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GENERALIZED BRAUER TREE ORDERS

BY

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Introduction. p-adic blocks of integral group rings with cyclic defect have projective resolutions which correspond to walks around the Brauer tree. Moreover, the embedded tree determines the structure of the block. Here we shall replace the tree by an arbitrary finite connected graph which in addition to the ordinary edges has also "truncated edges" with only one vertex attached. For such a graph we construct an order in such a way that "walks along" the graph give rise to projective resolutions. Truncated edges give rise to lattices of finite homological dimension, and the remaining cyclic walks give periodic resolutions. These orders have in the classical situation the property that almost split sequences coincide with projective resolutions. Moreover, these orders are even characterized by these projective resolutions.

In Section 2 locally embedded graphs with truncated edges are defined. In Section 3 we consider combinatorial walks along these graphs. Some are finite — these come from truncated edges, and some are periodic. In Section 4 we construct an order for such a graph, which generalizes directly blocks with cyclic defect. In Section 5 we present some examples to demonstrate the setup, and also to indicate how sensitive things are with respect to the local embedding. Section 6 describes projective resolutions, and in Section 7 we derive some elementary properties, which culminates in Section 8 in a combinatorial description of these orders, derived from the projective resolutions. In Section 9 we restrict to generalizations of blocks of defect p, to derive further properties; we describe the indecomposable lattices and the Auslander–Reiten quiver.

2. Truncated graphs

Assumption 2.1. Unless otherwise stated, we always assume that

- all graphs are finite and connected, and
- all rings are basic and connected.

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DEFINITION 2.2. A *truncated graph* G consists of a finite set of vertices $V = \{v_1, \ldots, v_n\}$ and a set of *edges* $E = E_1 \cup E_2$, where the edges in E_1 join two (not necessarily distinct) vertices — these are called *genuine edges*, and *truncated edges* in E_2, which are associated with only one vertex.

Next we choose a fixed local embedding of G into the plane; i.e. every vertex v together with the germs of edges at v is embedded into the plane. We then call G a *locally embedded truncated graph*.

Note 2.3. • In G we allow loops — a loop is then automatically a genuine edge.

• Though not every finite graph can be embedded into the plane, every graph can locally be embedded into the plane.

• A local embedding of a vertex v is given if we number the germs of edges and truncated edges at v as $\varepsilon(v) := (e_1(v), \ldots, e_{n_v}(v))$. We call n_v the *valency* of v.

• A cyclic permutation of $\varepsilon(v)$ gives rise to the same local embedding.

• If a local embedding is given, it gives rise to a unique numbering of the germs of edges and truncated edges at v modulo a cyclic permutation. This observation is important for the uniqueness of our construction later on.

• Sometimes we shall call a locally embedded truncated graph just a graph, if the meaning is clear from the context.

We shall next generalize "Green's walk" around the Brauer tree [Gr, 74]. Though Green's walk around the Brauer tree can be interpreted as a periodic resolution of certain modules in a block with cyclic defect, the walk eo ipso is just a combinatorial walk along the Brauer tree.

The aim is to define in a natural way walks along our graph which generalize Green's walk along a tree, in such a way that in the end

• we have passed each genuine edge twice, once in either direction,

• we have passed each truncated edge only once starting from its — only — vertex, which we call its *root*.

In case of a tree without truncated edges, there is just one walk, which gives rise to a periodic resolution.

3. Combinatorial walks around a graph. We shall first generalize Green's walk around a tree to the case where there might be truncated edges (cf. the graph G_2 in Section 5), before we generalize it to arbitrary graphs and then finally we pass to graphs with truncated edges.

DEFINITION 3.1 (The walks in case the graph is a truncated tree). We start with a truncated tree G which is embedded into the plane — note that a tree can always be embedded into the plane.

• If G is a tree with all edges genuine, then we have Green's walk around the tree, say clockwise: We start at a vertex v_0 and walk clockwise along the edges. Since G is a tree, we eventually return to v_0. We have only one walk, on which each edge is passed once in either direction. This is *Green's walk* around the Brauer tree. Since it is cyclic, it does not matter at which vertex we start.

• Let now G be a tree which has truncated edges. In this case there will be several walks. For Green's walk it was not essential at which vertex we started — for truncated trees it is.

— We start the walk w_0 at a truncated edge e_0 with vertex (root), say, v_0. We walk along the edge e_0 towards v_0. We then follow Green's walk along the locally embedded graph clockwise until we reach the root of a truncated edge e_1; the possibility $e_0 = e_1$ is not excluded.

— If $e_0 = e_1$, then we are done, and our walk W_0 is a cycle; however, it has a well determined starting point, the truncated edge, and it cannot be continued, since a truncated edge can only be walked along in one direction, the direction towards its root.

— Otherwise ($e_0 \neq e_1$) we start a new walk as above with the edge e_1. Continuing this way, we eventually reach again the root of e_0.

If the tree has no truncated edge, then there is exactly one periodic walk, which passes each edge once in either direction. If the tree has truncated edges, then every walk has finite length, and during all walks we have passed each truncated edge once in the direction towards its root, and we have passed each genuine edge once in either direction.

Definition 3.2 (The walks in case G has loops). We now assume that G is a locally embedded graph which is not a tree. We shall first remove the truncated edges.

