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ON A POSITIVE SINE SUM

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1. Introduction. We begin with a statement of our main result.

THEOREM. *For any positive integer n and for $0 \le \varphi \le \pi/2$ we have*

$$\frac{1}{4} + \sum_{k=1}^{n} \frac{\sin(4k+1)\varphi}{(4k+1)\sin\varphi} \ge 0. \tag{1.1}$$

The only case of equality in (1.1) *occurs when $n = 1$ and $\varphi = \arccos(\sqrt{6}/4)$.*

Note that the leading constant 1/4 in the above sum is best possible.

The weak version of (1.1) in which the constant 1/4 is replaced by 1 can be obtained using some more general results on positive trigonometric sums. In particular, Askey and Steinig have given in [2] an alternate version of the proof of a theorem originally published by Vietoris [9], which implies

$$\sum_{k=0}^{n} \alpha_k \sin(4k+1)\varphi > 0, \quad 0 < \varphi < \pi/2, \tag{1.2}$$

where $\alpha_k = 2^{-2k}\binom{2k}{k}$, $k = 0, 1, 2, \ldots$ Since the order of magnitude of α_k is $k^{-1/2}$, a summation by parts shows that (1.2) implies the inequality

$$\sum_{k=0}^{n} \frac{\sin(4k+1)\varphi}{(4k+1)\sin\varphi} > 0, \quad 0 < \varphi < \pi/2. \tag{1.3}$$

In [3], G. Brown and E. Hewitt proved, among other things, a result stronger than (1.2), replacing α_k by $\delta_k = 2^{2k}/(k+1)\binom{2k+1}{k}$, $k = 0, 1, 2, \ldots,$ so that

$$\sum_{k=0}^{n} \delta_k \sin(4k+1)\varphi > 0, \quad 0 < \varphi < \pi/2. \tag{1.4}$$

The order of magnitude of δ_k is also $k^{-1/2}$, nonetheless (1.2) can be derived by (1.4) by a summation by parts.

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Although (1.4) is strong enough to give the sharper version of (1.3) where the leading constant is 3/10, however, *it does not* imply (1.1) in which the constant 1/4 is, as already mentioned, best possible.

Substituting $\pi/2 - \varphi$ for φ in the above inequalities one obtains the corresponding result for cosine sums.

It should be noted that inequalities like (1.2) and (1.4), together with their cosine analogues, have a number of surprising applications, the most striking being estimates for the location of zeros of trigonometric polynomials whose coefficients grow in a certain manner (cf. [2] and [3]). More importantly, these inequalities can be incoporated into the context of more general orthogonal polynomials and this has been emphasised in [1] and [2].

In the present article, our aim is to give a direct proof of (1.1) and discuss a more general inequality involving ultraspherical polynomials (see Section 3) suggested by it.

2. Proof of the main result. We set $\varphi = \theta/2$ in (1.1) and we are concerned with proving that, for $0 < \theta \le \pi$,

$$\frac{1}{2}\sin\frac{\theta}{2} + \sum_{k=1}^{n} \frac{\sin\left(2k+\frac{1}{2}\right)\theta}{2k+\frac{1}{2}} > 0. \tag{2.1}$$

We observe, first of all, that this sum is positive when $0 < \theta \le \pi/(2n+1)$, because all its terms are positive for θ in this range.

Setting $u = \pi - \theta$, we see that inequality (2.1) becomes

$$\frac{1}{2}\cos\frac{u}{2} + \sum_{k=1}^{n} \frac{\cos\left(2k+\frac{1}{2}\right)u}{2k+\frac{1}{2}} > 0.$$

All terms in this last sum are positive for $0 < u \le \frac{\pi}{4n+2}$, hence the sum in (2.1) is positive for $\frac{4n+1}{4n+2}\pi \le \theta \le \pi$. Thus, we seek to prove inequality (2.1) for $\frac{\pi}{2n+1} < \theta < \frac{4n+1}{4n+2}\pi$.

