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THE PURE-PROJECTIVE IDEAL OF A MODULE CATEGORY

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Introduction. The Galois covering technique is an important method used successfully in contemporary representation theory of algebras over a field. Originally, it was invented for the study of module categories over representation-finite algebras (see [14], [1], [10]), afterwards it was generalized and adopted for the representation-infinite case (see [5], [7], [6], [13], [2]) and applied successfully (see [6], [20], [21], [22], [11]). It was also defined and investigated for matrix problems in [17], [8], [18], [19], [4].

The Galois covering reduction to stabilizers, proposed in [3], is a unification and a generalization of the approach in [10], [5], [6] and [2]. It allows us to reduce in good situations the study of certain subcategories of the module category over the quotient R/G, for a given Galois covering $F : R \to R/G$, to the study of the representation categories of stabilizers of certain indecomposable locally finite-dimensional R-modules called G-atoms (see [3]). An especially important role is played by the G-atoms with infinite cyclic stabilizers, since they are responsible for the appearance of 1-parameter families of R/G-modules. The effect of this reduction essentially depends on the behaviour of homomorphisms between G-atoms, in particular on a special splitting property of the Jacobson radical of the category $\mathrm{MOD}\,R$ of R-modules. In most cases where Galois coverings were efficiently used, each nonisomorphism between any two G-atoms factors through a direct sum of finite-dimensional R-modules.

The main aim of this paper is to define and investigate the pure-projective ideal $\mathcal{P}u$ of $\mathrm{MOD}\,R$ consisting of all homomorphisms having a factorization through a direct sum of finite-dimensional R-modules. In connection with Galois coverings we study mainly "local" properties of $\mathcal{P}u$ related with some stabilizer property of a free action of a group of K-linear automorphisms of R (see Theorem A).

The situation we deal with is the following. Let k be a field and R be a *lo-*

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cally bounded k-category, i.e. all objects of R have local endomorphism rings, different objects are nonisomorphic, and both sums $\sum_{y\in R}\dim_k R(x,y)$ and $\sum_{y\in R}\dim_k R(y,x)$ are finite for each $x\in R$. By an R-module we mean a contravariant k-linear functor from R to the category of k-vector spaces. An R-module M is *locally finite-dimensional* (resp. *finite-dimensional*) if $\dim_k M(x)$ is finite for each $x\in R$ (resp. $\sum_{x\in R}\dim_k M(x)$ is finite). We denote by $\mathrm{MOD}\,R$ the category of all R-modules, and by $\mathrm{Mod}\,R$ (resp. $\mathrm{mod}\,R$) the full subcategory of $\mathrm{MOD}\,R$ formed by all locally finite-dimensional (resp. finite-dimensional) R-modules.

By the *support* of any object M in $\mathrm{MOD}\,R$ we shall mean the full subcategory $\mathrm{supp}\,M$ of R formed by the set $\{x\in R : M(x)\neq 0\}$. If $f : M\to N$ is a homomorphism of R-modules then $\mathrm{supp}(\mathrm{Im}\,f)$ is called simply the support of f and shortly denoted by $\mathrm{supp}\,f$.

Given a full subcategory C of R and an R-module M we denote by $M_{|C}$ the C-module being the restriction of M to C. If $f : M\to N$ is an R-homomorphism we denote by $f_{|C} : M_{|C}\to N_{|C}$ the C-homomorphism being the restriction of f to C.

For each pair of R-modules M and N in $\mathrm{MOD}\,R$ we define two subspaces

$$(*)\qquad \mathcal{P}u(M,N)\subseteq \mathrm{Hom}_R(M,N)\quad\text{and}\quad \mathcal{F}(M,N)\subseteq \mathrm{Hom}_R(M,N)$$

of $\mathrm{Hom}_R(M,N)$ as follows. The vector space $\mathcal{P}u(M,N)$ consists of all R-homomorphisms $f : M\to N$ having a factorization through a direct sum of finite-dimensional modules. An R-homomorphism $f : M\to N$ belongs to $\mathcal{F}(M,N)$ if and only if for every finite full subcategory C of R there exists $f'\in \mathrm{Hom}_R(M,N)$, such that $\mathrm{supp}(\mathrm{Im}\,f')$ is finite and $f_{|C}=f'_{|C}$.

The inclusion $\mathcal{P}u(M,N)\subseteq \mathcal{F}(M,N)$ is shown in Corollary 1.4, for every pair M,N in $\mathrm{Mod}\,R$. In this case $\mathcal{P}u(M,N)$ consists of all R-homomorphisms $f : M\to N$ having a factorization through a direct sum of finite-dimensional modules, which is additionally a locally finite-dimensional R-module (see Corollary 1.2).

It is easy to see that the subspaces $(*)$ define two two-sided ideals

$$(**)\qquad \mathcal{P}u(\,\cdot\,,-)\subseteq \mathrm{Hom}_R(\,\cdot\,,-)\quad\text{and}\quad \mathcal{F}(\,\cdot\,,-)\subseteq \mathrm{Hom}_R(\,\cdot\,,-)$$

of the category $\mathrm{MOD}\,R$. Since $\mathrm{MOD}\,R$ is a Grothendieck k-category [9] and has a set of finite-dimensional generators, the direct sums of finite-dimensional R-modules are just pure-projective R-modules (see [15], [16]) and $\mathcal{P}u$ is the two-sided ideal of $\mathrm{MOD}\,R$ generated by all pure-projective R-modules. Following a suggestion of Daniel Simson we call $\mathcal{P}u$ the *pure-projective ideal* of $\mathrm{MOD}\,R$.

Let G be a group of k-linear automorphisms of R acting freely on objects of R. Then G acts on the category $\mathrm{MOD}\,R$ by translations ${}^g(-)$, which assign to each M in $\mathrm{MOD}\,R$ the R-module ${}^gM = M\circ g^{-1}$ and to each

$f : M \to N$ in $\mathrm{MOD}\, R$ the R-homomorphism ${}^g f : {}^g M \to {}^g N$ given by the family of k-linear maps $(f(g^{-1}(x)))_{x \in R}$. The ideals $\mathcal{P}u$ and $\mathcal{F}$ are invariant with respect to this action of G.

