COLLOQUIUM MATHEMATICUM

VOL. 71 1996 NO. 2

AUSLANDER–REITEN COMPONENTS FOR CONCEALED-CANONICAL ALGEBRAS

BY

HAGEN MELTZER (CHEMNITZ)

1. Introduction. We describe the structure of the Auslander–Reiten components of finite-dimensional modules for endomorphism rings Σ of tilting bundles and tilting sheaves on a weighted projective line. Such algebras were called in [10] concealed-canonical algebras and almost concealed-canonical algebras, respectively. Concealed-canonical algebras and almost concealed-canonical algebras are important classes of quasi-tilted algebras in the sense of Happel, Reiten and Smalø [3].

Our result generalizes theorems of Kerner [7] and [8] studying the case of tilted algebras and of Lenzing and de la Peña [11] considering the case of canonical algebras.

The representation type of Σ depends on the weight type, or equivalently, on the virtual genus g of the weighted projective line $\mathbb{X}$. If $g < 1$ then Σ is a tame concealed algebra and the Auslander–Reiten quiver is well known. If $g = 1$, the algebra Σ is a tubular algebra and the structure of the Auslander–Reiten components was described by Ringel in [13] (see also [9] for a classification using the geometrical approach).

Here we are interested in the case $g > 1$, i.e. $\mathbb{X}$ is wild. In this case Σ is strictly wild as was shown in [10]. In that paper we also gave a global view of the category $\mathrm{mod}(\Sigma)$ of finite-dimensional modules over Σ. Identifying the derived category of $\mathrm{mod}(\Sigma)$ with the derived category of coherent sheaves $\mathrm{coh}(\mathbb{X})$ and transporting the notions of rank and degree of sheaves to modules we have four typical parts of Σ-modules, $\mathrm{mod}_+(\Sigma)$, $\mathrm{mod}_0^+(\Sigma)$, $\mathrm{mod}_-(\Sigma)$ and $\mathrm{mod}_0^-(\Sigma)$ denoting respectively the additive closure of the indecomposable modules of positive rank, rank zero with positive degree, negative rank and rank zero with negative degree. The last part is finite and does not appear if we are dealing with a tilting bundle. The categories $\mathrm{mod}_+(\Sigma)$, $\mathrm{mod}_0^+(\Sigma)$ and $\mathrm{mod}_{\le}(\Sigma)$, which is the additive closure of $\mathrm{mod}_-(\Sigma)$ and $\mathrm{mod}_0^-(\Sigma)$, are closed under extensions and under the Auslander–Reiten translation. The components of $\mathrm{mod}_0^+(\Sigma)$ are tubes and form a separating family [10, Theorem 5.8].

1991 *Mathematics Subject Classification*: Primary 16G70; Secondary 14H60, 16G20.

Here we determine the shape of the Auslander–Reiten components in $\operatorname{mod}_+(\Sigma)$ and $\operatorname{mod}_{\le}(\Sigma)$ in the wild case. It is shown that a component in $\operatorname{mod}_{\le}(\Sigma)$ different from the preinjective component has as stable part a shape of type $\mathbb{Z}A_\infty$. Moreover, we can construct bijections between the following three sets:

- $\Omega_-(\Sigma)$ of components of $\operatorname{mod}_-(\Sigma)$,
- $\Omega(\mathbb{X})$ of components of the category of vector bundles over $\mathbb{X}$,
- $\Omega(\Sigma_I)$ of regular components of modules over a concealed wild algebra Σ_I defining the unique preinjective component of $\operatorname{mod}(\Sigma)$.

A similar result for $\operatorname{mod}_+(\Sigma)$ is true if Σ is a wild concealed-canonical algebra. For an almost concealed-canonical algebra Σ the part $\operatorname{mod}_+(\Sigma)$ can be "smaller", depending on the decomposition $T = T' \oplus T''$ of the tilting sheaf in a vector bundle T' and a sheaf of finite length T''.