• Let e_0 be a truncated edge. We start at this edge walking towards its root and then moving clockwise along the locally embedded graph on the path w_0. Then w_0 must end at the root of a truncated edge; note that this is the only possibility. We have thus constructed a finite walk w_0. (It is possible that the end of w_0 is the root of e_0.)

• We now remove the truncated edge e_0 from G to obtain the graph $G_1 := G \setminus e_0$. We keep track of the direction in which we have passed the edges on the walk $w_0 \setminus e_0$.

• If G_1 still has a truncated edge, we continue in the same fashion to obtain a walk w_1. Note that w_1 can be interpreted both as a walk on G and on G_1.

• We continue this way until, after t steps, we have reached a graph G_t

which does not have any truncated edge. We have now passed along the finite walks $W_0 := \{w_i : 0 \le i \le t-1\}$, which start and end at truncated edges.

• Let us recapitulate where we stand now:

— Altogether we have removed t truncated edges.
— We have the walks in W_0 on G and
— we have marked which genuine edge of G we have passed in which direction on the walks in W_0.

• If now we have already passed each genuine edge in G once in either direction, then we stop. All walks along G have finite length.

• So we may assume that there are some edges which we have not yet passed at all or only in one direction. Let η_0 be an edge which has not been passed in the direction δ_0. We now walk along η_0 in the direction δ_0, starting from the vertex v_0. Since G_t does not have truncated edges, we eventually come back to v_0 from the other direction, having completed a circular walk c_0. Note that the walk c_0 is at the same time a walk in G, since c_0 does not pass along an edge where there is a truncated edge on the path along this edge.

• With c_0 we have constructed a periodic resolution.

• We now add c_0 to the previous walks keeping track of the direction in which we have passed the edges along c_0 and the edges in the walks of W_0. If we have now passed each genuine edge of G (i.e. each edge of G_t) once in either direction, we are done. Otherwise we continue. Eventually we will finish, after having constructed the cycles $W_1 := \{c_j : 0 \le j \le \tau - 1\}$.

• Altogether we have constructed walks $W_0 \cup W_1$ which have the property:

— Each truncated edge is passed exactly once, in the direction towards its root.
— The walks in W_0 give rise to finite walks, starting and ending at roots of truncated edges.
— Each walk in W_1 is circular; it does not start at a truncated edge.
— Altogether we have passed each genuine edge once in either direction.

4. The order of a truncated graph. We shall now construct an order Λ depending on the locally embedded graph G such that certain modules have the above defined walks as projective resolutions. This will be done in two steps. Firstly, we associate with each locally embedded vertex an order which generalizes hereditary orders (cf. [Ha,63], [Br,63], [Ro,92]). We then use the edges to define certain congruences between the orders corresponding to the respective vertices and construct this way the order $\Lambda = \Lambda(G)$.

ASSUMPTION 4.1. • Let G be a finite connected locally embedded truncated graph (cf. Definition 2.2), and let v be a fixed locally embedded vertex with the edges (genuine and truncated) $\varepsilon(v) = (e_1(v), \dots, e_{n_v}(v))$ (cf. Note 2.3).

• Let R be a complete noetherian local integral domain with field of quotients K and residue field $\mathfrak{k}$.

We have the higher dimensional case in mind, in order to describe quantum groups and Hecke algebras in a subsequent paper.

Definition 4.2. An R-*order* Λ in a finite-dimensional K-algebra A is a unital subring of A such that

• $K \cdot \Lambda = A$,
• Λ is finitely generated as R-module.

Definition 4.3 (The order associated with the vertex v (cf. Assumption 4.1)).

• We associate with the vertex v a local R-order $\Omega := \Omega_v$ and a regular principal ideal $\omega := \omega_0 \cdot \Omega = \Omega \cdot \omega_0$ with $\omega_0 := \omega_0(v)$ and $\omega = \omega(v)$.

• The order associated with the locally embedded vertex v is then given by

$$\mathbb{H}_v := \begin{pmatrix} \Omega & \Omega & \Omega & \dots & \Omega & \Omega \\ \omega & \Omega & \Omega & \dots & \Omega & \Omega \\ \omega & \omega & \Omega & \dots & \Omega & \Omega \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \omega & \omega & \omega & \dots & \Omega & \Omega \\ \omega & \omega & \omega & \dots & \omega & \Omega \end{pmatrix}_{n_v} \tag{1}$$

with $\Omega := \Omega_v$ and $\omega = \omega(v)$.

• By Ω_v^j we denote the (j, j) entry of Ω_v in $\mathbb{H}_v$.
• For $1 \le j \le n_v$ we put

$$M_{v,j} := \begin{array}{cl} \begin{pmatrix} \Omega_v \\ \dots \\ \Omega_v \\ \omega_v \\ \omega_v \\ \dots \\ \omega_v \\ \omega_v \end{pmatrix}_{n_v} & \begin{array}{l} \\ \\ j\text{th row} \\ \\ \\ \\ \\ \\ \end{array} \end{array} \tag{2}$$

Then the modules $\{M_{v,j} : 1 \le j \le n_v\}$ constitute a complete set of non-isomorphic indecomposable projective $\mathbb{H}_v$-modules.

