi(x)|^2\,dx,$$

but L. Hörmander has shown that the multiplier $\exp(itP(\xi))$ is not bounded on $L^p(\mathbb{R}^N)$ if $p \ne 2$. See Theorem 1.14 in [4] and also [6]. Indeed, it is possible to generalize this result to the much larger class of rearrangement invariant Banach function spaces.

THEOREM 2. *Let* $P(\xi)$ *be a real polynomial in* N *variables of degree strictly greater than one*, *and let the operator* T *be defined on test functions on* $\mathbb{R}^N$ *via the Fourier transform by*

$$\widehat{T\phi}(\xi) = \exp(itP(\xi))\widehat{\phi}(\xi).$$

Then the only rearrangement invariant Banach function space $X(\mathbb{R}^N)$ *on which for a fixed time* t *this operator is bounded*, $\|T\phi\|_X \le c\|\phi\|_X$, *is the space* $L^2(\mathbb{R}^N)$.

Observe that the assumptions in the above theorem are necessary. If the symbol $P(\xi)$ is not purely real then the multiplier operator $\exp(itP(\xi))$ can be either bounded or unbounded on every rearrangement invariant Banach function space. An example is given by the heat multiplier $\exp(-4\pi^2 t|\xi|^2)$ or, more generally, by any multiplier of the form $\exp(-t|\xi|^2 + itP(\xi))$ with P real. In fact when $t > 0$ the kernel associated to the multiplier is a test function and the convolution with such a kernel gives a bounded operator. When $t < 0$ the multiplier grows exponentially and the associated operator is unbounded.

Observe also that for the multiplier operators considered in the above theorem one can have nontrivial "off diagonal" mapping properties from a rearrangement invariant Banach function space into a different one. If $P(\xi)$ is a "generic" real polynomial of degree two then the associated kernel at a fixed time $t \neq 0$ is a bounded function, so that the operator is bounded from $L^1(\mathbb{R}^N)$ into $L^\infty(\mathbb{R}^N)$. Since this operator is also bounded on $L^2(\mathbb{R}^N)$, by interpolation one obtains a Hausdorff–Young type inequality, that is, the operator is bounded from $L^p(\mathbb{R}^N)$ into $L^{p/(p-1)}(\mathbb{R}^N)$, $1 \leq p \leq 2$. A concrete example is given by the Schrödinger equation $\partial_t u = i\Delta_x u$, with associated multiplier $\exp(-4\pi^2 it|\xi|^2)$ and kernel $(4\pi it)^{-N/2}\exp(i|x|^2/(4t))$.

When $P(\xi)$ has degree greater than two, then the associated kernel may also have some decay at infinity and one can obtain different $L^p(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$ estimates. A concrete example is given by the multiplier $\exp(it8\pi^3\xi^3)$ associated with the Airy equation $\partial_t u = -\partial^3_{xxx}u$.

For this kind of estimates and for mixed norm estimates in the (t, x) variables see for example [10] and [5] and references therein.

Our next result deals with the wave equation in $\mathbb{R} \times \mathbb{R}^N$

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u(t,x) = \displaystyle\sum_{k=1}^N \dfrac{\partial^2}{\partial x_k^2}u(t,x), \\ u(0,x) = \phi(x), \\ \dfrac{\partial}{\partial t}u(0,x) = \psi(x). \end{cases}$$

By the principle of conservation of energy

$$E(t) = \int_{\mathbb{R}^N} \left\{ \left| \frac{\partial}{\partial t}u(t,x) \right|^2 + \sum_{k=1}^N \left| \frac{\partial}{\partial x_k}u(t,x) \right|^2 \right\} dx$$

is constant and equal to $E(0)=\|\psi\|^2_{L^2} + \|\nabla\phi\|^2_{L^2}$. W. Littman [8] has shown that there is no analogue of energy conservation if the $L^2(\mathbb{R}^N)$ norm is replaced with the $L^p(\mathbb{R}^N)$ norm when $N > 1$ and $p \neq 2$.

If ϕ and ψ are suitably smooth, the Fourier transform in the x variable of the solution of the wave equation is given by

$$\widehat{u}(t,\xi) = \widehat{\phi}(\xi)\cos(2\pi t|\xi|) + \widehat{\psi}(\xi)\frac{\sin(2\pi t|\xi|)}{2\pi|\xi|},$$

and since

$$\frac{\partial}{\partial t}\frac{\sin(2\pi t|\xi|)}{2\pi|\xi|} = \cos(2\pi t|\xi|),$$

any kind of conservation implies the boundedness of the multiplier operator

$$\widehat{W\phi}(\xi) = \cos(2\pi t|\xi|)\widehat{\phi}(\xi).$$

Of course in dimension one this operator is a sum of translations, hence it is trivially bounded on every translation invariant Banach function space. When the dimension of the space is greater than one it is possible to prove an analogue of Theorem 2, that is, the only rearrangement invariant Banach function space on which this operator is bounded is the space $L^2(\mathbb{R}^N)$. Indeed, it is possible to prove a much stronger result. Contrary to the case of the operators considered in Theorem 2, for the wave operator one cannot have nontrivial "off diagonal" estimates from one rearrangement invariant Banach function space $X(\mathbb{R}^N)$ into a different space $Y(\mathbb{R}^N)$.