The main results are similar to those in [8] following the general philosophy that the vector bundles in $\operatorname{coh}(\mathbb{X})$ have the same behaviour as regular modules over wild hereditary algebras. Some proofs, however, become easier in the geometrical situation. Moreover, in contrast to the situation of tilted algebras we can characterize special summands in the sense of Strauss using the rank and degree of vector bundles appearing in the wing decomposition. Note that a modified version of Theorem 5.3 for modules over tilting algebras can also be used for the inductive step of [8, Theorem 1].

I would like to thank Dieter Happel and Helmut Lenzing for many interesting discussions.

2. Notations

2.1. Throughout the paper we work over an algebraically closed field k and use the following notation. Let $\mathbf{p} = (p_1, \ldots, p_t)$ be a weight sequence of positive integers p_i and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_t)$ a parameter sequence of pairwise distinct elements of $\mathbb{P}^1(k)$ such that $\lambda_1 = \infty$, $\lambda_2 = 0$, $\lambda_3 = 1$ and $\mathbb{X} = \mathbb{X}(\mathbf{p}, \boldsymbol{\lambda})$ the attached weighted projective line in the sense of [1]. Using a graded theory Geigle and Lenzing introduced in [1] the category $\operatorname{coh}(\mathbb{X})$ of coherent sheaves on $\mathbb{X}$, with structure sheaf $\mathcal{O}$. We denote by $\operatorname{vect}(\mathbb{X})$ (resp. $\operatorname{coh}_0(\mathbb{X})$) the category of vector bundles (resp. finite length sheaves) on $\mathbb{X}$. The virtual genus $g_{\mathbb{X}}$ is defined by

$$g_{\mathbb{X}} = 1 + \frac{1}{2}\Big((t-2)p - \sum_{i=1}^{t} p/p_i\Big),$$

where $p = \text{l.c.m.}(p_1, \ldots, p_t)$. If not mentioned otherwise we assume that $g_{\mathbb{X}} > 1$, in this case $\operatorname{coh}(\mathbb{X})$ is wild. Recall that for sheaves F on $\mathbb{X}$ the notion of rank $\operatorname{rk}(F)$ and degree $\deg(F)$ are defined, moreover, the slope of F is given by $\mu(F) = \deg(F)/\mathrm{rk}(F)$. By a *tilting sheaf* (resp. *tilting bundle*) we mean a multiplicity-free object $T \in \mathrm{coh}(\mathbb{X})$ (resp. $T \in \mathrm{vect}(\mathbb{X})$) without self-extensions and generating the derived category $\mathcal{D}^{\mathrm{b}}(\mathrm{coh}(\mathbb{X}))$ of bounded complexes over $\mathrm{coh}(\mathbb{X})$. We call the endomorphism algebra of a tilting bundle (resp. tilting sheaf) a *concealed-canonical* (resp. *almost concealed-canonical*) *algebra*. The reason is that a tilting sheaf (resp. tilting bundle) can be alternatively viewed as a tilting module over the canonical algebra $\Lambda = \Lambda(\mathbf{p}, \boldsymbol{\lambda})$ attached to the data $\mathbf{p}$, $\boldsymbol{\lambda}$ [13], with the property that each indecomposable direct summand of T has rank ≥ 0 (resp> 0).