• We have natural inclusions — except the last, which is right multiplication by $\omega_0(v)$:

$$M_{v,1} \to M_{v,2} \to \dots \to M_{v,n_v-1} \to M_{v,n_v} \xrightarrow{\cdot\omega_0(v)} M_{v,1}. \tag{3}$$

• We identify $M_{v,j}$ with the germ of the edge $e_j(v)$ at the vertex v; then the above chain of inclusions represents the cycle ε_v from Note 2.3; it is the clockwise walk around v starting at $e_1(v)$.

• For notational convenience we shall identify $M_{v,i}$ with $M_{v,i+n_v}$ keeping in mind, though, that here multiplication with $\omega_0(v)$ is involved.

• Conjugation with the element

$$\underline{\omega} := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ \omega_0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}_{n_v} \tag{4}$$

cyclically permutes the indecomposable projective $\mathbb{H}_v$-modules $\{M_{v,j}\}$; it induces an automorphism α_v of $\mathbb{H}_v$.

• We have seen in Note 2.3 that the local embedding only determines $\varepsilon(v)$ up to a cyclic permutation; this can, however, be compensated by the automorphism α_v. Hence this construction of $\mathbb{H}_v$ only depends on the local embedding of v, the order Ω_v and the element $\omega_0(v)$.

Before we finally can define the order associated with G, we have to fix *some more notation*:

Assumption 4.4. • For each pair $\{v, w\}$ of vertices we have a fixed isomorphism

$$\Omega_v/\omega_v \simeq \Omega_w/\omega_w.$$

We identify all these rings and put $\overline{\Omega} = \Omega_v/\omega_v$ for every $v \in V$.

• By $\phi_v : \Omega_v \to \overline{\Omega}$ we denote a *fixed* epimorphism with kernel ω_v. Note that there may be many different such epimorphisms.

• We use the abbreviation $\Omega_v \overset{\omega}{—} \Omega_w$ to denote the pull-back

$$\begin{array}{ccc} \Omega_v \overset{\omega}{—} \Omega_w & \longrightarrow & \Omega_v \\ \big\downarrow & & \big\downarrow{\scriptstyle \phi_v} \\ \Omega_w & \xrightarrow{\phi_w} & \overline{\Omega} \end{array} \tag{5}$$

Note that $\Omega_v \overset{\omega}{—} \Omega_w$ changes — not only up to isomorphism — if the maps ϕ_v, ϕ_w are changed.

Definition 4.5 (The order associated with G). Let $\mathbb{H} := \bigtimes_{v\in V} \mathbb{H}_v$. We shall describe the order $\Lambda := \Lambda(G)$ as a subring of $\mathbb{H}$ spanning the same algebra. Let $v \in V$ be a vertex and let $e_i(v)$ be the germ of a genuine edge at v (cf. Note 2.3). Since $e_i(v)$ is a genuine edge, it is associated with a second vertex $e_j(w)$; note that $v = w$ is possible.

We now replace in $\mathbb{H}_v \times \mathbb{H}_w$ (in $\mathbb{H}_v$ if $v = w$) the product (cf. Definition 4.3) $\Omega_v^i \times \Omega_w^j$ by $\Omega_v \overset{\omega}{\text{——}} \Omega_w$. This means that we have the (i,i) entry of $\mathbb{H}_v$ identified with the (j,j) entry in $\mathbb{H}_w$ "modulo" ω. This we do for all genuine edges. $\Lambda = \Lambda(G)$ is then called the *order associated with the graph* G (with respect to Ω_v and ϕ_v).

Let us note some obvious properties of $\Lambda(G)$:

Note 4.6. • The indecomposable projective Λ-modules are in bijection with the "edges" of G (both genuine and truncated), so we label them $\{P_e\}_{e\in E}$.

• If $e = e_i(v)$ is a truncated edge with root v, then $P_e = M_{v,i}$ is a projective $\mathbb{H}_v$-module. In this case we have a short exact sequence

$$0 \to M_{v,i-1} \to P_e = M_{v,i} \to \overline{\Omega} \to 0.$$

• If e is a genuine edge, then P_e is the pull-back

$$\begin{tikzcd} P_e \arrow[r] \arrow[d] & M_{v,i} \arrow[d] \\ M_{w,j} \arrow[r] & \overline{\Omega} \end{tikzcd} \tag{6}$$

The case $v = w$ is not excluded. Moreover, in this case we have the following commutative diagram with exact rows and columns:

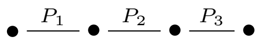
$$\begin{tikzcd} & 0 & 0 & 0 & \\ 0 \arrow[r] & M_{v,i-1} \arrow[r] \arrow[u] & M_{v,i} \arrow[r] \arrow[u] & \overline{\Omega} \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & M_{v,i-1} \arrow[r] \arrow[u] & P_e \arrow[r] \arrow[u] & M_{w,j} \arrow[r] \arrow[u] & 0 \\ 0 \arrow[r] & 0 \arrow[r] \arrow[u] & M_{w,j-1} \arrow[r] \arrow[u] & M_{w,j-1} \arrow[r] \arrow[u] & 0 \\ & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] & \end{tikzcd} \tag{7}$$

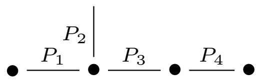
• It should be noted that the kernels of the projections $P_e \to M_{v,i}$ and $P_e \to M_{w,j}$ are projective $\mathbb{H}$-modules, and that they are local Λ-modules; this is the reason why we can interpret the "walks around the graph" as projective resolutions of some of the modules $\{M_{v,j}\}$.