Given M in $\mathrm{MOD}\, R$ the subgroup

$$G_M = \{g \in G : {}^g M \simeq M\}$$

of G is called the *stabilizer* of M.

By an *R-action* of a subgroup H of G_M on M we mean a family

$$\mu = (\mu_g : M \to {}^{g^{-1}}M)_{g \in H}$$

of R-homomorphisms such that $\mu_e = \mathrm{id}_M$, where e is the unit of H, and ${}^{g_1^{-1}}\mu_{g_2} \cdot \mu_{g_1} = \mu_{g_2 g_1}$ for all $g_1, g_2 \in H$ (see [10]).

Observe that if H is a free group then M admits an R-action of H (see [2, Lemma 4.1]).

Let M and N be R-modules and H be a subgroup of $G_M \cap G_N$. If μ is an R-action of H on M and ν is an R-action of H on N, then we define the induced group action

$$(***) \qquad \mathrm{Hom}_R(\mu, \nu) : H \times \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(M, N)$$

by the mapping $(h, f) \mapsto {}^h\nu_h \cdot {}^h f \cdot \mu_{h^{-1}}$ (see also [2, 2.4]). This defines a left kH-module structure on $\mathrm{Hom}_R(M, N)$, where kH is the group algebra of H over k. Observe that the subspaces $\mathcal{P}u(M, N)$, $\mathcal{F}(M, N)$ of $\mathrm{Hom}_R(M, N)$ are its kH-submodules.

The main goal of this paper is to prove the following result announced in [3].

THEOREM A. *Let G be a group of k-linear automorphisms of a locally bounded k-category R, acting freely on objects of R, and let M and N be R-modules in* $\mathrm{Mod}\, R$. *Assume that $G_M \cap G_N$ contains an infinite cyclic subgroup H such that* $\mathrm{supp}\, M \cap \mathrm{supp}\, N$ *is contained in a sum of finitely many H-orbits in R. Then the k-vector spaces $\mathcal{P}u(M, N)$ and $\mathcal{F}(M, N)$ have the following properties*:

(i) $\mathcal{P}u(M, N) = \mathcal{F}(M, N)$.

(ii) $\mathcal{P}u(M, N)$ *is summably closed* (*in the sense of* 1.2).

(iii) *All decompositions $M = \bigoplus_{s \in S} M_s$ and $N = \bigoplus_{t \in T} N_t$ of M and N into a direct sum of submodules induce the canonical embedding*

$$\mathcal{P}u(M, N) \to \prod_{s \in S} \prod_{t \in T} \mathcal{P}u(M_s, N_t),$$

which is an isomorphism.

(iv) *If μ is an R-action of H on M and ν is an R-action of H on N then $\mathcal{P}u(M, N)$ is a left kH-submodule of* $\mathrm{Hom}_R(M, N)$ *with respect to the action* $\mathrm{Hom}_R(\mu, \nu)$ (*see* (***)) *and $\mathcal{P}u(M, N)$ is an injective kH-module.*

For applications of this result we refer to [3, Theorem 5.2].

Let C_1 and C_2 be full subcategories of a locally bounded k-category R. Then $C_1 \cup C_2$ (resp. $C_1 \cap C_2$ and $C_1 \setminus C_2$) is the full subcategory of R formed by the union (resp. intersection and difference) of the sets of objects $\mathrm{ob}\, C_1$ and $\mathrm{ob}\, C_2$. The subcategories C_1 and C_2 are *disjoint* (resp. *orthogonal*) if $\mathrm{ob}\, C_1 \cap \mathrm{ob}\, C_2 = \emptyset$ (resp. $R(x,y) = 0 = R(y,x)$ for any $x \in \mathrm{ob}\, C_1$ and $y \in \mathrm{ob}\, C_2$). The union $\bigcup_{i\in I} C_i$ of the family of full subcategories C_i of R, $i \in I$, is said to be a *disjoint union*, and then denoted by $\bigvee_{i\in I} C_i$, if the subcategories C_i, $i \in I$, are pairwise disjoint. The union of the family C_i, $i \in I$, of pairwise orthogonal full subcategories of R (then $\bigcup_{i\in I} = \bigvee_{i\in I} C_i$) is naturally isomorphic to the coproduct of this family and will be denoted by $\coprod_{i\in I} C_i$.

Throughout the paper by an ideal of a category we mean a two-sided ideal.

The paper is organized as follows. In Section 1 the properties of summable families of homomorphisms and summably closed ideals are discussed. The main result of this paper is proved in Section 2.

Some of the results of this paper with the proofs in brief outline have been announced in [3], and have been presented at the Cocoyoc Conference ICRA VII in Mexico, August 1994, and at Paderborn University, June 1995.

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1. Summable families and summably closed ideals. The category $\mathrm{MOD}\, R$ is equipped with an additional structure given by the partial operation of forming sums of infinite families of homomorphisms. Properties of this operation are especially interesting if the subcategory $\mathrm{Mod}\, R$ (for R infinite) is considered. Observe that in $\mathrm{Mod}\, R$ the decomposition into a direct sum of submodules is in fact the decomposition into their direct product. In this section we discuss summably closed ideals in $\mathrm{Mod}\, R$, i.e. ideals which are closed with respect to this partial operation.

1.1. DEFINITION. Let M and N be R-modules in $\mathrm{MOD}\, R$. A family $(f_i)_{i\in I}$ of homomorphisms from $\mathrm{Hom}_R(M,N)$ is said to be *summable* if for each $x \in R$ and $m \in M(x)$, $f_i(x)(m) = 0$ for almost all $i \in I$. In this case the well defined R-homomorphism $f = \sum_{i\in I} f_i : M \to N$, given by $f(x)(m) = \sum_{i\in I} f_i(x)(m)$ for any $x \in R, m \in M(x)$, is called the *sum* of the family $(f_i)_{i\in I}$.