2.2. For a finite-dimensional algebra A we denote by $\mathrm{mod}(A)$ the category of finite-dimensional right A-modules. Let T be a tilting sheaf and $\Sigma = \mathrm{End}(T)$. Then by [1, Theorem 3.2], $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}(\Sigma))$, the derived category of bounded complexes over $\mathrm{mod}(\Sigma)$, is triangle-equivalent to $\mathcal{D}^{\mathrm{b}}(\mathrm{coh}(\mathbb{X}))$. Let $\mathrm{coh}_+(T)$ (resp. $\mathrm{coh}_0^+(T)$) be the full subcategory of $\mathrm{vect}(\mathbb{X})$ (resp. $\mathrm{coh}_0(\mathbb{X})$) consisting of all F satisfying the condition $\mathrm{Ext}^1(T, F) = 0$. Similarly we denote by $\mathrm{coh}_-(T)$ (resp. $\mathrm{coh}_0^-(T)$) the full subcategory of $\mathrm{vect}(\mathbb{X})$ (resp. $\mathrm{coh}_0(\mathbb{X})$) consisting of all F satisfying the condition $\mathrm{Hom}(T, F) = 0$. Furthermore, let $\mathrm{coh}_{\geq}(T)$ (resp. $\mathrm{coh}_{\leq}(T)$) be the additive closure of $\mathrm{coh}_+(T) \cup \mathrm{coh}_0^+(T)$ (resp. $\mathrm{coh}_-(T) \cup \mathrm{coh}_0^-(T)$). Then by [10, Theorem 5.1] under the identification $\mathcal{D}^{\mathrm{b}}(\mathrm{mod}(\Sigma)) \simeq \mathcal{D}^{\mathrm{b}}(\mathrm{coh}(\mathbb{X}))$ each indecomposable Σ-module is in one of the four subcategories $\mathrm{coh}_+(T)$, $\mathrm{coh}_0^+(T)$, $\mathrm{coh}_-(T)[1]$, $\mathrm{coh}_0^-(T)[1]$, where $[1]$ denotes the translation in the derived category. We denote these four parts of the module category respectively by $\mathrm{mod}_+(\Sigma)$, $\mathrm{mod}_0^+(\Sigma)$, $\mathrm{mod}_-(\Sigma)$, $\mathrm{mod}_0^-(\Sigma)$, accordingly to the fact that for an indecomposable module M we have

- $M \in \mathrm{mod}_+(\Sigma)$ iff $\mathrm{rk}(M) > 0$,
- $M \in \mathrm{mod}_0^+(\Sigma)$ iff $\mathrm{rk}(M) = 0$ and $\deg(M) > 0$,
- $M \in \mathrm{mod}_-(\Sigma)$ iff $\mathrm{rk}(M) < 0$,
- $M \in \mathrm{mod}_0^-(\Sigma)$ iff $\mathrm{rk}(M) = 0$ and $\deg(M) < 0$.

Finally, we denote by $\mathrm{mod}_{\geq}(\Sigma)$ (resp. $\mathrm{mod}_{\leq}(\Sigma)$) the additive closure of $\mathrm{mod}_+(\Sigma) \cup \mathrm{mod}_0^+(\Sigma)$ (resp. $\mathrm{mod}_-(\Sigma) \cup \mathrm{mod}_0^-(\Sigma)$). For the sake of simplicity we often write $\mathrm{coh}_0(T)$ and $\mathrm{mod}_0(\Sigma)$ instead of $\mathrm{coh}_0^+(T)$ and $\mathrm{mod}_0^+(\Sigma)$.

2.3. Assume that an object Z belongs to an Auslander–Reiten component $\mathcal{C}$ of $\mathrm{coh}(\mathbb{X})$ or $\mathrm{mod}(\Sigma)$. Then the *τ-cone* $(\to Z)$ (resp. the *τ^--cone* $(Z \to)$) consists of all objects of $\mathcal{C}$ which are predecessors (resp. successors) of Z. If necessary we will distinguish the Auslander–Reiten translations in $\mathrm{coh}(\mathbb{X})$ and $\mathrm{mod}(\Sigma)$ and denote them by $\tau_{\mathbb{X}}$ and τ_{Σ} respectively. Recall that $\tau_{\mathbb{X}}$ is given by a line bundle shift with the canonical sheaf $\omega = \mathcal{O}(\vec{\omega})$ where $\vec{\omega} = (t-2)\vec{c} - \sum_{i=1}^{t} \vec{x}_i$ [1, Corollary 2.3].

3. Regular components. In this section T denotes a tilting sheaf on a weighted projective line $\mathbb{X}$ of arbitrary type. Let $\Sigma = \mathrm{End}(T)$ be the attached almost concealed-canonical algebra. Here we describe the regular components in $\mathrm{mod}(\Sigma)$, i.e. the components without projective and injective modules.

3.1. Similarly to [1, 3.5] for each $F \in \mathrm{coh}(\mathbb{X})$ there is a short exact sequence

$$0 \to F_+ \to F \to F_- \to 0 \quad \text{with } F_+ \in \mathrm{coh}_{\geq}(T),\ F_- \in \mathrm{coh}_{<}(T).$$

In fact, F_+ is the largest subsheaf of F belonging to $\mathrm{coh}_{\geq}(T)$.