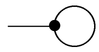
• Actually, there is no reason to require that A is separable; the whole construction works if Ω_v are finitely generated (as modules) local torsion-free R-algebras.

5. Examples. Let us demonstrate the situation with some examples. Let R be a complete Dedekind domain with maximal ideal π generated by π_0, residue field $\mathfrak{k}$ and field of fractions K.

We shall be considering the following graphs:

G_1 : 

G_2 : 

G_3 : 

The *first graph* G_1 corresponds to the order

$$\Lambda_1 := \begin{pmatrix} R & 0 & 0 & 0 & 0 & 0 \\ 0 & R & R & 0 & 0 & 0 \\ 0 & \pi & R & 0 & 0 & 0 \\ 0 & 0 & 0 & R & R & 0 \\ 0 & 0 & 0 & \pi & R & 0 \\ 0 & 0 & 0 & 0 & 0 & R \end{pmatrix}$$

with the following congruences mod π:

$$\lambda_{1,1} \equiv \lambda_{2,2}, \quad \lambda_{3,3} \equiv \lambda_{4,4}, \quad \lambda_{5,5} \equiv \lambda_{6,6}.$$

In this and the following examples we shall label the module in the jth column by M_j. The walk around the graph is given by

$$w_1 : \ldots \to P_1 \to P_2 \to P_3 \to P_3 \to P_2 \to P_1 \xrightarrow{\delta_1} P_1 \to \ldots,$$

and we have a periodic resolution of length 6. Here $\operatorname{coker}(\delta_1) = M_2$; this then uniquely determines the other homomorphisms in w_1.

The *second graph* G_2 corresponds to the order

$$\Lambda_2 := \begin{pmatrix} R & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & R & R & R & 0 & 0 & 0 \\ 0 & \pi & R & R & 0 & 0 & 0 \\ 0 & \pi & \pi & R & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & R & R & 0 \\ 0 & 0 & 0 & 0 & \pi & R & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & R \end{pmatrix}$$

with the following congruences mod π:

$$\lambda_{1,1} \equiv \lambda_{2,2}, \quad \lambda_{4,4} \equiv \lambda_{5,5}, \quad \lambda_{6,6} \equiv \lambda_{7,7}.$$

The walk around the graph is given by

$$w_2 : 0 \to P_2 \to P_3 \to P_4 \to P_4 \to P_3 \to P_1 \to P_1 \to M_2 \to 0, \tag{8}$$

and we have a single finite projective resolution; as a matter of fact, Λ_2 has finite global dimension 7. This shows that by introducing truncated edges, an order of infinite global dimension can be made into an order of finite global dimension. Note that $\mathrm{End}_{\Lambda_2}(P_1 \oplus P_3 \oplus P_4)$ has infinite global dimension.

The *third graph* G_3 corresponds to the order

$$\Lambda_3 := \begin{pmatrix} R & R & R \\ \pi & R & R \\ \pi & \pi & R \end{pmatrix}$$

with the congruence $\lambda_{2,2} \equiv \lambda_{3,3} \bmod \pi$. There are now two walks around the graph, which are given by

$$w_{3,1} : 0 \to P_1 \to P_2 \to M_3 \to 0,$$

a finite projective resolution, and

$$w_{3,2} : \ldots \to P_2 \to P_2 \to M_2 \to 0,$$

which is a periodic resolution of length one.

The next example is to demonstrate how sensitive the situation is with respect to the local embedding into the plane:

Our graphs G_4 and G_5 have three vertices each, $\{u, v, w\}$.

- u has one truncated edge $e_1(u)$ and two genuine edges $e_2(u)$ and $e_3(u)$.
- v has three genuine edges $e_1(v)$, $e_2(v)$ and $e_3(v)$.
- w has one genuine edge $e_1(w)$.
- P_0 corresponds to the truncated edge $e_1(u)$.
- The following vertices are joined by genuine edges:

  — $P_1 = (e_2(u), e_1(v))$,
  — $P_2 = (e_3(u), e_2(v))$ and
  — $P_3 = (e_3(v), e_1(w))$.

For the local embedding for G_4 we choose the permutations $\varepsilon_u := (e_1(u), e_2(u), e_3(u))$ and $\varepsilon_v = (e_1(v), e_2(v), e_3(v))$.

For the local embedding for G_5 we choose the permutations $\varepsilon_u := (e_1(u), e_2(u), e_3(u))$ and $\varepsilon_v = (e_2(v), e_1(v), e_3(v))$.