THEOREM 3. *Let $X(\mathbb{R}^N)$ and $Y(\mathbb{R}^N)$ be rearrangement invariant Banach function spaces over $\mathbb{R}^N$, $N \geq 2$, and let W be the operator defined on test functions on $\mathbb{R}^N$ via the Fourier transform by*

$$\widehat{W\phi}(\xi) = \cos(2\pi t|\xi|)\widehat{\phi}(\xi).$$

If for a fixed time t this operator is bounded from $X(\mathbb{R}^N)$ into $Y(\mathbb{R}^N)$, $\|W\phi\|_Y \leq c\|\phi\|_X$, then the space $X(\mathbb{R}^N)$ is contained in $L^2(\mathbb{R}^N)$ and $Y(\mathbb{R}^N)$ contains $L^2(\mathbb{R}^N)$, that is, we have the continuous imbeddings $X(\mathbb{R}^N) \subseteq L^2(\mathbb{R}^N) \subseteq Y(\mathbb{R}^N)$. In particular, $L^2(\mathbb{R}^N)$ is the only rearrangement invariant Banach function space on which the wave operator W is bounded.

The above theorem implies that the multipliers $\exp(\pm 2\pi it|\xi|)$ are bounded from $X(\mathbb{R}^N)$ into $Y(\mathbb{R}^N)$ if and only if $X(\mathbb{R}^N) \subseteq L^2(\mathbb{R}^N) \subseteq Y(\mathbb{R}^N)$. The same holds for multipliers of the form $\exp(it\vartheta(\xi))$, with ϑ real, smooth and with an asymptotic expansion $\vartheta(\xi) = \alpha|\xi| + \beta + \gamma|\xi|^{-1} + \ldots$ as $|\xi| \to \infty$. In fact one can prove that $\exp(it\alpha|\xi|) = \chi(t\xi)\exp(it\vartheta(\xi))$, where χ is the Fourier transform of a finite Borel measure. These considerations can be applied for example to the multipliers $\exp(\pm it\sqrt{\lambda^2 + 4\pi^2|\xi|^2})$ associated with the Klein–Gordon equation $\partial^2_{tt}u = (\Delta_x - \lambda^2)u$.

The above theorem does not extend to function spaces which are not rearrangement invariant. For example, since waves propagate with finite speed and the operator W is bounded on $L^2(\mathbb{R}^N)$, it follows that W is also bounded on the amalgam of $L^2(Q)$ with some solid sequence space on $\mathbb{Z}^N$.

The fact that Theorem 3 does not hold in dimension one and does not extend to spaces which are not rearrangement invariant suggests that perhaps one may obtain some positive results for the wave equation with radial boundary data. Indeed, we have the following.

THEOREM 4. *Let $X(\mathbb{R}^3)$ be a rearrangement invariant Banach function space which is contained in $L^2_{\mathrm{local}}(\mathbb{R}^3)$. Then the wave operator W is bounded from the subspace of radial functions in $X(\mathbb{R}^3)+L^2(\mathbb{R}^3)$ into $X(\mathbb{R}^3)+L^2(\mathbb{R}^3)$.*

In particular, it follows from the above result that W is bounded from the radial functions in $L^2(\mathbb{R}^3)+L^p(\mathbb{R}^3)$, $2\le p\le\infty$, into $L^2(\mathbb{R}^3)+L^p(\mathbb{R}^3)$. The above theorem holds true in any dimension, but the easy 3-dimensional proof gets more complicated.

We conclude by mentioning that D. Müller and A. Seeger have recently proved in [9] that for radial solutions of the wave equation with $\frac{\partial}{\partial t}u(0,x)=0$ one has the mixed norm estimate

$$\left\{\frac{1}{2T}\int_{-T}^{+T}\int_{\mathbb{R}^N}|u(t,x)|^p\,dx\,dt\right\}^{1/p}\le c\left\{\int_{\mathbb{R}^N}|u(0,x)|^p\,dx\right\}^{1/p},$$

where $2\le p<2N/(N-1)$ and the constant c is independent of T.

We point out that in dimension $N=3$ this result is a simple consequence of the representation formula for radial solutions of the wave equation which is used in the proof of Theorem 4. Moreover, at the critical index $p=3$ one can prove a weak type result.

1. Rearrangement invariant Banach function spaces. Let (M,Σ,μ) be a measure space. A *function norm* is a map that associates with every measurable function in (X,Σ,μ) a nonnegative number with the following properties:

1) $\|\lambda\phi\|=|\lambda|\cdot\|\phi\|$, $\|\phi+\psi\|\le\|\phi\|+\|\psi\|$,
2) $\|\phi\|=0\Leftrightarrow\phi(x)=0$ μ-almost everywhere,
3) $\|\phi\|=\|\,|\phi|\,\|$,
4) $0\le\psi(x)\le\phi(x)$ μ-almost everywhere $\Rightarrow\|\psi\|\le\|\phi\|$,
5) $0\le\phi_n(x)\nearrow\phi(x)$ μ-almost everywhere $\Rightarrow\|\phi_n\|\nearrow\|\phi\|$,
6) $\mu(E)<\infty\Rightarrow\|\chi_E\|<\infty$,
7) $\mu(E)<\infty\Rightarrow\int_E|\phi(x)|\,d\mu(x)<c(E)\|\phi\|<\infty$.