Since

$$\frac{\sin\left(2k+\frac{1}{2}\right)\theta}{2k+\frac{1}{2}} = \int_0^{\theta} \cos\left(2k+\tfrac{1}{2}\right)t\,dt \tag{2.2}$$

and by a direct summation

$$\sum_{k=1}^{n} \cos\left(2k+\tfrac{1}{2}\right)t = \frac{\sin\left(2n+\frac{3}{2}\right)t - \sin\frac{3}{2}t}{2\sin t},$$

it can be easily checked that (2.1) is equivalent to

$$(2.3) \qquad -6\sin\frac{\theta}{2} + 2\ln\left(\frac{1+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right) + \int_0^\theta \frac{\sin(2n+1)t}{\sin\frac{t}{2}}\,dt + \int_0^\theta \frac{\cos(2n+1)t}{\cos\frac{t}{2}}\,dt > 0.$$

In what follows we shall denote

$$f(\theta) = -6\sin\frac{\theta}{2} + 2\ln\left(\frac{1+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}\right),$$

$$I_n(\theta) = \int_0^\theta \frac{\sin(2n+1)t}{\sin\frac{t}{2}}\,dt, \qquad J_n(\theta) = \int_0^\theta \frac{\cos(2n+1)t}{\cos\frac{t}{2}}\,dt,$$

$$S_n(\theta) = f(\theta) + I_n(\theta) + J_n(\theta).$$

So, in view of (2.3), it suffices to establish the positivity of $S_n(\theta)$ in $\left(\frac{\pi}{2n+1}, \frac{4n+1}{4n+2}\pi\right)$. For this purpose, we consider the following cases:

The interval $\frac{4n-3}{4n+2}\pi \le \theta < \frac{4n+1}{4n+2}\pi$, $n \ge 5$. Let

$$\sigma(k) = \int_0^\pi \frac{\sin t}{t+k\pi}\,dt, \qquad k = 0, 1, 2, \dots,$$

and

$$p(x) = \frac{x}{\sin x}.$$

We observe that for θ lying in this interval we have

$$(2.4) \qquad I_n(\theta) > \int_0^{6\pi/(2n+1)} \frac{\sin(2n+1)t}{\sin\frac{t}{2}}\,dt = 2\int_0^{6\pi} \frac{\sin t}{t} p\left(\frac{t}{4n+2}\right) dt$$

$$\ge 2\left\{\sigma(0) - \sigma(1)p\left(\frac{\pi}{2n+1}\right) + \sigma(2) - \sigma(3)p\left(\frac{2\pi}{2n+1}\right) + \sigma(4) - \sigma(5)p\left(\frac{3\pi}{2n+1}\right)\right\}$$

$$\ge 2\left\{\sigma(0) - \sigma(1)\frac{\pi}{11\sin\frac{\pi}{11}} + \sigma(2) - \sigma(3)\frac{2\pi}{11\sin\frac{2\pi}{11}} + \sigma(4) - \sigma(5)\frac{3\pi}{11\sin\frac{3\pi}{11}}\right\}.$$

Numerical integration using Maple V (see [6]) gives

$$\sigma(0) = 1.851937\dots, \qquad \sigma(1) = 0.433785\dots,$$
$$\sigma(2) = 0.25661\dots, \qquad \sigma(3) = 0.1826\dots,$$
$$\sigma(4) = 0.1418\dots, \qquad \sigma(5) = 0.11593\dots,$$

so that in view of (2.4) above we get

$$I_n(\theta) > 2.9725. \tag{2.5}$$

It can be easily seen that in this case

$$J_n(\theta) \geq \int_0^{3\pi/(4n+2)} \frac{\cos(2n+1)t}{\cos\frac{t}{2}}\,dt + \int_{3\pi/(4n+2)}^{(4n-1)\pi/(4n+2)} \frac{\cos(2n+1)t}{\cos\frac{t}{2}}\,dt. \tag{2.6}$$

Clearly,

$$\begin{aligned} \int_0^{3\pi/(4n+2)} \frac{\cos(2n+1)t}{\cos\frac{t}{2}}\,dt &= \frac{1}{2n+1}\int_0^{3\pi/2} \frac{\cos t}{\cos\frac{t}{4n+2}}\,dt \\ &\geq \frac{1}{2n+1}\left(1 - \frac{2}{\cos\frac{3\pi}{8n+4}}\right). \end{aligned} \tag{2.7}$$