The order Λ_4 corresponding to G_4 is then given by

$$\Lambda_4 := \begin{pmatrix} R & R & R & 0 & 0 & 0 & 0 \\ \pi & R & R & 0 & 0 & 0 & 0 \\ \pi & \pi & R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R & R & R & 0 \\ 0 & 0 & 0 & \pi & R & R & 0 \\ 0 & 0 & 0 & \pi & \pi & R & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & R \end{pmatrix}$$

with the following congruences mod π:

$$\lambda_{2,2} \equiv \lambda_{4,4}, \quad \lambda_{3,3} \equiv \lambda_{5,5}, \quad \lambda_{6,6} \equiv \lambda_{7,7}.$$

We have the following projective resolutions: a finite projective resolution of length 3

$$w_{4,1} : 0 \to P_0 \to P_1 \to P_2 \to M_3 \to 0,$$

and a periodic resolution

$$w_{4,2} : 0 \to M_7 \to P_3 \to P_1 \to P_2 \to P_3 \to M_7 \to 0.$$

The order Λ_5 corresponding to G_5 is then given by

$$\Lambda_5 := \begin{pmatrix} R & R & R & 0 & 0 & 0 & 0 \\ \pi & R & R & 0 & 0 & 0 & 0 \\ \pi & \pi & R & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & R & R & R & 0 \\ 0 & 0 & 0 & \pi & R & R & 0 \\ 0 & 0 & 0 & \pi & \pi & R & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & R \end{pmatrix}$$

with the following congruences mod π:

$$\lambda_{2,2} \equiv \lambda_{5,5}, \quad \lambda_{3,3} \equiv \lambda_{4,4}, \quad \lambda_{6,6} \equiv \lambda_{7,7}.$$

We have the following projective resolutions: a finite projective resolution of length 5

$$w_{5,1} : 0 \to P_0 \to P_1 \to P_3 \to P_3 \to P_2 \to M_3 \to 0,$$

and a periodic resolution

$$w_{5,2} : 0 \to M_4 \to P_1 \to P_2 \to M_4 \to 0.$$

Finally, we give an example with a higher dimensional ring. Let R be the completion of $\mathbb{Z}[q, q^{-1}]$ completed at the maximal ideal $\langle 3, 1 + q + q^2 \rangle$. Then the Hecke algebra of the symmetric group on three letters with respect to R is given by

$$\mathcal{H}(S_3) := \begin{pmatrix} R & R & 0 & 0 \\ \pi & R & 0 & 0 \\ 0 & 0 & R & 0 \\ 0 & 0 & 0 & R \end{pmatrix}$$

with

$$\lambda_{1,1} \equiv \lambda_{3,3} \bmod (1+q+q^2) \quad \text{and} \quad \lambda_{2,2} \equiv \lambda_{4,4} \bmod (1+q+q^2).$$

6. Projective resolutions. G is as usual a locally embedded graph (with truncated edges) with given local orders (cf. Definition 4.3). We keep the notation of Note 2.3 and the assumptions from 4.4. We shall now consider the order $\Lambda(G)$ from Definition 4.5, and we shall show that with the walks W in G (cf. Definition 3.2), we can associate in a natural way projective resolutions, as is done for blocks with cyclic defect.

Recall from Section 4 (Definition 4.3) that with each vertex $v \in V$ we have associated the order $\mathbb{H}_v$ (cf. Definition 4.3) together with the modules $\{M_{v,j}\}_{1\le j\le n_v}$ (cf. (2)), which we have identified with the germs of the edges $\varepsilon(v) = (e_1(v), \ldots, e_{n_v}(v))$ (cf. Note 2.3), thus incorporating the local embedding of v into the structure of $\mathbb{H}_v$.

Recall from Section 4 (Definition 4.5) that we have associated the indecomposable projective modules $\{P_e\}$ with the edges $\{e\}$, both genuine and truncated; the latter correspond to certain $\mathbb{H}$-lattices.

We shall first discuss Green's walk around a tree which might have truncated edges (cf. the graph G_2 in Section 5), before we generalize it to arbitrary graphs with truncated edges.

Definition 6.1 (The graph is a truncated tree). We start with a truncated tree G which is embedded into the plane.

• If G is a tree with all edges genuine then we have Green's walk around the tree, say clockwise: We start with a vertex, say $v_0 =: 1$, with valency one (i.e. v is a leaf) and module $M_{1,1}$ of $\mathbb{H}_1$ — this is possible, since G is a tree. Next we pass from v_0 along the edge e_1 corresponding to the projective module P_1 to the vertex $v_1 =: 2$. Since G is a tree we have $v_0 \neq v_1$. The edge e_1 gives rise to a germ $e_1(v)$ of v_1 associated with the module $M_{2,1}$. (Note that here we have used the automorphism α_v; cf. (4) in Definition 4.3.) In our bookkeeping we mark that we have passed e_1 once in the "positive direction". We then have the projective cover sequence

$$(9) \qquad 0 \to M_{1,1} \to P_1 \to M_{2,1} \to 0.$$

We now turn clockwise around the vertex 2 and reach the edge e_2, corresponding to the projective module P_2. We now walk along the edge e_2;

note that G is embedded into the plane and we are walking clockwise. As for the second germ of e_2 we have to distinguish two different cases:

— *Either* the vertex at the end of the edge e_2 is $v_2 = 3$, different from the previous ones. In this case we associate with it the module $M_{3,1}$ and we have the exact sequence

$$0 \to M_{2,1} \to P_2 \to M_{3,1} \to 0. \tag{10}$$

— *Or* the vertex at the end of the edge e_2 is again $v_0 = 1$, in which case we obtain the exact sequence

$$0 \to M_{2,1} \to P_2 \to M_{1,1} \to 0 \tag{11}$$

and we are done.