The following result is similar to a result of Hoshino [5] concerning relative Auslander–Reiten sequences for torsion pairs in module categories.

PROPOSITION 3.1. (a) *For each indecomposable module* $M \in \mathrm{mod}_{\geq}(\Sigma)$ *we have* $\tau_\Sigma M = (\tau_{\mathbb{X}} M)_+$.

(b) *For each indecomposable module* $M \in \mathrm{mod}_{<}(\Sigma)$ *we have* $\tau_\Sigma^- M = (\tau_{\mathbb{X}}^- M)_-$.

Proof. (a) was proved in [9, 5.1] if T is the canonical tilting sheaf; the general case follows easily.

(b) For $M, N \in \mathrm{mod}_{<}(\Sigma)$ we have $\mathrm{Hom}_\Sigma(\tau_\Sigma^- M, N) \simeq \underline{\mathrm{Hom}}_\Sigma(\tau_\Sigma^- M, N) \simeq \overline{\mathrm{Hom}}_\Sigma(M, \tau_\Sigma N)$, where $\underline{\mathrm{Hom}}_\Sigma(X,Y)$ (resp. $\overline{\mathrm{Hom}}_\Sigma(X,Y)$) denotes the group $\mathrm{Hom}_\Sigma(X,Y)$ modulo the subgroup consisting of all Σ-homomorphisms from X to Y which factor through projective (resp. injective) modules. This follows from the facts that all projective modules are in $\mathrm{mod}_{\geq}(\Sigma)$, in particular $\tau_\Sigma M$ and N have no nonzero projective direct summand, and that there are no nonzero homomorphisms from $\mathrm{mod}_{<}(\Sigma)$ to $\mathrm{mod}_{\geq}(\Sigma)$.

Now, invoking the Auslander–Reiten formula and the Serre duality for $\mathrm{coh}(\mathbb{X})$ we obtain $\mathrm{Hom}_\Sigma(\tau_\Sigma^- M, N) \simeq \mathrm{DExt}^1_\Sigma(N, M) \simeq \mathrm{DExt}^1_{\mathbb{X}}(N, M) \simeq \mathrm{Hom}_{\mathbb{X}}(M, N(\vec{\omega})) \simeq \mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}}^- M, N)$.

Applying the functor $\mathrm{Hom}(-, N)$ to the exact sequence

$$0 \to (\tau_{\mathbb{X}} M)_+ \to \tau_{\mathbb{X}} M \to (\tau_{\mathbb{X}} M)_- \to 0,$$

we see that $\mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}}^- M, N) \simeq \mathrm{Hom}_{\mathbb{X}}((\tau_{\mathbb{X}}^- M)_-, N)$ since there are no nonzero homomorphisms from $\mathrm{coh}_{\geq}(T)$ to $\mathrm{coh}_{<}(T)$. The last term equals $\mathrm{Hom}_\Sigma((\tau_{\mathbb{X}}^- M)_-, N)$ because both are modules in $\mathrm{mod}_{<}(\Sigma)$. Therefore we obtain isomorphisms $\mathrm{Hom}_\Sigma(\tau_\Sigma^- M, N) \simeq \mathrm{Hom}_\Sigma((\tau_{\mathbb{X}}^- M)_-, N)$, which are functorial in $N \in \mathrm{mod}_{<}(\Sigma)$, and consequently $\tau_\Sigma^- M \simeq (\tau_{\mathbb{X}}^- M)_-$.

COROLLARY 3.2. (i) *For each indecomposable module* $M \in \mathrm{mod}_{<}(\Sigma)$ *we have* $\mathrm{rk}(\tau_\Sigma^- M) \geq \mathrm{rk}(M)$.

(ii) *Let* M *be indecomposable in* $\mathrm{mod}_-(\Sigma)$. *Then* $\mathrm{rk}(\tau_\Sigma^- M) = \mathrm{rk}(M)$ *if and only if* $\tau_\Sigma^- M = \tau_{\mathbb{X}}^- M$.