The collection of all measurable functions with finite norm is a complete normed space and it is called a *Banach function space*.

The *associated space* X^* of a Banach function space X is the Banach function space defined by the function norm

$$\|\psi\|_{X^*} = \sup\Big\{ \int_M |\phi(x)\psi(x)|\, d\mu(x) : \|\phi\|_X \le 1\Big\}.$$

The *distribution function* and the *nonincreasing rearrangement* of a measurable function are defined for every $t \ge 0$ by

$$\mu(\phi,t) = \mu\{x \in M : |\phi(x)| > t\},$$
$$\phi^*(t) = \inf\{s \ge 0 : \mu(\phi,s) \le t\}.$$

A Banach function space X is *rearrangement invariant* if functions with the same nonincreasing rearrangement have the same norm: if $\phi^* = \psi^*$ then $\|\phi\|_X = \|\psi\|_X$.

It turns out that if a Banach function space X is rearrangement invariant, then also the associate space X^* is rearrangement invariant. If a rearrangement invariant Banach function space is contained in another, $X \subseteq Y$, then the inclusion is continuous, $\|\phi\|_Y \le c\|\phi\|_X$.

Since in a rearrangement invariant Banach function space the norm depends only on the nonincreasing rearrangement, it makes sense to speak of the "same" space over different measure spaces. In the sequel we shall write $X(\mathbb{R}^N)$ for a rearrangement invariant Banach function spaces over $\mathbb{R}^N$ equipped with the Lebesgue measure, and we shall use the same letter X to denote two spaces $X(\mathbb{R}^N)$ and $X(\mathbb{R}^D)$ over different measure spaces but with the same image under the nonincreasing rearrangement mapping.

As a general reference on function spaces see the books [2] and [7].

The operators considered in this paper are defined, via the Fourier transform, by

$$\widehat{T\phi}(\xi) = m(\xi)\widehat{\phi}(\xi).$$

When such an operator is bounded from $X(\mathbb{R}^N)$ into $Y(\mathbb{R}^N)$ we call the function $m(\xi)$ a *multiplier* from $X(\mathbb{R}^N)$ into $Y(\mathbb{R}^N)$.

In the sequel we shall need the following elementary properties of multipliers.

LEMMA 5. *Let $m(\xi)$ and $n(\xi)$ be multipliers on a rearrangement invariant Banach function space $X(\mathbb{R}^N)$, let α be a vector in $\mathbb{R}^N$ and let ϱ be an $N\times N$ real matrix with determinant ± 1. Then $\overline{m(\xi)}$, $m(\xi\varrho+\alpha)$, $m(\xi)\exp(2\pi i\alpha\cdot\xi)$, and $m(\xi)n(\xi)$, are again multipliers on $X(\mathbb{R}^N)$.*

Proof. The proof for the spaces $L^p(\mathbb{R}^N)$ is well known, and the proof for rearrangement invariant Banach function spaces is the same. See [4]. ∎

LEMMA 6. *Let $\xi = (\zeta,\eta)$, with $\zeta \in \mathbb{R}^D$ and $\eta \in \mathbb{R}^{N-D}$, and let $\ell(\xi) = m(\zeta)\, n(\eta)$ be a nonzero multiplier of $X(\mathbb{R}^N)$. Then $m(\zeta)$ and $n(\eta)$ are multipliers of $X(\mathbb{R}^D)$ and $X(\mathbb{R}^{N-D})$ respectively.*

Proof. Again the proof for the spaces $L^p(\mathbb{R}^N)$ is well known. Let M and N be the operators associated with the multipliers $m(\zeta)$ and $n(\eta)$. There exists a test function $\psi(z)$ in the Schwartz space $\mathcal{S}(\mathbb{R}^{N-D})$ such that

$$|\{z \in \mathbb{R}^{N-D} : |N\psi(z)| > 1\}| > 1,$$

and it is immediate to verify that if $\phi(y)$ is in $X(\mathbb{R}^D)$ then the product $\phi(y)\,\psi(z)$ is in $X(\mathbb{R}^N)$, and

$$|\{y \in \mathbb{R}^D : |M\phi(y)| > t\}| \leq |\{(y,z) \in \mathbb{R}^N : |M\phi(y)N\psi(z)| > t\}|.$$

The lemma follows from this inequality for distribution functions. ∎

2. Fourier transform

Proof of Theorem 1(i). If the Fourier transform is bounded from a rearrangement invariant Banach function space $X(\mathbb{R}^N)$ into $L^1(Q)$, where Q is the cube $\{x \in \mathbb{R}^N : -1/2 \leq x_j < 1/2\}$, then one has the vector-valued extension

$$\Big\|\Big\{\sum_j |\mathcal{F}\psi_j|^2\Big\}^{1/2}\Big\|_{L^1(Q)} \leq c\Big\|\{\sum_j |\psi_j|^2\Big\}^{1/2}\Big\|_X.$$

See Theorem 1.f.14 in [7].