Below we list a collection of basic facts concerning summable families of homomorphisms.

LEMMA. *Let $(f_i)_{i\in I}$ be a family of R-homomorphisms from M to N, where M, N is a pair of R-modules.*

(i) *The family $(f_i)_{i\in I}$ is summable if and only if the image of the map $M \to \prod_I N$ induced by $(f_i)_{i\in I}$ is contained in $\bigoplus_I N$ (equivalently, in $\bigoplus_{i\in I} \operatorname{Im} f_i$). In this case the sum $f = \sum_{i\in I} f_i : M \to N$ factors through $\bigoplus_I N$ (equivalently $\bigoplus_{i\in I} \operatorname{Im} f_i$).*

(ii) *If $f = f''f'$ is the composed R-homomorphism $M \xrightarrow{f'} \bigoplus_{i\in I} Z_i \xrightarrow{f''} N$, where f' and f'' are defined by families $(f'_i)_{i\in I}$ and $(f''_i)_{i\in I}$, then the family $(f''_i f'_i)_{i\in I}$ is summable and its sum equals f.*

(iii) *The family $(f_i)_{i\in I}$ is summable if and only if there exist factorizations $M \xrightarrow{f'_i} Z_i \xrightarrow{f''_i} N$ of all f_i's, $i \in I$, such that the image of the R-homomorphism $M \to \prod_{i\in I} Z_i$ induced by $(f'_i)_{i\in I}$ is contained in $\bigoplus_{i\in I} Z_i$.*

(iv) *If $(f_i)_{i\in I}$ and $(f'_i)_{i\in I}$ are summable families of R-homomorphisms from M to N, with sums $f = \sum_{i\in I} f_i$ and $f' = \sum_{i\in I} f'_i$, then for any homomorphisms $h : M' \to M$, $g : N \to N'$ the family $(gf_ih + gf'_ih)_{i\in I}$ is summable and its sum is equal to $gfh + gf'h$.*

(v) *Assume that $M = \bigoplus_{s\in S} M_s$ and $N = \prod_{t\in T} N_t$. For each $i \in I$, let $(f_i^{(s,t)})_{(s,t)\in S\times T}$ be a family of component homomorphisms $f_i^{(s,t)} \in \operatorname{Hom}_R(M_s, N_t)$, $s \in S, t \in T$, defining the homomorphism f_i. If the family $(f_i)_{i\in I}$ is summable then for each $(s,t) \in S \times T$ the family $(f_i^{(s,t)})_{i\in I}$ is summable and the sum $\sum_{i\in I} f_i$ is defined by the family of component homomorphisms $(\sum_{i\in I} f_i^{(s,t)})_{(s,t)\in S\times T}$. If additionally T is finite or M is locally finite-dimensional then the converse implication holds true.*

(vi) *Assume that M is locally finite-dimensional. Then the family $(f_i)_{i\in I}$ is summable if and only if for every $x \in R$, $f_i(x) = 0$ for almost all $i \in I$.*

Proof. Use the definition. ∎

COROLLARY. *Let M and N be R-modules. If M is locally finite-dimensional then $\mathcal{P}u(M, N)$ consists of all R-homomorphisms $f : M \to N$ having a factorization through a direct sum of finite-dimensional modules, which is additionally a locally finite-dimensional R-module.*

Proof. Let f be the composed R-homomorphism $M \xrightarrow{f'} \bigoplus_{i\in I} Z_i \xrightarrow{f''} N$, where f' and f'' are defined by families $(f'_i)_{i\in I}$ and $(f''_i)_{i\in I}$, and all Z_i, $i \in I$, are finite-dimensional. Then f factors through the module $\bigoplus_{i\in I} \operatorname{Im}(f''_i f'_i)$, which satisfies the required conditions. ∎

Remark. If $\big((f_{s,i})_{i\in I_s}\big)_{s\in S}$ is a family of summable families of R-homomorphisms from M to N and $I = \coprod_{s\in S} I_s$, then the family $(f_{s,i})_{(s,i)\in I}$ need not be summable.

1.2. Let $(M_s)_{s\in S}$ and $(N_t)_{t\in T}$ be families of R-modules. From now on we will identify $\mathrm{Hom}_R(\bigoplus_{s\in S} M_s, \prod_{t\in T} N_t)$ and $\prod_{s\in S}\prod_{t\in T}\mathrm{Hom}_R(M_s,N_t)$. Then by Lemma 1.1, $\mathrm{Hom}_R(\bigoplus_{s\in S} M_s, \prod_{t\in T} N_t)$ consists of all families $(f^{(s,t)})_{(s,t)\in S\times T}$ such that the family $(w_t f^{(s,t)} p_s)_{s\in S, t\in T}$ is summable, where $p_s : \bigoplus_{s\in S} M_s \to M_s$, $s\in S$, (resp. $w_t : N_t \to \bigoplus_{t\in T} N_t$, $t\in T$) denote the canonical projections (resp. embeddings). The identification in this situation is given by the sum operator.

If $\mathcal{I}$ is an ideal of a full subcategory $\mathcal{C}$ of $\mathrm{MOD}\,R$, then any pair of families $(M_s)_{s\in S}$ and $(N_t)_{t\in T}$ of R-modules in $\mathcal{C}$ such that both $\bigoplus_{s\in S} M_s$ and $\bigoplus_{t\in T} N_t$ belong to $\mathcal{C}$, induces the canonical embedding

$$(****)\qquad \mathcal{I}\Big(\bigoplus_{s\in S} M_s, \bigoplus_{t\in T} N_t\Big) \subset \prod_{s\in S}\prod_{t\in T}\mathcal{I}(M_s,N_t).$$

We will discuss the natural problem when this inclusion is an equality.