P r o o f. (i) The inequality follows from Proposition 3.1 and the exact sequence
$$0 \to (\tau_{\mathbb{X}}^- M)_+ \to \tau_{\mathbb{X}}^- M \to \tau_{\Sigma}^- M \to 0$$
and the fact that the application of $\tau_{\mathbb{X}}^-$ does not change the rank. Note that $\mathrm{rk}((\tau_{\Sigma}^- M)_+) \le 0$.

(ii) Suppose that M is indecomposable in $\mathrm{mod}_-(\Sigma)$ and $\mathrm{rk}(\tau_{\Sigma}^- M) = \mathrm{rk}(M)$. From the exact sequence above we infer that $\mathrm{rk}((\tau_{\mathbb{X}}^- M)_+) = 0$. By our assumption we have $\tau_{\Sigma}^- M = F[1]$ for some $F \in \mathrm{vect}(\mathbb{X})$. Because there are no nonzero morphisms from finite length sheaves to vector bundles it follows that $(\tau_{\mathbb{X}}^- M)_+ = 0$.

COROLLARY 3.3. (i) *For each indecomposable module* $M \in \mathrm{mod}_{\ge}(\Sigma)$ *we have* $\mathrm{rk}(\tau_\Sigma M) \le \mathrm{rk}(M)$.

(ii) *Assume in addition that* T *is a tilting bundle and let* M *be indecomposable in* $\mathrm{mod}_+(\Sigma)$. *Then* $\mathrm{rk}(\tau_\Sigma M) = \mathrm{rk}(M)$ *if and only if* $\tau_\Sigma M \simeq \tau_{\mathbb{X}} M$.

P r o o f. (i) The inequality follows from Proposition 3.1 and the exact sequence
$$0 \to \tau_\Sigma M \to \tau_{\mathbb{X}} M \to (\tau_{\mathbb{X}} M)_- \to 0.$$

(ii) Let M be indecomposable in $\mathrm{mod}_+(\Sigma)$ and assume that $\mathrm{rk}(\tau_\Sigma M) = \mathrm{rk} M$. Then $\mathrm{rk}((\tau_{\mathbb{X}} M)_-) = 0$. Because for a tilting bundle $\mathrm{coh}_{\le}(T)$ does not contain sheaves of rank zero we obtain $(\tau_{\mathbb{X}} M)_- = 0$, consequently $\tau_\Sigma M \simeq \tau_{\mathbb{X}} M$.

THEOREM 3.4. *Let* Σ *be an almost concealed-canonical algebra and* $\mathcal{C}$ *be an Auslander–Reiten component in* $\mathrm{mod}_{\le}(\Sigma)$ *different from a preinjective component. Then there exists an indecomposable* $Z \in \mathcal{C}$ *such that the* τ_Σ^--*cone* $(Z \to)$ *in* $\mathcal{C}$ *is a full subquiver of a component in* $\mathrm{vect}(\mathbb{X})[1]$.

P r o o f. Applying Corollary 3.2(i) and the assumption that $\mathcal{C}$ is not a preinjective component, we can find an indecomposable $Z \in \mathcal{C}$ such that the τ_Σ-orbit of Z does not contain an injective Σ-module and $0 > \mathrm{rk}(Z) = \mathrm{rk}(\tau_\Sigma^{-t} Z)$ for all $t \ge 0$. Let
$$(*) \qquad 0 \to Z \xrightarrow{\alpha} Y_1 \oplus Y_2 \xrightarrow{\beta} \tau_{\mathbb{X}}^- Z \to 0$$
be the Auslander–Reiten sequence in $\mathrm{vect}(\mathbb{X})[1]$. Applying Corollary 3.2(i) we infer that $\tau_{\mathbb{X}}^- Z \simeq \tau_\Sigma^- Z$, in particular $\tau_{\mathbb{X}}^- Z \in \mathrm{mod}_{\le}(\Sigma)$. Applying the functor $\mathrm{Hom}(T, -)$ we see that also $Y_1 \oplus Y_2$ is in $\mathrm{mod}_{\le}(\Sigma)$.