Assume that $X(\mathbb{R}^N)$ is not contained in $L^1 + L^2(\mathbb{R}^N)$. Since functions in a rearrangement invariant Banach function space are locally integrable, there must exist functions which are in $X \cap L^\infty(\mathbb{R}^N)$ but not in $L^2(\mathbb{R}^N)$. Hence there exists a sequence $\{\alpha_j\}_{j\in\mathbb{Z}^N}$ with $\sum_{j\in\mathbb{Z}^N} |\alpha_j|^2 = \infty$, and with $\sum_{j\in\mathbb{Z}^N} \alpha_j\chi_{j+Q}$ in $X(\mathbb{R}^N)$. Then

$$\Big\|\Big\{\sum_{j\in\mathbb{Z}^N} |\alpha_j\chi_{j+Q}|^2\Big\}^{1/2}\Big\|_X = \Big\|\sum_{j\in\mathbb{Z}^N} \alpha_j\chi_{j+Q}\Big\|_X < \infty,$$

but $\mathcal{F}\chi_{j+Q}(\xi) = \exp(-2\pi ij\cdot\xi)\mathcal{F}\chi_Q(\xi)$, so that for almost every ξ,

$$\Big\{\sum_{j\in\mathbb{Z}^N} |\mathcal{F}[\alpha_j\chi_{j+Q}](\xi)|^2\Big\}^{1/2} = \Big\{\sum_{j\in\mathbb{Z}^N} |\alpha_j|^2\Big\}^{1/2}|\mathcal{F}\chi_Q(\xi)| = \infty. \ ∎$$

Proof of Theorem 1(ii). It is possible to give a direct proof of this part of the theorem, but we have been suggested a shorter approach which is based on part (i) of the theorem.

If the Fourier transform is bounded on the rearrangement invariant Banach function space $X(\mathbb{R}^N)$ then, by (i), $X(\mathbb{R}^N)$ is contained in $L^1+L^2(\mathbb{R}^N)$ and the image via the Fourier transform of $X(\mathbb{R}^N)$ is contained in $L^2 + L^\infty(\mathbb{R}^N)$. Since $\mathcal{F}\mathcal{F}\phi(x) = \phi(-x)$, it also follows that $X(\mathbb{R}^N)$ is contained in $L^2 + L^\infty(\mathbb{R}^N)$.

By splitting a function into $\phi\chi_{\{|\phi|>1\}} + \phi\chi_{\{|\phi|\leq 1\}}$ one easily checks that

$$(L^1 + L^2(\mathbb{R}^N)) \cap (L^2 + L^\infty(\mathbb{R}^N)) = L^2(\mathbb{R}^N)$$

and therefore $X(\mathbb{R}^N) \subseteq L^2(\mathbb{R}^N)$.

Since $\int_{\mathbb{R}^N} \mathcal{F}\phi(x)\psi(x)\,dx = \int_{\mathbb{R}^N} \phi(x)\mathcal{F}\psi(x)\,dx$, the boundedness of the Fourier transform on $X(\mathbb{R}^N)$ implies the boundedness also on the associated space $X^*(\mathbb{R}^N)$, and so $X^*(\mathbb{R}^N) \subseteq L^2(\mathbb{R}^N)$. This yields $X(\mathbb{R}^N) = L^2(\mathbb{R}^N)$. Moreover, one can show, directly or by general properties of inclusions between rearrangement invariant Banach function spaces, that the norm of $X(\mathbb{R}^N)$ is equivalent to the norm of $L^2(\mathbb{R}^N)$. ■

3. Evolution equations. To prove Theorem 2, first we shall show that if $P(\xi)$ is a real polynomial in N variables of degree strictly greater than one and if $\exp(itP(\xi))$ is a multiplier on $X(\mathbb{R}^N)$, then the multiplier $\exp(-4\pi^2 it|\zeta|^2)$ associated with the Schrödinger equation in $\mathbb{R}\times\mathbb{R}$ is a multiplier on $X(\mathbb{R})$. Then we shall show that the Schrödinger multiplier is bounded only on $L^2(\mathbb{R}^N)$.

LEMMA 7. *If $P(\xi)$, $\xi \in \mathbb{R}^N$, is a real polynomial in N variables of degree strictly greater than one, and if for some t the multiplier $\exp(itP(\xi))$ is bounded on $X(\mathbb{R}^N)$, then for some t the multiplier $\exp(-4\pi^2 it|\zeta|^2)$ is bounded on $X(\mathbb{R})$.*