Definition. (i) Let M and N be R-modules in $\mathrm{MOD}\,R$. A subspace $\mathcal{W}$ of $\mathrm{Hom}_R(M,N)$ is said to be *summably closed* if for any summable family $(f_i)_{i\in I}$ of R-homomorphisms from $\mathcal{W}$ the sum $\sum_{i\in I} f_i$ belongs to $\mathcal{W}$.

(ii) An ideal $\mathcal{I}$ of a full subcategory $\mathcal{C}$ of $\mathrm{MOD}\,R$ is said to be *summably closed* if the subspace $\mathcal{I}(M,N)$ of $\mathrm{Hom}_R(M,N)$ is summably closed for each pair M,N of R-modules in $\mathcal{C}$.

Lemma. *Let $\mathcal{I}$ be an ideal of a full subcategory $\mathcal{C}$ of* $\mathrm{MOD}\,R$, *and M,N be R-modules in $\mathcal{C}$. If the subspace $\mathcal{I}(M,N)$ of $\mathrm{Hom}_R(M,N)$ is summably closed then any decompositions $M=\bigoplus_{s\in S} M_s$ and $N=\bigoplus_{t\in T} N_t$ in $\mathcal{C}$ yield the equality*

$$\mathcal{I}(M,N) = \prod_{s\in S}\prod_{t\in T}\mathcal{I}(M_s,N_t)\cap \mathrm{Hom}_R(M,N).$$

Proof. To prove the nontrivial inclusion $\supseteq$ take any $f\in\mathrm{Hom}_R(M,N)$ whose components $f^{(s,t)}\in\mathrm{Hom}_R(M_s,N_t)$ belong to $\mathcal{I}(M_s,N_t)$ for every $s\in S$, $t\in T$. The family $(w_t f^{(s,t)} p_s)_{s\in S,t\in T}$ is a summable family of R-homomorphims from $\mathcal{I}(M,N)$ and therefore, by assumption, its sum f also belongs to $\mathcal{I}(M,N)$. ■

Corollary. *If $\mathcal{I}$ is an ideal of* $\mathrm{Mod}\,R$ *and M,N are R-modules in* $\mathrm{Mod}\,R$, *satisfying the conditions of Lemma* 1.2, *then*

$$\mathcal{I}(M,N) = \prod_{s\in S}\prod_{t\in T}\mathcal{I}(M_s,N_t).$$

Proof. Apply Lemma 1.2 to $\mathcal{C}=\mathrm{Mod}\,R$ and use the equality $\bigoplus_{t\in T} N_t = \prod_{t\in T} N_t$. ■

1.3. Proposition. *Let $\mathcal{I}$ be an ideal of the category* $\mathrm{Mod}\,R$. *Then $\mathcal{I}$ is summably closed if and only if $\mathcal{I}$ satisfies the following two conditions*:

(i) *The space* $\mathcal{I}(M,N)$ *is summably closed for all indecomposable* M, N *in* $\operatorname{Mod} R$.

(ii) *For each pair* M, N *in* $\operatorname{Mod} R$ *there exist decompositions* $M = \bigoplus_{s\in S} M_s$ *and* $N = \bigoplus_{t\in T} N_t$ *into direct sums of indecomposable submodules such that* $\mathcal{I}(M,N) = \prod_{s\in S}\prod_{t\in T}\mathcal{I}(M_s,N_t)$.

Proof. Since each M in $\operatorname{Mod} R$ has a decomposition into a direct sum of indecomposables (see [6, Lemma 2.1]) the summable closedness of $\mathcal{I}$ by Corollary 1.2 implies the condition (ii) and obviously (i).

Take now any M,N in $\operatorname{Mod} R$ and a summable family $(f_i)_{i\in I}$ of R-homomorphisms from M to N such that all f_i, $i\in I$, belong to $\mathcal{I}(M,N)$. Let $M = \bigoplus_{s\in S} M_s$ and $N = \bigoplus_{t\in T} N_t$ satisfy (ii). By Lemma 1.1(v) for each $(s,t)\in S\times T$ the family $(f_i^{(s,t)})_{i\in I}$ of (s,t)-component maps is a summable family of homomorphisms from $\mathcal{I}(M_s,N_t)$, and $(\sum_{i\in I} f_i)^{(s,t)} = \sum_{i\in I} f_i^{(s,t)}$. Now from (i) and (ii) we conclude that $\sum_{i\in I} f_i$ belongs to $\mathcal{I}(M,N)$ and the proof is finished. ∎

Remark. An ideal $\mathcal{I}$ of $\operatorname{Mod} R$ is usually not determined by its values on pairs of indecomposable objects.

EXAMPLE. Let R be an infinite locally-support finite locally bounded k-category (see [5]). Denote by $\mathcal{I}$ the ideal of $\operatorname{Mod} R$ consisting of all homomorphisms with a finite support. Since each indecomposable R-module in this case is finite-dimensional, the summably closed ideal Hom_R and the ideal $\mathcal{I}$ take the same values on pairs of indecomposable R-modules. The identity homomorphism id_M does not belong to $\mathcal{I}(M,M)$ for any infinite-dimensional R-module M in $\operatorname{Mod} R$.

1.4. Given an ideal $\mathcal{I}$ of the category $\operatorname{MOD} R$ and a full subcategory $\mathcal{C}$ of $\operatorname{MOD} R$ we denote by $\mathcal{I}_{\mathcal{C}}$ the restriction of $\mathcal{I}$ to $\mathcal{C}^{op}\times\mathcal{C}$.