Moreover, if $f : Z \to U$ is a morphism in $\mathrm{mod}(\Sigma)$ which is not a split monomorphism, then there is a morphism $g : Y_1 \oplus Y_2 \to U$ in $\mathrm{mod}(\Sigma)$ such that $f = g \circ \alpha$. Indeed, in case $U \in \mathrm{mod}_{\le}(\Sigma)$ we can use the Auslander-Reiten factorization property in $\mathrm{coh}(\mathbb{X})[1]$, and in case $U \in \mathrm{mod}_{\ge}(\Sigma)$, f is zero. Thus $(*)$ is also an Auslander–Reiten sequence in $\mathrm{mod}(\Sigma)$.

Repeating this argument, first for the Auslander–Reiten sequence

$$0 \to \tau_{\mathbb{X}}^{-} Z \to \tau_{\mathbb{X}}^{-} Y_1 \oplus \tau_{\mathbb{X}}^{-} Y_2 \to \tau_{\mathbb{X}}^{-2} Z \to 0,$$

then for the meshes adjacent to the two already studied and continuing this process we see that the whole $\tau_{\mathbb{X}}^{-}$-cone $(Z \to)$ consists of Auslander–Reiten sequences in $\operatorname{mod}(\Sigma)$. Therefore the τ^{-}-cones $(Z \to)$ in $\mathcal{C}$ and in $\operatorname{vect}(\mathbb{X})$ coincide.

Remark. It follows from the results in Section 4 that $\operatorname{mod}(\Sigma)$ has a unique preinjective component.

COROLLARY 3.5. *Let Σ be a wild almost concealed-canonical algebra and $\mathcal{C}$ a regular Auslander–Reiten component in* $\operatorname{mod}_{-}(\Sigma)$. *Then $\mathcal{C}$ is of type $\mathbb{Z}A_{\infty}$.*

Proof. Let $Z \in \mathcal{C}$ be such that the τ^{-}-cones $(Z \to)$ in $\mathcal{C}$ and $\operatorname{vect}(\mathbb{X})[1]$ coincide. The application of τ_{Σ} does not produce projective Σ-modules, thus the result follows from the fact that all regular components in $\operatorname{vect}(\mathbb{X})$ are of shape $\mathbb{Z}A_{\infty}$ [11].

3.6. Recall from [2] that for a system of objects $\mathcal{S}$ in an abelian category $\mathcal{A}$ the *right perpendicular category* $\mathcal{S}^{\perp}$ (resp. the *left perpendicular category* ${}^{\perp}\mathcal{S}$) is defined as the full subcategory of $\mathcal{A}$ consisting of all objects $A \in \mathcal{A}$ satisfying the following two conditions:

(i) $\operatorname{Hom}(S, A) = 0$ (resp. $\operatorname{Hom}(A, S) = 0$) for all $S \in \mathcal{S}$,
(ii) $\operatorname{Ext}^1(S, A) = 0$ (resp. $\operatorname{Ext}^1(A, S) = 0$) for all $S \in \mathcal{S}$.

Now, let $T = T' \oplus T''$ be a tilting sheaf on $\mathbb{X}$ where $T' \in \operatorname{vect}(\mathbb{X})$ and $T'' \in \operatorname{coh}_0(\mathbb{X})$. Denote $\Sigma' = \operatorname{End}(T')$. We know from [10] that T' is a tilting bundle on a weighted projective line $\mathbb{X}'$ with the property that in $\operatorname{coh}(\mathbb{X})$ the right perpendicular category to all simple composition factors of the objects of T'' is equivalent to $\operatorname{coh}(\mathbb{X}')$. Moreover, $\operatorname{mod}_{+}(\Sigma)$ coincides with $\operatorname{mod}_{+}(\Sigma')$. Using these notations we have

THEOREM 3.6. *Let Σ be an almost concealed-canonical algebra and $\mathcal{C}$ be an Auslander–Reiten component in* $\operatorname{mod}_{+}(\Sigma)$ *different from a preprojective component. Then there exists an indecomposable $Z \in \mathcal{C}$ such that the τ_{Σ}-cone $(\to Z)$ in $\mathcal{C}$ is a full subquiver of a component of* $\operatorname{vect}(\mathbb{X}')$.