Proof. Observe that if $\exp(itP(\xi_1,\ldots,\xi_N))$ is a multiplier on $X(\mathbb{R}^N)$ then, by Lemma 5, the product

$$\begin{aligned}\exp(itP(\xi_1,\ldots,\xi_j+1,\ldots,\xi_N))&\overline{\exp(itP(\xi_1,\ldots,\xi_j,\ldots,\xi_N))}\\ &= \exp(it(P(\xi_1,\ldots,\xi_j+1,\ldots,\xi_N) - P(\xi_1,\ldots,\xi_j,\ldots,\xi_N)))\end{aligned}$$

is again a multiplier on $X(\mathbb{R}^N)$. This "differentiation" procedure decreases the degree of the polynomial involved in the multiplier, and iterating one can reduce the polynomial to a nonzero quadratic form plus a linear term. Again, by Lemma 5, we can discard the linear term, which corresponds to a translation, and with a rotation we can diagonalize the quadratic form. We thus deduce that

$$\exp(it(\alpha_1\xi_1^2 + \ldots + \alpha_N\xi_N^2)) = \prod_{k=1}^{N} \exp(it\alpha_k\xi_k^2)$$

is a multiplier on $X(\mathbb{R}^N)$. Finally, by Lemma 6, we conclude that $\exp(it\alpha_k\xi_k^2)$ is a multiplier on $X(\mathbb{R})$. ■

LEMMA 8. *Let the operator S be defined by*

$$\widehat{S\phi}(\xi) = \exp(-4\pi^2 it|\xi|^2)\widehat{\phi}(\xi).$$

Then the only rearrangement invariant Banach function space $X(\mathbb{R}^N)$ on which this operator is bounded is the space $L^2(\mathbb{R}^N)$.

Proof. Since the kernel associated with the multiplier $\exp(-4\pi^2 it|\xi|^2)$ is the gaussian $(4\pi it)^{-N/2}\exp(i|x|^2/(4t))$, we have

$$\begin{aligned} S\phi(x) &= (4\pi it)^{-N/2} \int\limits_{\mathbb{R}^N} \exp\left(\frac{i|x-y|^2}{4t}\right)\phi(y)\,dy \\ &= (4\pi it)^{-N/2}\exp\left(\frac{i|x|^2}{4t}\right) \\ &\quad\times \int\limits_{\mathbb{R}^N} \exp\left(\frac{-ix\cdot y}{2t}\right)\left[\exp\left(\frac{i|y|^2}{4t}\right)\phi(y)\right]dy \\ &= (4\pi it)^{-N/2}\exp\left(\frac{i|x|^2}{4t}\right)\mathcal{F}\left[\exp\left(\frac{i|y|^2}{4t}\right)\phi(y)\right]\left(\frac{x}{4\pi t}\right). \end{aligned}$$

Hence the boundedness of the Schrödinger operator S on a rearrangement invariant Banach function space is equivalent to the boundedness of the Fourier transform $\mathcal{F}$, and the lemma follows from Theorem 1. ■

4. The wave equation. In the sequel we shall consider the wave operator at time $t = 1$, that is,

$$\widehat{W\phi}(\xi) = \cos(2\pi|\xi|)\widehat{\phi}(\xi).$$

To prove Theorem 3 we test this operator on characteristic functions, and for this we appeal to the intuitive understanding of waves. The idea is that a circular wave gets higher when it moves towards the center, while it gets smaller moving away. The technical details are particularly simple for waves in three dimensions, as the following lemma shows. We point out that in order to prove the theorem we do not strictly need this lemma, but the analogous lemma in two dimensions. However, since the proof in three dimensions is much more transparent we report this case as well.

LEMMA 9. *Consider the wave equation in $\mathbb{R}\times\mathbb{R}^3$. If ε is suitably small and $0<\delta<\varepsilon<2\delta$, then*

(i) $W\chi_{\{\delta<|x|<\varepsilon\}}(x) > \varepsilon/4$ *in the annulus* $\{1+\delta<|x|<1+\varepsilon\}$,
(ii) $W\chi_{\{1+\delta<|x|<1+\varepsilon\}}(x) > 1/(2\varepsilon)$ *in the annulus* $\{\delta<|x|<\varepsilon\}$.

Proof. The radial part of the Laplace operator Δ in $\mathbb{R}^3$ is the differential operator

$$\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}.$$

Therefore the radial solutions of the wave equation $(\partial^2/\partial t^2)u(t,x) = \Delta u(t,x)$ also satisfy the one-dimensional wave equation $(\partial^2/\partial t^2)[ru(t,r)] =$

$(\partial^2/\partial r^2)[ru(t,r)]$. The solution of the Cauchy problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u(t,x) = \sum_{i=1}^{3} \dfrac{\partial^2}{\partial x_i^2}u(t,x), \\ u(0,x) = \phi(x), \\ \dfrac{\partial}{\partial t}u(0,x) = \psi(x), \end{cases}$$

with radial initial data $\phi(x) = \Phi(|x|)$ and $\psi(x) = \Psi(|x|)$, Φ and Ψ even functions on $\mathbb{R}$, is thus given by the d'Alembert formula

$$u(t,x) = \frac{(t+|x|)\Phi(t+|x|) - (t-|x|)\Phi(t-|x|)}{2|x|} + \frac{1}{2|x|}\int_{|x|-t}^{|x|+t} s\Psi(s)\,ds.$$