LEMMA. *The ideal* $\mathcal{F}_{\operatorname{Mod} R}$ *of* $\operatorname{Mod} R$ *is summably closed.*

Proof. Take any M,N in $\operatorname{Mod} R$, a summable family $(f_i)_{i\in I}$ of homomorphisms such that $f_i\in\mathcal{F}(M,N)$, for each $i\in I$, and a full finite subcategory C of R. By Lemma 1.1(vi) the set I' consisting of all $i\in I$ such that $f_{i|C}\neq 0$, is finite. For every $i\in I'$ we fix some $\bar f_i\in\operatorname{Hom}_R(M,N)$ such that $\operatorname{supp}\bar f_i$ is finite and $\bar f_{i|C} = f_{i|C}$. Then $(\sum_{i\in I} f_i)_{|C} = (\sum_{i\in I}\bar f_i)_{|C}$, the support $\operatorname{supp}(\sum_{i\in I}\bar f_i)$ is finite and consequently $\sum_{i\in I} f_i$ belongs to $\mathcal{F}(M,N)$. ∎

COROLLARY. *The ideal* $\mathcal{P}u_{\operatorname{Mod} R}$ *is contained in* $\mathcal{F}_{\operatorname{Mod} R}$.

Proof. Take any M,N in $\operatorname{Mod} R$ and a composed map $f = f''f' : M\to N$, where $f' : M\to\bigoplus_{i\in I} Z_i$ and $f'' : \bigoplus_{i\in I} Z_i\to N$ are given by families $(f_i)_{i\in I}$ and $(f_i'')_{i\in I}$, and where all R-modules Z_i, $i\in I$, are finite-dimensional. By Lemma 1.1(ii), $(f_i'' f_i')_{i\in I}$ is a summable family of homomorphisms from $\mathcal{F}(M,N)$ and therefore by Lemma 1.4, $f=\sum_{i\in I} f_i'' f_i'$ belongs to $\mathcal{F}(M,N)$. ■

We do not know whether $\mathcal{P}u_{\operatorname{Mod} R}$ is always summably closed or is always equal to $\mathcal{F}_{\operatorname{Mod} R}$; see Theorem A for a complete answer in an important special case.

2. Injective submodules. We shall prove some general facts on injective submodules of the kH-module $\operatorname{Hom}_R(M,N)$, where M,N are in $\operatorname{Mod} R$ and H is an infinite cyclic subgroup contained in $G_M\cap G_N$ (see Theorem B, Section 2.2). The proof of Theorem A will follow rather easily from this result and will be completed in 2.6.

The formulation of Theorem B needs some preparation.

2.1. Let C be a full subcategory of R. Following [5] we denote by $\widehat{C}$ the full subcategory of all $y\in R$ such that $R(x,y)\neq 0$ or $R(y,x)\neq 0$ for some $x\in C$. For any M,N in MOD R we denote by $\operatorname{Hom}_R^{\mathcal{C}}(M,N)$ the k-subspace of all $f\in\operatorname{Hom}_R(M,N)$ such that $\operatorname{supp} f$ is contained in C.

Lemma. *Let M,N be a pair of R-modules and C be a full subcategory of R. Then a homomorphism $f'\in\operatorname{Hom}_C(M_{|C},N_{|C})$ can be extended by $f''\in\operatorname{Hom}_{R\setminus C}(M_{|R\setminus C},N_{|R\setminus C})$ to a homomorphism from $\operatorname{Hom}_R(M,N)$ if and only if f' can be extended by $f''_{|\widehat{C}\setminus C}$ to a homomorphism from $\operatorname{Hom}_{\widehat{C}}(M_{|\widehat{C}}, N_{|\widehat{C}})$. In particular, any homomorphism $f\in\operatorname{Hom}_{\widehat{C}}^C(M|_{\widehat{C}},N_{|\widehat{C}})$ can be extended by the zero map to the homomorphism $(f;0)\in\operatorname{Hom}_R(M,N)$ and therefore the restriction to $\widehat{C}$ induces an isomorphism $\operatorname{Hom}_R^C(M,N)\simeq\operatorname{Hom}_{\widehat{C}}^C(M_{|\widehat{C}},N_{|\widehat{C}})$.*

Proof. Obvious. ■

Corollary. *Let C_i, $i\in I$, be a family of pairwise orthogonal full subcategories of R, $C=\bigsqcup_{i\in I} C_i$ and $C_j^*=\bigsqcup_{i\in I\setminus\{j\}} C_i$ for each $j\in I$. Then the natural inclusions $\operatorname{Hom}_R^{C_i}(M,N)\subset\operatorname{Hom}_R^C(M,N)$ and the maps $(-_{|\widehat{C}_i};0)$: $\operatorname{Hom}_R^C(M,N)\to\operatorname{Hom}_R^{C_i}(M,N)$, $i\in I$, induce an isomorphism $\operatorname{Hom}_R^C(M,N)\simeq\prod_{i\in I}\operatorname{Hom}_R^{C_i}(M,N)$, which, in case I is finite, turns into the equality $\operatorname{Hom}_R^C(M,N)=\bigoplus_{i\in I}\operatorname{Hom}_R^{C_i}(M,N)$. In particular, we have $\operatorname{Hom}_R^C(M,N)=\operatorname{Hom}_R^{C_j}(M,N)\oplus\operatorname{Hom}_R^{C_j^*}(M,N)$ for every $j\in I$.*

Proof. Since C_i, $i\in I$, are pairwise disjoint, $\prod_{i\in I}\operatorname{Hom}_R^{C_i}(M,N)$ consists only of summable families and the natural inclusions define correctly the map $\prod_{i\in I}\operatorname{Hom}_R^{C_i}(M,N)\to\operatorname{Hom}_R^C(M,N)$ by attaching to any $(f_i)_{i\in I}$ its sum $\sum_{i\in I} f_i$. The map $\operatorname{Hom}_R^C(M,N)\to\prod_{i\in I}\operatorname{Hom}_R^{C_i}(M,N)$ assigning to any f the family $\big((f_{|\widehat{C}_i};0)\big)_{i\in I}$ is also well defined (see Lemma 2.1), because the $C_i, i \in I$, are pairwise orthogonal. The two maps are mutually inverse and the remaining assertions follow trivially. ∎

2.2. Let $\mathcal{W}$ be a k-linear subspace of $\mathrm{Hom}_R(M,N)$. We say that $\mathcal{W}$ is *f-summably closed* if for any summable family $(f_i)_{i\in I}$ of R-homomorphisms from $\mathcal{W}$ such that $\operatorname{supp} f_i$ is finite for every $i \in I$, the sum $\sum_{i\in I} f_i$ belongs to $\mathcal{W}$.