Proof. Similarly to the proof of Theorem 3.4 there is an indecomposable $Z \in \mathcal{C}$ with the property that the τ_{Σ}-orbit of Z does not contain a projective Σ-module and $0 < \operatorname{rk}(Z) = \operatorname{rk}(\tau_{\Sigma}^{t} Z)$ for all $t \geq 0$. Then for the Auslander–Reiten sequence

$$0 \to \tau_{\Sigma'} Z \to Y \to Z \to 0$$

in $\mathrm{vect}(\mathbb{X}')$ we have $\mathrm{rk}(\tau_{\Sigma'}Z) = \mathrm{rk}(\tau_{\mathbb{X}'}Z) = \mathrm{rk}(Z)$. Therefore by Corollary 3.3(ii), $\tau_{\mathbb{X}'}Z = \tau_{\Sigma'}Z \in \mathrm{mod}_+(\Sigma') = \mathrm{mod}_+(\Sigma)$. Now one can follow the dual of the arguments of the proof of Theorem 3.4.

COROLLARY 3.7. *Let Σ be a wild concealed-canonical algebra and $\mathcal{C}$ a regular component in* $\mathrm{mod}_+(\Sigma)$. *Then $\mathcal{C}$ is of type $\mathbb{Z}A_\infty$.*

Remark 3.8. If T is a tilting sheaf on a wild weighted projective line $\mathbb{X}$, then $\mathbb{X}'$ can be wild, tubular or domestic, thus for the almost concealed-canonical algebra Σ a regular component in $\mathrm{mod}_+(\Sigma)$ can be of type $\mathbb{Z}A_\infty$, a stable tube or of type $\mathbb{Z}\Delta$ for an extended Dynkin graph Δ.

4. The wing decomposition of a tilting bundle

4.1. In this section we assume that $\mathbb{X}$ is wild and T is a tilting bundle on $\mathbb{X}$. The following theorem is the analogue of the result of Strauss [14] concerning tilting modules without nonzero preinjective direct summands over connected (wild) hereditary algebras. We use the fact that for a vector bundle $E \in \mathrm{vect}(\mathbb{X})$ without self-extensions the right perpendicular category $E^\perp$ formed in $\mathrm{coh}(\mathbb{X})$ is equivalent to a module category over a hereditary algebra [6], thus it makes sense to speak about $E^\perp$-preprojective objects. The proof of the following theorem can be given using the arguments of [14].

THEOREM 4.1. *Let T be a tilting bundle over a wild weighted projective line $\mathbb{X}$. Then there exists a decomposition*

$$T = T_P \oplus T_1$$

which satisfies the following conditions:

(i) *The category $T_1^\perp$ is equivalent to the module category of a connected wild hereditary algebra.*

(ii) *T_P is $T_1^\perp$-preprojective.*

(iii) *The preprojective component of the algebra $\Sigma_P = \mathrm{End}(T_P)$ is a full component of the Auslander–Reiten quiver of Σ. Moreover*, *this is the only preprojective component for Σ.*

4.2. Let $T = T_P \oplus T_1$ be the decomposition of a tilting bundle from the Theorem above. Now we apply the results of [12] and [8] in order to obtain a wing decomposition for T. By [11] all components of the Auslander–Reiten quiver of $\mathrm{vect}(\mathbb{X})$ are of the form $\mathbb{Z}A_\infty$, thus the indecomposable direct summands of T_1 determine wings in the sense of [13, (3.3)]. For an indecomposable vector bundle W on $\mathbb{X}$ with quasi-length s and quasi-socle X, which is contained in a component $\mathcal{C}$, the *wing* $\mathcal{W}(W)$ of W is defined to be the mesh-complete full subquiver given by the vertices $\tau_{\mathbb{X}}^{-t}X(r)$ with $1 \le r \le m$, $0 \le t \le s - r$, where $X(r)$ is the indecomposable with quasi-length r and quasi-socle X.