We thus may assume the first situation. We now walk along the next edge e_3. Here again we have to distinguish different cases:

— The edge e_3 is different from e_2, and hence its end point is $v_3 = 4$. Then we have a new projective module P_3 and a new module $M_{4,1}$ and continue in the above explained fashion.

— The edge e_3 is our edge e_2. Then we do not add a new projective module. However, we add a new module $M_{2,2}$ associated with the vertex 2. We then get the projective cover sequence

$$0 \to M_{3,1} \to P_2 \to M_{2,2} \to 0. \tag{12}$$

Continuing this way, we finally return to the vertex 1. We have thus passed each edge exactly twice with only one walk — this was Green's walk around the Brauer tree.

• Let now G be a tree which has truncated edges. Now there will be several walks.

We start the walk of W_1 at a truncated edge e_0 with vertex, say, $v_0 = 1$ as in Definition 3.1. Note that the vertices are embedded into the plane, and that we can cyclically permute the germs of the edges leaving a certain vertex.

The importance of a truncated edge is that the module at the end of this edge is a projective module; here $M_{1,1}$ is projective. The resolution which we construct is thus a resolution of a module of finite homological dimension.

We start at the vertex $v_0 = 1$ with the projective indecomposable module $M_{1,1}$. We now follow Green's walk of projective resolutions until we reach a vertex v_1, where (note that the graph is locally embedded into the plane) the final edge on the walk is a truncated edge e_1. Here we stop. If $e_0 = e_1$ then we are done. Otherwise we start a new walk as above with the edge e_1. Continuing this way we eventually reach again the vertex v_0 at a position where the next (clockwise) edge would be e_0. Then we are done.

We note that, G being a tree, all the non-projective modules at the vertices have finite homological dimension.

Definition 6.2 (G has loops). We now assume that G is a locally embedded graph which is not a tree.

• We first consider modules of finite homological dimension which correspond to walks in W_0 (cf. Definition 3.2). For each vertex v we have the fixed arrangement of the germs of edges $\varepsilon(v) = (e_1(v), \ldots, e_{n_v}(v))$ and the corresponding projective $\mathbb{H}_v$-modules $\{M_{v,j}\}$. As above, each walk $w_i \in W_0$ can thus be identified with a finite projective resolution, starting with the truncated edge with which w_i starts and ending before the truncated edge at which w_i ends. We shall denote the projective resolution by $\mathbb{P}_{w_i}$. Its length coincides with the length of w_i. After we have passed all truncated edges, we have exhausted all the walks in W_0, i.e. all the finite projective resolutions.

• We now turn to the cyclic walks in W_1 (cf. Definition 3.2). As above, we can associate with each cyclic walk c_j a periodic resolution $\mathbb{P}_{c_j}$.

• Let $\mathbb{P}_0 = \{\mathbb{P}_{w_i}\}_{w_i \in W_0}$ and $\mathbb{P}_1 = \{\mathbb{P}_{c_j}\}_{c_j \in W_1}$ and put $\mathbb{P} := \mathbb{P}_0 \cup \mathbb{P}_1$.

Note 6.3. • Every module $M_{v,j}$ occurs in one of the projective resolutions in $\mathbb{P}$.

• The projective modules in the projective resolutions in $\mathbb{P}$ are all indecomposable — the modules $M_{v,j}$ are local.

• Every projective corresponding to a truncated edge occurs exactly once among the projective resolutions in $\mathbb{P}$.

• Every projective corresponding to a genuine edge occurs exactly twice among the projective resolutions in $\mathbb{P}$.

7. Elementary properties of graph orders.

Let us note some obvious properties of $\Lambda(G) =: \Lambda$.

Note 7.1. • The indecomposable projective Λ-modules are in bijection with the "edges" of G, so we have labeled them P_e.

• If $e = e_i(v)$ is a truncated edge with vertex v, then $P_e = M_{v,i}$. In this case we have a short exact sequence

$$0 \to M_{v,i-1} \to P_e = M_{v,i} \to \overline{\Omega} \to 0.$$

• If e is a genuine edge, then P_e is the pull-back

$$\begin{tikzcd} P_e \arrow[r] \arrow[d] & M_{v,i} \arrow[d] \\ M_{w,j} \arrow[r] & \overline{\Omega} \end{tikzcd}$$

Moreover, in this case we have the following commutative diagram with exact rows and columns:

$$
\tag{13}
\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & M_{v,i-1} \arrow[r] \arrow[u] & M_{v,i} \arrow[r] \arrow[u] & \overline{\Omega} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & M_{v,i-1} \arrow[r] \arrow[u] & P_e \arrow[r] \arrow[u] & M_{w,j} \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[u] & M_{w,j-1} \arrow[r] \arrow[u] & M_{w,j-1} \arrow[r] \arrow[u] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
$$

• It should be noted that the kernels of the projections $P_e \to M_{v,i}$ and $P_e \to M_{w,j}$ are projective $\mathbb{H}$-modules, and that they are local Λ-modules; this is the reason why it was possible to define the "walk around the graph".