For any full subcategory C of R we denote by $\mathcal{W}^C$ the intersection $\mathcal{W}\cap \mathrm{Hom}_R^C(M,N)$. We say that $\mathcal{W}$ is *homogeneous with respect to a pair of orthogonal full subcategories* A and B of R if the natural inclusions induce the equality $\mathcal{W}^A \oplus \mathcal{W}^B = \mathcal{W}^{A\sqcup B}$ (see Corollary 2.1).

Theorem B. *Let G be a group of k-linear automorphisms of a locally bounded k-category R, acting freely on objects of R, and let M and N be R-modules in* $\mathrm{Mod}\, R$. *Assume that $G_M \cap G_N$ contains an infinite cyclic subgroup H such that* $\operatorname{supp} M \cap \operatorname{supp} N$ *is contained in a sum of finitely many H-orbits in R. Suppose that μ is an R-action of H on M, ν is an R-action of H on N and $\mathcal{W}$ is a k-subspace of* $\mathrm{Hom}_R(M,N)$ *satisfying the following conditions*:

(i) *$\mathcal{W}$ is H-invariant with respect to the action* $\mathrm{Hom}_R(\mu,\nu)$,

(ii) *$\mathcal{W}$ is f-summably closed*,

(iii) *$\mathcal{W}$ is homogeneous with respect to any pair of full orthogonal subcategories of R such that one of them is finite*,

(iv) *for any $f \in \mathcal{W}$ and any finite full subcategory C of R there exists $\bar f \in \mathcal{W}$ such that the support* $\operatorname{supp} \bar f$ *is finite and $f_{|C} = \bar f_{|C}$.*

Then the kH-module $\mathcal{W}$, with the structure defined by the action $\mathrm{Hom}_R(\mu,\nu)$, *is injective.*

The major part of this section will be devoted to the proof of the above theorem.

2.3. Denote by $\mathrm{fs}(R)$ the class of all finite full subcategories of R.

Lemma. *Let M,N and H be as in Theorem* B, *μ (resp. ν) be an R-action of H on M (resp. N) and let $L = \operatorname{supp} M \cap \operatorname{supp} N$. Assume additionally that $\mathcal{W}$ is a k-subspace of* $\mathrm{Hom}_R(M,N)$ *which is H-invariant with respect to the action* $\mathrm{Hom}_R(\mu,\nu)$ *and moreover satisfies the condition* (iv) *of Theorem* B. *Then there exists an H-equivariant function $u : \mathrm{fs}(R) \to \mathrm{fs}(L)$ with the property that for any $C \in \mathrm{fs}(C)$ and $f \in \mathcal{W}$, the restriction $f_{|C}$ coincides with $\bar f_{|C}$ for some $\bar f \in \mathcal{W}^{u(C)}$.*

Proof. Fix any set $\mathrm{fs}_0(R)$ of representatives of H-orbits in $\mathrm{fs}(R)$. First we define a function $u_0 : \mathrm{fs}_0(R) \to \mathrm{fs}(L)$. Take any $C \in \mathrm{fs}_0(R)$. Then the set $\mathcal{V}$ of all $f_{|C}$, where $f \in \mathcal{W}$, is a finite-dimensional k-vector space. Fix any basis $f_{1|C}, \ldots, f_{n|C}$ of $\mathcal{V}$, where $n = \dim_k \mathcal{V}$. Since $\mathcal{W}$ satisfies the condition (iv) of Theorem B we can assume that all f_i, $i = 1, \ldots, n$, have finite support. Then we set $u_0(C) = \operatorname{supp} f_1 \cup \ldots \cup \operatorname{supp} f_n$. Since the action of H on $\operatorname{fs}(R)$ is free the map u_0 can be uniquely extended to an H-equivariant function $u : \operatorname{fs}(R) \to \operatorname{fs}(L)$. It is easy to check that u has the required property. ∎

2.4. Throughout this section L denotes a locally bounded k-category.

LEMMA. *Let $\cdot : H \times L \to L$ be a free action of an infinite cyclic group H, with a generator h, on a locally bounded k-category L such that L/H is finite. Then there exists a trisection of L into a disjoint union $L = L^- \vee L^0 \vee L^+$ of full subcategories satisfying the following conditions*:

(i) L^0 *is finite*,
(ii) $L(x, y) = 0 = L(y, x)$ *for each* $x \in L^-$ *and* $y \in L^+$,
(iii) $L^+ \subset \bigcup_{n \geq 0} h^n L^0$ *and* $L^- \subset \bigcup_{n \leq 0} h^{-n} L^0$,
(iv) $h^n(L^0 \cup L^+) \subset L^+$ *and* $h^{-n}(L^0 \cup L^-) \subset L^-$ *for* $n \gg 0$.

Proof. Take any set D of representatives of H-orbits in L. Since by assumption D is finite, $\widehat{D}$ is also finite. The action of H on L is free, therefore there exist only finitely many $i \in \mathbb{N}$ such that $h^i D \cap \widehat{D} \neq \emptyset$. Denote by m the maximum of all $i \in \mathbb{N}$ with the above property and set $L^0 = \bigcup_{i=0}^{m-1} h^i D$, $L^+ = \bigcup_{i \in \mathbb{N}} h^i L^0 \setminus L^0 = \bigcup_{i \geq m} h^i D$ and $L^- = \bigcup_{i \in \mathbb{N}} h^{-i} L^0 \setminus L^0 = \bigcup_{i \leq 0} h^i D$. Now one can easily check the required conditions. ∎

Let $L = L^- \vee L^0 \vee L^+$ satisfy the assertions of Lemma 2.4. Then we set $L'_n = L^+ \cap h^n L^-$, $L_{n,m} = h^{mn} L^0$, $L^+_{n,m} = h^{mn} L^+$, $L^-_{n,m} = h^{mn} L^-$, $L'_{n,m} = h^{mn} L'_n$ for any $n \in \mathbb{N}$ and $m \in \mathbb{Z}$.