We write

$$A_n = \int_{3\pi/(4n+2)}^{(4n-1)\pi/(4n+2)} \frac{\cos(2n+1)t}{\cos\frac{t}{2}}\,dt = \sum_{k=1}^{n-1}\int_{(4k-1)\pi/(4n+2)}^{(4k+3)\pi/(4n+2)} \frac{\cos(2n+1)t}{\cos\frac{t}{2}}\,dt$$

and observe that

$$\begin{aligned} &\int_{(4k-1)\pi/(4n+2)}^{(4k+3)\pi/(4n+2)} \frac{\cos(2n+1)t}{\cos\frac{t}{2}}\,dt \\ &= \frac{1}{2n+1}\int_{(4k-1)\pi/2}^{(4k+1)\pi/2} \left\{\frac{1}{\cos\frac{t}{4n+2}} - \frac{1}{\cos\left(\frac{t}{4n+2}+\frac{\pi}{4n+2}\right)}\right\}\cos t\,dt \\ &\geq \frac{2}{2n+1}\left\{\frac{1}{\cos\frac{4k+1}{8n+4}\pi} - \frac{1}{\cos\frac{4k+3}{8n+4}\pi}\right\}. \end{aligned} \tag{2.8}$$

It follows from this that

$$\begin{aligned} A_n &\geq -\frac{2}{2n+1}\sum_{k=1}^{n-1}\left(\frac{1}{\cos\frac{4k+3}{8n+4}\pi} - \frac{1}{\cos\frac{4k+1}{8n+4}\pi}\right) \\ &= -\frac{2}{2n+1}\sum_{k=2}^{2n-1}(-1)^{k-1}\frac{1}{\cos\frac{2k+1}{8n+4}\pi} \\ &= -\frac{2}{2n+1}\sum_{k=1}^{2n-2}(-1)^{k+1}\frac{1}{\sin\frac{2k+1}{8n+4}\pi} \\ &> -\frac{8}{\pi}\sum_{k=1}^{2n-2}(-1)^{k+1}\frac{1}{2k+1}. \end{aligned}$$

Since

$$\sum_{k=0}^{\infty}(-1)^k\frac{1}{2k+1}=\frac{\pi}{4}, \tag{2.9}$$

we deduce from the above that

$$A_n > 2-\frac{8}{\pi}\qquad\text{for all } n.$$

Hence, from this, (2.6) and (2.7) we obtain

$$\begin{aligned} J_n(\theta) &> \frac{1}{2n+1}\left(1-\frac{2}{\cos\frac{3\pi}{8n+4}}\right)+2-\frac{8}{\pi} \\ &\ge \frac{1}{11}\left(1-\frac{2}{\cos\frac{3\pi}{44}}\right)+2-\frac{8}{\pi}=-0.64164\ldots \end{aligned} \tag{2.10}$$

Since $\frac{17\pi}{22}\le\frac{4n-3}{4n+2}\pi$ for $n\ge5$ and the function $f(\theta)$ is strictly increasing on $\left[\frac{17\pi}{22},\pi\right]$ we have

$$f(\theta)\ge f\left(\frac{17\pi}{22}\right)=-2.19676\ldots,$$

which in combination with (2.5) and (2.10) yields $S_n(\theta)>0.134$.

The interval $\frac{4\pi}{2n+1}<\theta\le\frac{4n-3}{4n+2}\pi$, $n\ge4$. In a similar way, for any θ in this interval we have

$$\begin{aligned} I_n(\theta) &\ge \int_0^{4\pi/(2n+1)}\frac{\sin(2n+1)t}{\sin\frac{t}{2}}\,dt \\ &\ge 2\left(\sigma(0)-\sigma(1)\frac{\pi}{9\sin\frac{\pi}{9}}+\sigma(2)-\sigma(3)\frac{2\pi}{9\sin\frac{2\pi}{9}}\right)>2.935. \end{aligned} \tag{2.11}$$

We also have

$$J_n(\theta)\ge\int_0^{3\pi/(4n+2)}\frac{\cos(2n+1)t}{\cos\frac{t}{2}}\,dt+\int_{3\pi/(4n+2)}^{(4n-5)\pi/(4n+2)}\frac{\cos(2n+1)t}{\cos\frac{t}{2}}\,dt.$$