LEMMA 7.2. *Let G be the locally embedded graph with associated order $\Lambda := \Lambda(G)$ and underlying order $\mathbb{H} = \bigtimes_{v\in V} \mathbb{H}_v$ with indecomposable (non-isomorphic) projective $\mathbb{H}_v$-modules $\{M_{v,j}\}_{j=1}^{n_v}$. Then*

• *The syzygies of $M_{v,j}$ all are indecomposable local Λ-modules of the form $M_{w,j}$.*

• *Each module $M_{v,j}$ has as Λ-module a periodic resolution if and only if G does not have truncated edges if and only if all walks along G are periodic. The period of the projective resolution coincides with the period of the walk.*

• *The walks starting at a truncated edge are finite and they give rise to modules of finite homological dimension. The homological dimension is bounded by the length of the walk.*

• *Λ has finite homological dimension if and only if there are no cycles in G, or equivalently, every walk is finite.*

• *Λ has homological dimension bounded by two if and only if every genuine edge has as successor a truncated edge and as predecessor also a truncated edge* ([1]).

Proof. These are direct consequences of the definition of $\Lambda = \Lambda(G)$. ■

([1]) The notions of predecessor and successor are to be understood with respect to a local embedding and walking clockwise.

8. A characterization of graph orders. In order to prove that blocks of cyclic defect are Green orders (cf. [Ro,92]) one had to give an internal characterization of Green orders: *The structure follows directly from the "walk around the tree"*. A similar result can be proved for graph orders. Above in Definition 4.5 we have constructed an order Λ and an order $\mathbb{H}$, given a locally embedded graph G and orders Ω_v together with selected homomorphisms $\Omega_v \to \overline{\Omega}$, which had as kernel a regular principal ideal. From these data we have then derived projective resolutions of projective Λ-lattices.

The combinatorial description of "graph" orders starts with an order and projective resolutions of certain lattices, which correspond to walks in W, to conclude that Λ is then the graph order of a locally embedded graph. In [Ro,92] Green orders were defined for a tree without truncated edges. Here we extend this definition to graphs with truncated edges.

DEFINITION 8.1. Let Λ be a connected basic R-order which is Cohen–Macaulay. Assume that there are local Cohen–Macaulay modules $\{M_w\}_{w\in W}$ which have minimal projective resolutions $\mathbb{P}_w$ consisting of indecomposable projective modules.

• Let $\{\mathbb{P}_w : w \in W_0\}$ be those projective resolutions which are finite, say P_w ends in the projective module Q_w for $w \in W_0$.

• Let $\{\mathbb{P}_w : w \in W_1\}$ be the periodic projective resolutions of minimal period p_w for $w \in W_1$.

Assume that in the projective resolutions $\bigcup_{w\in W} \mathbb{P}_w$,

• the indecomposable projective modules $\{Q_w\}_{w\in W_0}$ occur exactly once,

• the other indecomposable projective modules occur exactly twice.

Then Λ is called a *generalized Green order*.

THEOREM 8.2. *Let Λ be a generalized Green order. Then there exists a graph G with truncated edges such that Λ is the graph order of G with respect to some set of local orders Ω_v and homomorphisms $\Omega_v \to \overline{\Omega}$ as in Definition* 4.5, *and its projective resolutions are given by walks in the graph according to Section* 6.

Proof. The proof will be published in a subsequent paper and essentially follows the arguments of the proof of Theorem 2.3 in [Ro,92]. Let us briefly *indicate the idea*:

According to the hypothesis, we have the set of maximal projective resolutions, some of which are periodic, and some terminating in a projective indecomposable module. Moreover, each projective occurring in such a resolution is indecomposable, and all projectives except the terminating ones occur with multiplicity two. So let $\{P_e\}_{e\in E_1}$ be those indecomposable projectives occurring twice, and let $\{P_e\}_{e\in E_2}$ be those occurring only once (at the end of a projective resolution).

Since each indecomposable projective module P_e for an edge $e \in E_1$, e a genuine edge, occurs twice, we can associate with it a commutative diagram like (7). Let us change the notation of diagram (7) slightly, and write $P_e^-(1) := M_{v,i}$ and $P_e^-(2) := M_{w,j}$ and for the radicals we have $P_e^+(1) := M_{v,i-1}$ and $P_e^+(2) := M_{w,j-j}$ and write $\overline{\Omega}_e = \overline{\Omega}$. We now put

$$\mathbb{H} := \operatorname{End}_\Lambda\Big(\bigoplus_{e\in E_1}(P_e^-(1)\oplus P_e^-(2))\oplus\Big(\bigoplus_{e\in E_3} P_e\Big)\Big).$$

Then one can show that the hypotheses of Definition (1.1) of [Ro,92] are satisfied. An application of Theorem 1.5 of [Ro,92] then shows that $\mathbb{H}$ has the form of the order $\bigoplus_{v\in V}\mathbb{H}_v$, where $\mathbb{H}_v$ is described in Definition 4.3.

The structure of Λ now follows, by analyzing the given projective resolutions. ∎

9. Graph orders of type "defect p". The situation becomes much more lucid if we make the following assumptions:

ASSUMPTION 9.1. • R is a complete discrete rank one valuation ring with field of fractions K, parameter π_0, maximal ideal π and residue field $\mathfrak{k}$.

• Ω_v is the maximal order in a skew-field D_v.

• $\pi_v := \pi_{v,0}\cdot\Omega_v = \Omega_v\cdot\pi_{v,0} = \operatorname{rad}(\Omega_v)$.