COROLLARY. *Under the notation above*, *the category L decomposes into a disjoint union*

$$L = \bigvee_{m \in \mathbb{Z}} L_{n,m} \vee \bigsqcup_{m \in \mathbb{Z}} L'_{n,m}$$

of finite full subcategories, *for sufficiently large* $n \in \mathbb{N}$.

Proof. Fix any $n \in \mathbb{N}$ satisfying the condition (iv) of Lemma 2.4. The finiteness of the $L'_{n,m}$'s, $m \in \mathbb{Z}$, follows from the property (iii) and the fact that the action is free. For the proof of the decomposition one shows first inductively, using the property (iv), that

$$L^+_n = \bigvee_{m=1}^{p} L_{n,m} \vee \bigvee_{m=0}^{p-1} L'_{n,m} \vee L^+_{n,p}$$

for every $p \in \mathbb{N}$. Observe that since the action is free, by (iii) we obtain

$\bigcap_{p\in\mathbb{N}} L^+_{n,p} = \emptyset$, and therefore

$$L^+_n = \bigvee_{m>0} L_{n,m} \vee \bigvee_{m\geq 0} L'_{n,m}.$$

Similarly one shows

$$L^-_n = \bigvee_{m<0} L_{n,m} \vee \bigvee_{m<0} L'_{n,m} \quad \text{and} \quad L = \bigvee_{m\in\mathbb{Z}} L_{n,m} \vee \bigvee_{m\in\mathbb{Z}} L'_{n,m}.$$

The orthogonality of the $L'_{n,m}$'s follows from the property (ii) and the inclusions $L'_{n,m} \subset L^+_{n,s}$, $L'_{n,m} \subset L^-_{n,t}$ proved above for any $s, t \in \mathbb{Z}$ such that $m \geq s$ and $m < t$.

2.5. Let M, N be a pair of R-modules in $\operatorname{Mod} R$, H be an infinite cyclic subgroup of $G_M \cap G_N$ and $\mathcal{W}$ a submodule of the kH-module $\operatorname{Hom}_R(M,N)$ with the kH-module structure defined by the action $\operatorname{Hom}_R(\mu,\nu)$, for fixed R-actions μ and ν of H on M and N respectively (see $(***)$). For any $C \in \operatorname{fs}(R)$ we denote by Q^C the kH-module $\prod_{g\in H} \mathcal{W}^{gC}$. Since C is finite, $\mathcal{W}^C$ is finite-dimensional, the kH-module Q^C is isomorphic to the k-dual of a finitely generated free kH-module and therefore Q^C is injective. The inclusions $\mathcal{W}^{gC} \subset \operatorname{Hom}_R(M,N)$ induce a map $\pi^C : Q^C \to \operatorname{Hom}_R(M,N)$ given by the formula $\pi^C\big((f_g)_{g\in H}\big) = \sum_{g\in H} f_g$, where $(f_g)_{g\in H} \in Q^C$. The map π^C is well defined since G acts freely on R and therefore Q^C consists only of summable families of homomorphisms. Denote by

$$\pi : \bigoplus_{C\in\operatorname{fs}(C)} Q^C \to \operatorname{Hom}_R(M,N)$$

the kH-homomorphism induced by the maps π^C, $C \in \operatorname{fs}(R)$. Observe that $\operatorname{Im}\pi$ is an injective kH-module since kH is a principal ideal domain. Moreover, $\operatorname{Im}\pi \subset \mathcal{P}u(M,N)$ by Lemma 1.1(i).

Proof of Theorem B. For the proof of the injectivity of kH-module $\mathcal{W}$ satisfying the assumptions (i)–(iv) it is enough to show that $\mathcal{W} = \operatorname{Im}\pi$, where π is the map defined above. By (ii) we obtain $\operatorname{Im}\pi^C \subset \mathcal{W}$ for any $C \in \operatorname{fs}(R)$ and therefore $\mathcal{W} \subset \operatorname{Im}\pi$. Let $L = \operatorname{supp} M \cap \operatorname{supp} N$ and fix some generator h of H. Then L/H is finite, hence there exists a trisection $L = L^- \vee L^0 \vee L^+$ of L satisfying the assertions of Lemma 2.4. Denote by $\overline{L}$ the union $L^0 \cup u(L^0)$ (see Lemma 2.3). Since G acts freely on R and $\overline{L}$ is finite, by Corollary 2.5 there exists $n \in \mathbb{N}$ such that $L = \bigvee_{m\in\mathbb{Z}} L_{n,m} \vee \bigsqcup_{m\in\mathbb{Z}} L'_{n,m}$ and $\bigcup_{m\in\mathbb{Z}} \overline{L}_{n,m} = \bigvee_{m\in\mathbb{Z}} \overline{L}_{n,m}$, where $L_{n,m} = h^{mn}L^0$, $L'_{n,m} = h^{mn}L^+ \cap h^{n(m+1)}L^-$ and $\overline{L}_{n,m} = h^{mn}\overline{L}$, for each $m \in \mathbb{Z}$. Fix some $n \in \mathbb{N}$ as above, and take now an arbitrary $f \in \mathcal{W}$. By Lemma 2.3 for each $m \in \mathbb{Z}$, there exists $\bar{f}_m \in \mathcal{W}^{\bar{L}_{n,m}}$ such that $f_{|L_{n,m}} = \bar{f}_{m|L_{n,m}}$. Obviously the family $(\bar{f}_m)_{m\in\mathbb{Z}}$ is summable and $\bar{f} = \sum_{m\in\mathbb{Z}} \bar{f}_m \in \operatorname{Im}\pi^{\bar{L}}$. By (ii) the map $f' = f - \bar{f}$ belongs to