Now using again (2.8) and (2.9) we get

$$\begin{aligned} \int_{3\pi/(4n+2)}^{(4n-5)\pi/(4n+2)}\frac{\cos(2n+1)t}{\cos\frac{t}{2}}\,dt &= \sum_{k=1}^{n-2}\int_{(4k-1)\pi/(4n+2)}^{(4k+3)\pi/(4n+2)}\frac{\cos(2n+1)t}{\cos\frac{t}{2}}\,dt \\ &\ge -\frac{2}{2n+1}\sum_{k=1}^{n-2}\left(\frac{1}{\cos\frac{4k+3}{8n+4}\pi}-\frac{1}{\cos\frac{4k+1}{8n+4}\pi}\right) \end{aligned}$$

$$= -\frac{2}{2n+1}\sum_{k=3}^{2n-2}(-1)^{k+1}\frac{1}{\sin\frac{2k+1}{8n+4}\pi}$$

$$> -\frac{8}{\pi}\sum_{k=3}^{2n-2}(-1)^{k+1}\frac{1}{2k+1} > 2 - \frac{104}{15\pi}.$$

From (2.7) and the above it follows that

$$J_n(\theta) > \frac{1}{2n+1}\left(1 - \frac{2}{\cos\frac{3\pi}{8n+4}}\right) + 2 - \frac{104}{15\pi} \tag{2.12}$$

$$\geq \frac{1}{9}\left(1 - \frac{2}{\cos\frac{\pi}{12}}\right) + 2 - \frac{104}{15\pi} = -0.325898\dots$$

Now by (2.11), (2.12) and the fact that the function $f(\theta)$ attains its absolute minimum in $[0,\pi]$ at $\theta_0 = 2\arccos\left(\sqrt{3}/3\right) = 1.9106\dots$, so that $f(\theta_0) = -2\sqrt{6} + 2\ln\left(\sqrt{2}+\sqrt{3}\right) = -2.6065478\dots$, we obtain $S_n(\theta) > 0.0025$ in the interval under consideration.

The interval $\frac{\pi}{2n+1} < \theta \leq \frac{4\pi}{2n+1}$, $n \geq 4$. Here we follow again the same argument as in the proof of the two previous cases. In particular, for θ in this range we have

$$I_n(\theta) \geq \int_0^{2\pi/(2n+1)} \frac{\sin(2n+1)t}{\sin\frac{t}{2}}\,dt \tag{2.13}$$

$$\geq 2\left(\sigma(0) - \sigma(1)\frac{\pi}{9\sin\frac{\pi}{9}}\right) > 2.81843.$$

Plainly, in this case

$$J_n(\theta) \geq \int_0^{3\pi/(4n+2)} \frac{\cos(2n+1)t}{\cos\frac{t}{2}}\,dt + \int_{3\pi/(4n+2)}^{7\pi/(4n+2)} \frac{\cos(2n+1)t}{\cos\frac{t}{2}}\,dt.$$

On account of (2.8),

$$\int_{3\pi/(4n+2)}^{7\pi/(4n+2)} \frac{\cos(2n+1)t}{\cos\frac{t}{2}}\,dt \geq \frac{2}{2n+1}\left(\frac{1}{\cos\frac{5\pi}{8n+4}} - \frac{1}{\cos\frac{7\pi}{8n+4}}\right).$$

It follows from (2.7) and the above that

$$J_n(\theta) \geq \frac{1}{2n+1}\left(1 - \frac{2}{\cos\frac{3\pi}{8n+4}} + \frac{2}{\cos\frac{5\pi}{8n+4}} - \frac{2}{\cos\frac{7\pi}{8n+4}}\right) \tag{2.14}$$

$$> -0.1451 \quad \text{for } n \geq 4.$$

Observe also that in this case $\theta < 4\pi/9$ and the function $f(\theta)$ is strictly decreasing on $[0, 4\pi/9]$, so that

$$f(\theta) \geq f\left(\frac{4\pi}{9}\right) = -2.330906\ldots$$

and hence by (2.13) and (2.14) we now obtain $S_n(\theta) > 0.3424$.