• $\phi_v : \Omega \to \Omega_v/\pi_v \simeq \mathfrak{f}$ is a fixed epimorphism, where $\mathfrak{f}$ is a finite extension of $\mathfrak{k}$, which is independent of v.

• $G = (V, E)$ is a graph as above.

• $\mathbb{H} = \bigtimes_{v\in V}\mathbb{H}_v$ is a product of hereditary orders $\mathbb{H}_v$ of size n_v over Ω_v.

• $\Lambda := \Lambda(G)$ is the associated order.

We shall keep the notation from the previous sections.

LEMMA 9.2. *The indecomposable Λ-lattices are*

• *the projective indecomposable Λ-lattices, which are in bijection with the edges of G, both genuine and truncated,*

• *the non-projective indecomposable Λ-lattices which are indecomposable $\mathbb{H}$-lattices. There is an overlap, coming from the truncated edges. Let $M_{v,j}$ be an indecomposable $\mathbb{H}$-lattice which is not a projective Λ-lattice. Then the projective cover sequence* (*cf.* (7))

$$0 \to \Omega_1(v,j) \to P_{v,j} \to M_{v,j} \to 0$$

that arises from the corresponding walk in the graph is at the same time the almost split sequence [RoSch,76] *of $M_{v,i}$.*

Note 9.3. It is easy to describe the Auslander–Reiten quiver [RoSch,76] from Lemma 9.2 and (7).

Proof of Lemma 9.2. The statements will follow if we can show that the almost split sequences are identical with the projective cover sequences. For a Λ-lattice X we define

$$\operatorname{rad}_\Lambda^-(X) := \operatorname{Hom}_R(\operatorname{rad}_\Lambda(\operatorname{Hom}_R(X,R)),R).$$

Because of the assumption about $\overline{\Omega}$ — it is simple — we have for an indecomposable projective Λ-lattice P_e corresponding to a genuine edge (using the notation of (7))

$$\operatorname{rad}_\Lambda(P_e) = M_{v,i-1} \oplus M_{w,j-1} \quad \text{and} \quad \operatorname{rad}_\Lambda^-(P_e) = M_{v,i} \oplus M_{w,j}.$$

The natural injections $\operatorname{rad}_\Lambda(P_e) \to P_e$ and $P_e \to \operatorname{rad}_\Lambda^-(P_e)$ are direct sums of irreducible maps. Thus the exact sequences

$$\mathcal{E}_1(e) : 0 \to M_{v,i-1} \to P_e \to M_{w,j} \to 0$$

and

$$\mathcal{E}_2(e) : 0 \to M_{w,j-1} \to P_e \to M_{v,i} \to 0.$$

have all maps irreducible. Since rationally the sequences coincide with the almost split sequences, we conclude that $\mathcal{E}_i(e)$ are almost split sequences. Since this way we have constructed a finite connected component of the Auslander–Reiten quiver, the statements of the lemma follow. ∎

We have seen above that truncated edges give rise to lattices of finite homological dimension.

This can be used to modify graph orders by inserting truncated edges and thus modifying the homological dimension of lattices, say from ∞ to finite. In this process, only new *projective lattices* are introduced.

This can be used for example to show that graph orders are stably equivalent to orders of global dimension two:

PROPOSITION 9.4 ([Ro,95]). *Let Λ be a graph order of a graph G which does not have any truncated edges, and let $\mathbb{H}$ be the associated hereditary order. Then the order*

$$\Delta = \Delta(\Lambda, \mathbb{H}) := \begin{pmatrix} \Lambda & \mathbb{H} \\ \operatorname{rad}(\Lambda) & \mathbb{H} \end{pmatrix} \tag{14}$$

has global dimension two and is stably equivalent to Λ.

Proof. We indicate the arguments of the proof. Since Λ does not have any truncated edges, and since $\mathbb{H}$ is hereditary, we conclude that

$$\operatorname{rad}(\Lambda) = \operatorname{rad}(\mathbb{H}) \simeq \mathbb{H}.$$

Hence by making $\mathbb{H}$-lattices projective, in Δ we have reached that $\operatorname{rad}(P_\Delta)$ is projective, where P_Δ is the image of the indecomposable projective Λ-lattice P in Δ. So these modules have homological dimension 1. Indecomposable $\mathbb{H}$-modules have in Δ homological dimension 2. ∎

REFERENCES

[Br,63] A. Brumer, *Structure of hereditary orders*, Bull. Amer. Math. Soc. 69 (1963), 721–729.

[Da,66] E. C. Dade, *Blocks with cyclic defect groups*, Ann. of Math. 84 (1966), 20-48.

[Gr,74] J. A. Green, *Walking around the Brauer tree*, J. Austral. Math. Soc. 17 (1974), 197–213.

[Ha,63] M. Harada, *Structure of hereditary orders over local rings*, Osaka J. Math. 14 (1963), 1–22.

[Ro,92] K. W. Roggenkamp, *Blocks with cyclic defect and Green-orders*, Comm. Algebra 20 (1992), 1715–1734.

[Ro,95] —, *Green orders and orders of global dimension two*, Far East J. Math. Sci. 3 (2) (1995), 101–197.

[RoSch,76] K. W. Roggenkamp and J. Schmidt, *Almost split sequences for integral group rings and orders*, Comm. Algebra 4 (1976), 893–917.

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