$\mathcal{W}$ and since $\operatorname{supp} f' \subset \coprod_{m\in\mathbb{Z}} L'_{n,m}$ it follows that $f' = \sum_{m\in\mathbb{Z}} f'_m$ for some summable family $(f'_m)_{m\in\mathbb{Z}}$, where $f'_m \in \mathrm{Hom}^{L'_{n,m}}(M,N)$ for each $m \in \mathbb{Z}$ (see Corollary 2.1). Invoking the assumption (iii), we get $f'_m \in \mathcal{W}^{L'_{n,m}}$ for every $m \in \mathbb{Z}$, and therefore $f' \in \operatorname{Im}\pi^{L'}$, where $L' = L'_{n,0}$. In this way the inclusion $\mathcal{W} \subset \operatorname{Im}\pi$ is shown and the proof of the injectivity of the kH-module $\mathcal{W}$ is completed. ∎

As a consequence of the above proof we get the following.

COROLLARY. *Under the assumptions of Theorem B and the notation of the above proof, the following inclusions hold*:

$$\mathcal{W} \subset \operatorname{Im}\pi^{\bar{D}} + \operatorname{Im}\pi^{E} \subset \operatorname{Im}\pi \subset \mathcal{W} \quad \textit{and} \quad \operatorname{Im}\pi \subset \mathcal{P}u(M,N).$$

2.6. Proof of Theorem A. Let M, N and H satisfy the assumptions of Theorem A. Since H is a free group there always exists an R-action of H on any R-module. Let us fix a pair of such actions, μ on M and ν on N. Then the subspace $\mathcal{F}(M,N)$ of $\mathrm{Hom}_R(M,N)$ is H-invariant with respect to the action $\mathrm{Hom}_R(\mu,\nu)$ and satisfies the remaining assumptions of Theorem B. The assumption (ii) is satisfied since by Lemma 1.4 the subspace $\mathcal{F}(M,N)$ is summably closed. The assumptions (iii) and (iv) are trivially satisfied by the definition of $\mathcal{F}$. Therefore by Theorem B the kH-module $\mathcal{F}(M,N)$ is injective, and by Corollaries 2.5 and 1.4 we get $\mathcal{P}u(M,N) = \mathcal{F}(M,N)$. The remaining assertion now follows easily from the above equality, the inclusion $(****)$, Lemma and Corollaries 1.4 and 1.2. ∎

REFERENCES

[1] K. Bongartz and P. Gabriel, *Covering spaces in representation theory*, Invent. Math. 65 (1982), 331–378.

[2] P. Dowbor, *On the category of modules of second kind for Galois coverings*, Fund. Math. 149 (1996), 31–54.

[3] —, *Galois covering reduction to stabilizers*, Bull. Polish Acad. Sci. Math. (1996), to appear.

[4] P. Dowbor and S. Kasjan, *Non-simply connected posets of polynomial growth*, preprint 1996.

[5] P. Dowbor and A. Skowroński, *On Galois coverings of tame algebras*, Arch. Math. (Basel) 44 (1985), 522–529.

[6] —, —, *Galois coverings of representation-infinite algebras*, Comment. Math. Helv. 62 (1987), 311–337.

[7] P. Dowbor, H. Lenzing and A. Skowroński, *Galois coverings of algebras by locally support-finite categories*, Proc. ICRA IV, Ottawa 1984, Lecture Notes in Math. 1177, Springer, 1986, 91–93.

[8] Yu. A. Drozd, S. A. Ovsienko and B. Yu. Furcin, *Categorical construction in representation theory*, in: Algebraic Structures and their Applications, University of Kiev, Kiev UMK VO, 1988, 43–73 (in Russian).

[9] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France 90 (1962), 323–448.
[10] —, *The universal cover of a representation-finite algebra*, in: Proc. ICRA III, Puebla 1980, Lecture Notes in Math. 903, Springer, 1982, 68–105.
[11] C. Geiss and J. A. de la Peña, *An interesting family of algebras*, Arch. Math. (Basel) 60 (1993), 25–35.
[12] E. L. Green, *Group-graded algebras and the zero relation problem*, in: Proc. ICRA III, Puebla 1980, Lecture Notes in Math. 903, Springer, 1981, 106–115.
[13] Z. Pogorzały, *Regularly biserial algebras*, in: Topics in Algebra, Part 1: Rings and Representations of Algebras, Banach Center Publ. 26, PWN, Warszawa, 1990, 371–384.
[14] Ch. Riedtmann, *Algebren, Darstellungsköcher, Überlagerungen und zurück*, Comment. Math. Helv. 55 (1980), 199–224.
[15] D. Simson, *On pure semisimple Grothendieck categories. I*, Fund. Math. 100 (1978), 211–222.
[16] —, *On pure semisimple Grothendieck categories. II*, ibid. 110 (1980), 107–116.
[17] —, *Socle reduction and socle projective modules*, J. Algebra 108 (1986), 18–68.
[18] —, *Representations of bounded stratified posets, coverings and socle projective modules*, in: Topics in Algebra, Part 1: Rings and Representations of Algebras, Banach Center Publ. 26, PWN, Warszawa, 1990, 499–533.
[19] —, *Right peak algebras of two-separate stratified posets, their Galois coverings and socle projective modules*, Comm. Algebra 20 (1992), 3541–3591.
[20] A. Skowroński, *Selfinjective algebras of polynomial growth*, Math. Ann. 285 (1989), 177–193.
[21] —, *Criteria for polynomial growth of algebras*, Bull. Polish Acad. Sci. Math. 42 (1994), 173–183.
[22] —, *Tame algebras with simply connected Galois coverings*, preprint, 1995.

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