In order to establish (1.1) for the remaining cases $n = 1, 2, 3, 4$, we set $x = \cos\varphi$ and recall that

$$\frac{\sin(4k+1)\varphi}{\sin\varphi} = U_{4k}(x)$$

is the Chebyshev polynomial of second kind and degree $4k$, in x. Then we define the polynomials

$$g_n(x) = \frac{1}{4} + \sum_{k=1}^{n} \frac{1}{4k+1} U_{4k}(x).$$

The positivity of the polynomials $g_n(x)$, $n = 2, 3, 4$, in $[0, 1]$ can be easily checked by a straightforward computation. For example, by the method of Sturmian sequences one can verify that these polynomials have no zeros in $[0, 1]$ and since $g_n(0) > 0$, it follows that $g_n(x) > 0$, $0 \leq x \leq 1$. Finally, an elementary computation yields $g_1(x) = \frac{16}{5}x^4 - \frac{12}{5}x^2 + \frac{9}{20} \geq 0$, $0 \leq x \leq 1$.

The proof of (1.1) is now complete.

3. Ultraspherical sums. Let $C_n^\lambda(x)$ be the ultraspherical polynomial of degree n and order λ, $\lambda > 0$, defined by the generating function

$$(1 - 2xr + r^2)^{-\lambda} = \sum_{n=0}^{\infty} C_n^\lambda(x) r^n, \qquad |x| < 1.$$

Recalling that

$$\frac{C_n^1(\cos\theta)}{C_n^1(1)} = \frac{\sin(n+1)\theta}{(n+1)\sin\theta},$$

we see that (1.3) is the special case $\lambda = 1$ of the inequality

$$\sum_{k=0}^{n} \frac{C_{4k}^\lambda(\cos\varphi)}{C_{4k}^\lambda(1)} > 0, \qquad 0 < \varphi < \pi/2, \tag{3.1}$$

which holds for all $\lambda \geq \lambda_0$, where λ_0 is the unique root in (0,1) of the equation

$$\int_0^{3\pi/2} \frac{\cos t}{t^\lambda}\, dt = 0$$

($\lambda_0 = 0.308443\ldots$). This is obtained from our results in [4].

Inequality (1.1) suggests that a sharper version of (3.1) may be true. This is

$$\frac{3}{(\lambda+3)(2\lambda+1)}+\sum_{k=1}^{n}\frac{C_{4k}^{\lambda}(\cos\varphi)}{C_{4k}^{\lambda}(1)}\geq 0,\qquad 0<\varphi<\pi/2. \tag{3.2}$$

Clearly, when $\lambda=1$, (3.2) is the inequality (1.1).

The leading constant $\frac{3}{(\lambda+3)(2\lambda+1)}$ is best possible, because the equality in (3.2) occurs when $n=1$ and $\varphi=\arccos\big(\frac{\sqrt{6(\lambda+3)}}{2\lambda+6}\big)$.

Numerical evidence suggests that (3.2) should be also true for the range $\lambda\geq\lambda_0$. The natural method to prove this is to use the integral representation of ultraspherical polynomials given by the Dirichlet–Mehler formula, see [7, 10.9, 32], (whose (2.2) itself is the special case $\lambda=1$) and then estimate the corresponding integrals in a manner similar to that demonstrated in [4]. However, it appears to be quite laborious to achieve a proof of (3.2) in this way. The reason (3.2) is interesting is that it can be used to prove the positivity of some quadrature schemes by the method developed in [5].

Finally, we note that neither (3.1) nor (3.2) holds for $\lambda<\lambda_0$. Indeed, it is well known that (see, for example, [8, p. 192])

$$\lim_{n\to\infty}\frac{C_n^{\lambda}\big(\cos\frac{z}{n}\big)}{C_n^{\lambda}(1)}=2^{\alpha}\Gamma(\alpha+1)\cdot z^{-\alpha}J_{\alpha}(z),$$

where $\alpha=\lambda-1/2$, J_α being the Bessel function of the first kind and order α. Using this and the fact that

$$J_{-1/2}(t)=\left(\frac{2}{\pi t}\right)^{1/2}\cos t,$$

we obtain

$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{C_{4k}^{\lambda}\big(\cos\big(\frac{\pi}{2}+\frac{\theta}{4n}\big)\big)}{C_{4k}^{\lambda}(1)}$$

$$=\lim_{n\to\infty}\left(\frac{2n}{\theta}\right)^{1-\lambda}\frac{\Gamma\big(\lambda+\frac{1}{2}\big)}{2\sqrt{\pi}}\int_0^{\theta}\frac{\cos t}{t^{\lambda}}\,dt=-\infty\qquad\text{for }\lambda<\lambda_0,\ \theta=3\pi/2.$$

See also the discussion in [10, V, 2.29].

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