Using this formula we obtain

$$W\chi_{\{\delta<|x|<\varepsilon\}}(x) = \frac{|x|-1}{2|x|}$$

for every point in the annulus $\{1+\delta<|x|<1+\varepsilon\}$ and (i) follows. Similarly

$$W\chi_{\{1+\delta<|x|<1+\varepsilon\}}(x) = \frac{1+|x|}{2|x|}$$

in the annulus $\{\delta<|x|<\varepsilon\}$ and also (ii) follows. ∎

LEMMA 10. *Consider the wave equation in $\mathbb{R}\times\mathbb{R}^2$. There exists a positive constant c such that if ε is suitably small and $0<\delta<\varepsilon<2\delta$, then*

(i) $W\chi_{\{\delta<|x|<\varepsilon\}}(x) > c\sqrt{\varepsilon}$ *in the annulus* $\{1+\delta<|x|<1+\varepsilon\}$,
(ii) $W\chi_{\{1+\delta<|x|<1+\varepsilon\}}(x) > c/\sqrt{\varepsilon}$ *in the annulus* $\{\delta<|x|<\varepsilon\}$.

Proof. One has the following integral representation for the solution of the wave equation in $\mathbb{R}\times\mathbb{R}^2$:

$$u(t,x) = \frac{\partial}{\partial t}\left\{\frac{t}{2\pi}\int_{\{|y|<1\}} \frac{\phi(x+ty)}{\sqrt{1-|y|^2}}\,dy\right\} + \frac{t}{2\pi}\int_{\{|y|<1\}} \frac{\psi(x+ty)}{\sqrt{1-|y|^2}}\,dy.$$

Observe that

$$\int_0^{2\pi} \chi_{\{|z|<\varepsilon\}}(r+tse^{i\theta})\,d\theta = 2\,\mathrm{Arcos}\left(\frac{s^2t^2+r^2-\varepsilon^2}{2rst}\right).$$

Hence if $t=1$, $|x|=r$, and $1+\delta<r<1+\varepsilon$, after some painful computations one obtains

$$\begin{aligned} W\chi_{\{\delta<|x|<\varepsilon\}}(x) &= W\chi_{\{|x|<\varepsilon\}}(x) \\ &= 2\int_{r-\varepsilon}^{1} \frac{\operatorname{Arcos}\left(\frac{r^2+s^2-\varepsilon^2}{2rs}\right)}{\sqrt{1-s^2}}\, s\, ds \\ &\quad + 2\int_{r-\varepsilon}^{1} \frac{s(r^2-s^2-\varepsilon^2)}{\sqrt{[(r+\varepsilon)^2-s^2][s^2-(r-\varepsilon)^2][1-s^2]}}\, ds. \end{aligned}$$

The contribution of the first integral is much smaller than $\sqrt{\varepsilon}$. Now observe that if ε is suitably small then $r^2-s^2-\varepsilon^2 \approx \varepsilon$ and $(r+\varepsilon)^2-s^2 \approx \varepsilon$, so that the second integral is of the order of

$$\sqrt{\varepsilon}\int_{r-\varepsilon}^{1} \frac{ds}{\sqrt{[s-(r-\varepsilon)][1-s]}}\, ds = \sqrt{\varepsilon}\,\pi.$$

This proves (i). The proof of (ii) is similar.

Since

$$\int_0^{2\pi} \chi_{\{|z|>1+\delta\}}(r+tse^{i\theta})\, d\theta = 2\operatorname{Arcos}\left(\frac{(1+\delta)^2-r^2-s^2t^2}{2rst}\right),$$

if $t=1$, $|x|=r$, and $\delta<r<\varepsilon$, we have

$$\begin{aligned} W\chi_{\{1+\delta<|x|<1+\varepsilon\}}(x) &= W\chi_{\{|x|>1+\delta\}}(x) \\ &= 2\int_{1+\delta-r}^{1} \frac{\operatorname{Arcos}\left(\frac{(1+\delta)^2-r^2-s^2}{2rs}\right)}{\sqrt{1-s^2}}\, s\, ds \\ &\quad + 2\int_{1+\delta-r}^{1} \frac{s[(1+\delta)^2+s^2-r^2]}{\sqrt{[(r+1+\delta)^2-s^2][s^2-(1+\delta-r)^2][1-s^2]}}\, ds. \end{aligned}$$

The first integral is bounded independently of ε and the second integral is of the order of $1/\sqrt{\varepsilon}$. ■

LEMMA 11. *Consider the wave equation in* $\mathbb{R}\times\mathbb{R}^N$. *Let* α *and* β *be positive numbers. Then there exists a set* A *of measure* α *such that* $W\chi_A(x)>\beta$ *for all* x *in a set of measure greater than* $c\alpha\beta^{-2}$.

Proof. The case $N=2$ follows from the previous lemma. Consider first β small. Let $\{x_j\}_{j=1}^m$ be a sequence of points of $\mathbb{R}^2$ such that $|x_i-x_j|>4$, and let $A=\bigcup_{j=1}^m\{\delta<|x-x_j|<\varepsilon\}$, with $0<\delta<\varepsilon<2\delta<1$. Then by the previous lemma $\mathrm{W}\chi_A(x)>c\sqrt{\varepsilon}$ in the set $\bigcup_{j=1}^m\{1+\delta<|x-x_j|<1+\varepsilon\}$.

Since

$$\Big|\bigcup_{j=1}^{m}\{\delta<|x-x_j|<\varepsilon\}\Big| = m\pi(\varepsilon^2-\delta^2)\approx m\varepsilon(\varepsilon-\delta),$$

$$\Big|\bigcup_{j=1}^{m}\{1+\delta<|x-x_j|<1+\varepsilon\}\Big| = m\pi[(1+\varepsilon)^2-(1+\delta)^2]\approx m(\varepsilon-\delta),$$

we choose $\varepsilon=\beta^2/c^2$ and δ and m such that $m\varepsilon(\varepsilon-\delta)=\alpha$.

Consider now the case β large. Let $A=\bigcup_{j=1}^{m}\{1+\delta<|x-x_j|<1+\varepsilon\}$, with $0<\delta<\varepsilon<2\delta<1$ and $|x_i-x_j|>6$. Then by the previous lemma $W\chi_A(x)>c/\sqrt{\varepsilon}$ in the set $\bigcup_{j=1}^{m}\{\delta<|x-x_j|<\varepsilon\}$. Now we choose $\varepsilon=c^2/\beta^2$ and δ and m such that $m(\varepsilon-\delta)=\alpha$.

The case $N>2$ follows from the case $N=2$ via the method of descent. Let W_2 be the 2-dimensional wave operator and W_N be the N-dimensional one. Let $\widetilde{A}\subset\mathbb{R}^2$ be such that $W_2\chi_{\tilde{A}}(x_1,x_2)>\beta$ in a set $\widetilde{B}\subset\mathbb{R}^2$, and let

$$A=\{(x_1,x_2,\ldots,x_N)\in\mathbb{R}^N:(x_1,x_2)\in\widetilde{A},\ -2<x_j<2,\ 3\le j\le N\}.$$

Since waves propagate with finite speed it is easy to see that

$$W_N\chi_A(x_1,x_2,\ldots,x_N)=W_2\chi_{\tilde{A}}(x_1,x_2)>\beta$$

for every $(x_1,x_2,\ldots,x_N)$ in the set

$$B=\{(x_1,x_2,\ldots,x_N)\in\mathbb{R}^N:(x_1,x_2)\in\widetilde{A},\ -1<x_j<1,\ 3\le j\le N\}. \ \blacksquare$$

LEMMA 12. *There exist three positive constants c_1,c_2,c_3 such that if $\sum_{k=1}^{n}\beta_k\chi_{B_k}$ is a simple function with $\{B_k\}$ pairwise disjoint, then there exists a set A with*

$$c_1\sum_{k=1}^{n}|\beta_k|^2|B_k|\le|A|\le c_2\sum_{k=1}^{n}|\beta_k|^2|B_k|$$

and for nonincreasing rearrangements we have

$$\Big[\sum_{k=1}^{n}\beta_k\chi_{B_k}\Big]^*(t)\le c_3[W\chi_A]^*(t).$$

Proof. Since waves propagate with finite speed, one easily verifies that it is enough to consider simple functions of the form $\beta\chi_B$. The proof then follows from Lemma 11. $\blacksquare$

LEMMA 13. *Let $X(\mathbb{R}^N)$ and $Y(\mathbb{R}^N)$ be rearrangement invariant Banach function spaces. If the operator W is bounded from $X(\mathbb{R}^N)$ into $Y(\mathbb{R}^N)$, then $Y(\mathbb{R}^N)$ contains $L^2(\mathbb{R}^N)$.*

Proof. Let $\sum_{k=1}^{n}\beta_k\chi_{B_k}$ be a simple function with $\{B_k\}$ pairwise disjoint and normalized by $\sum_{k=1}^{n}|\beta_k|^2|B_k|=1$. Then by Lemma 12 there exists a set A with $|A| \approx 1$ and with nonincreasing rearrangement $[\sum_{k=1}^{n} \beta_k \chi_{B_k}]^*(t) \le c[W\chi_A]^*(t)$. Hence

$$\Big\| \sum_{k=1}^{n} \beta_k \chi_{B_k} \Big\|_Y \le c\|W\chi_A\|_Y \le c\|\chi_A\|_X \le c \le c\Big\| \sum_{k=1}^{n} \beta_k \chi_{B_k} \Big\|_{L^2}.$$

Since the inequality $\| \sum_{k=1}^{n} \beta_k \chi_{B_k} \|_Y \le c\| \sum_{k=1}^{n} \beta_k \chi_{B_k} \|_{L^2}$ holds for every simple function in $L^2(\mathbb{R}^N)$ with norm equal to 1, it also holds for every simple function and every limit of simple functions. Hence we have the imbedding $L^2(\mathbb{R}^N) \subseteq Y(\mathbb{R}^N)$. ■

Proof of Theorem 3. If the wave operator W is bounded from $X(\mathbb{R}^N)$ into $Y(\mathbb{R}^N)$ then by Lemma 13, $L^2(\mathbb{R}^N)$ is contained in $Y(\mathbb{R}^N)$.

Since $\int_{\mathbb{R}^N} W\phi(x)\psi(x)\,dx = \int_{\mathbb{R}^N} \phi(x)W\psi(x)\,dx$, the boundedness of W from $X(\mathbb{R}^N)$ into $Y(\mathbb{R}^N)$ implies the boundedness of W from the associated space $Y^*(\mathbb{R}^N)$ into $X^*(\mathbb{R}^N)$. As before we see that $L^2(\mathbb{R}^N)$ is contained in $X^*(\mathbb{R}^N)$, and by duality we can conclude that $X(\mathbb{R}^N)$ is contained in $L^2(\mathbb{R}^N)$. ■

Proof of Theorem 4. We have seen in the proof of Lemma 9 that the three-dimensional wave operator at time $t = 1$ on radial functions $\phi(x) = \Phi(|x|)$ is given by

$$W\phi(x) = \frac{(1 + |x|)\Phi(1 + |x|) - (1 - |x|)\Phi(1 - |x|)}{2|x|}.$$

Let $X(\mathbb{R}^3)$ be a rearrangement invariant Banach function space which is contained in $L^2_{\mathrm{local}}(\mathbb{R}^3)$. We have to prove that the operator W is bounded from the subspace of radial functions in $X(\mathbb{R}^3)+L^2(\mathbb{R}^3)$ into $X(\mathbb{R}^3)+L^2(\mathbb{R}^3)$, and since W is already bounded on $L^2(\mathbb{R}^3)$, it is enough to show that W is bounded from $X(\mathbb{R}^3)$ into $X(\mathbb{R}^3) + L^2(\mathbb{R}^3)$.

Let B be the ball $\{x \in \mathbb{R}^3 : |x| < 3\}$ and decompose the operator W into $W\phi = W[\chi_B\phi] + W[\chi_{\mathbb{R}^3 - B}\phi]$.

The support of $W[\chi_B\phi]$ is contained in the ball $\{x \in \mathbb{R}^3 : |x| < 4\}$, and

$$\|W[\chi_B\phi]\|_{L^2} \le \|\chi_B\phi\|_{L^2} \le c\|\phi\|_X.$$

The support of $W[\chi_{\mathbb{R}^3 - B}\phi]$ is contained in $\{x \in \mathbb{R}^3 : |x| \ge 2\}$ and it is easy to check that this operator is bounded from $X(\mathbb{R}^3)$ into $X(\mathbb{R}^3)$. Indeed, if $|x| \ge 2$,

$$|W[\chi_{\mathbb{R}^3 - B}\phi](x)| \le |\Phi(1 + |x|)| + |\Phi(1 - |x|)|,$$

and

$$|\{|x| > 2,\ |\Phi(1 \pm |x|)| > t\}| \le c|\{|\Phi(|x|)| > t\}|. \ ■$$

Let us say a few words about the proof of Theorem 4 when $N \ne 3$.

The Fourier transform of a radial function can be expressed in terms of the Hankel or Fourier–Bessel transform. Using the asymptotic expansion of

Bessel functions it can be shown that the solutions of the wave equation in $\mathbb{R}^N$ with radial boundary data are given for $|x| > t$ by

$$u(t,x) = \frac{(|x|+t)^{(N-1)/2}\Phi(|x|+t) + (|x|-t)^{(N-1)/2}\Phi(|x|-t)}{2|x|^{(N-1)/2}} + \text{negligible error.}$$

Using this estimate, the proof of the theorem when $N \neq 3$ follows as in the case $N = 3$.

REFERENCES

[1] C. Bennett, *Banach function spaces and interpolation methods III. Hausdorff–Young estimates*, J. Approx. Theory 13 (1975), 267–275.

[2] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, 1988.

[3] J. J. F. Fournier and J. Stewart, *Amalgams of L^p and ℓ^q*, Bull. Amer. Math. Soc. 13 (1985), 1–21.

[4] L. Hörmander, *Estimates for translation invariant operators on L^p spaces*, Acta Math. 104 (1960), 93–140.

[5] C. E. Kenig, G. Ponce and L. Vega, *Oscillatory integrals and regularity of dispersive equations*, Indiana Univ. Math. J. 40 (1991), 33–69.

[6] V. Lebedev and A. Olevskiĭ, *C^1 changes of variables: Beurling–Helson type theorem and Hörmander conjecture on Fourier multipliers*, Geom. Funct. Anal. 4 (1994), 213–235.

[7] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II*, Springer, 1979.

[8] W. Littman, *The wave operator and L^p norms*, J. Math. Mech. 12 (1963), 55–68.

[9] D. Müller and A. Seeger, *Inequalities for spherically symmetric solutions of the wave equation*, Math. Z. 218 (1995), 417–426.

[10] R. S. Strichartz, *Restriction of Fourier transforms to quadratic surfaces and decay of solutions of wave equations*, Duke Math. J. 44 (1977), 705–714.

[11] P. Szeptycki, *Some remarks on the extended domain of Fourier transform*, Bull. Amer. Math. Soc. 73 (1967), 398–402.

Dipartimento di Matematica
Unversità degli Studi di Milano
Via Saldini 50
20133 Milano, Italy
E-mail: brandolini@vmimat.mat.unimi.it
leonardo@vmimat.mat.unimi.it

Received 3 January 1996