Now, if W is an indecomposable direct summand of T of quasi-length s, then in the wing $\mathcal{W}(W)$ there are s indecomposable direct summands of T and they form a branch in the sense of [13]. Further, for an indecomposable direct summand W of T_1 no summand of T_P is contained in $\mathcal{W}(W)$. If W_1 and W_2 are summands of T_1 such that $\mathcal{W}(W_i) \nsubseteq \mathcal{W}(W_j)$ for $i \neq j$, then $\mathcal{W}(W_1) \cap \mathcal{W}(W_2) = \emptyset$. Therefore T has a decomposition $T = T_P \oplus \bigoplus_{i=1}^{l} T(M_i)$ such that $T(M_i)$ is a tilting object, hence a branch, in the wing $\mathcal{W}(M_i)$ and furthermore the wings $\mathcal{W}(M_i)$ are pairwise disjoint. Observe that M_i is a direct summand of $T(M_i)$.

Finally, we want to distinguish the branches $T(M_i)$ which do not allow nonzero morphisms to other branches. Define $T'(M_j) = T_P \oplus \bigoplus_{i\neq j} T(M_i)$. Since the quiver of T has no oriented cycles there exists an M_j such that $T'(M_j) \in T^{\perp}(M_j)$.

Let $\{W_1, \ldots, W_r\}$ be the set of these M_j's and $\{V_1, \ldots, V_s\}$ be the others. Then we have

$$T = T_P \oplus \bigoplus_{i=1}^{s} T(V_i) \oplus \bigoplus_{j=1}^{r} T(W_j).$$

We call this decomposition the *wing decomposition* of T.

Observe that for each V_i there exists a W_j and a sequence of nonzero maps $(f_l)_{1\leq l\leq t}$:

$$(*) \qquad V_i = V_{i_1} \xrightarrow{f_1} V_{i_2} \xrightarrow{f_2} \ldots \to V_{i_t} \xrightarrow{f_t} V_{i_{t+1}} = W_j.$$

It follows from [4] that each f_l is either a monomorphism or an epimorphism. In fact we have

Lemma 4.2. *In the sequence* $(*)$ *above every morphism is an epimorphism.*

Proof. Assume that some $f_l : V_{i_l} \to V_{i_{l+1}}$ is a monomorphism. Denote by T'' the direct sum of all branches $T(M_j)$ where $M = W$ or $M = V$ such that $M_j \neq V_{i_l}$ and there is a chain of nonzero maps

$$V_{i_l} \to M_{k_1} \to M_{k_2} \to \ldots \to M_{k_u} = M_j$$

and let T' be the complement of T'' in T. Now, the perpendicular category $(T'')^{\perp}$ is by [6] equivalent to a module category $\mathrm{mod}(H)$ over a hereditary algebra. Observe that T' is in $(T'')^{\perp}$. Since f_l is a monomorphism we have an embedding $V_l \hookrightarrow T''$. Now, V_{i_l} is projective in $(T'')^{\perp}$. Indeed, if Z is an arbitrary object in $(T'')^{\perp}$, then $\mathrm{Ext}^1_{\mathbb{X}}(T'', Z) = 0$ and therefore $\mathrm{Ext}^1_{(T'')^{\perp}}(V_{i_l}, Z) = \mathrm{Ext}_{\mathbb{X}}(V_{i_l}, Z) = 0$. Then V_{i_l} is preprojective in $\mathrm{mod}(\Sigma')$ where $\Sigma' = \mathrm{End}(T')$. Since T_P is contained in T', V_{i_l} is also preprojective in $\mathrm{mod}(\Sigma_P)$ hence in $\mathrm{mod}(\Sigma)$. Consequently, V_{i_l} is a direct summand from T_P by Theorem 4.1, a contradiction.

4.3. If T is a tilting sheaf with wing decomposition

$$T = T_P \oplus \bigoplus_{i=1}^{s} T(V_i) \oplus \bigoplus_{j=1}^{r} T(W_j)$$

then we consider

$$\overline{T} = T_P \oplus \bigoplus_{i=1}^{s} \overline{V}_i \oplus \bigoplus_{j=1}^{r} \overline{W}_j,$$

where $\overline{V}_i$ (resp. $\overline{W}_j$) is the direct sum of the projectives in the wing $\mathcal{W}(V_i)$ (resp. $\mathcal{W}(W_j)$). By [13, (4.4)], $\overline{T}$ is a tilting sheaf again and

$$\overline{T} = T_P \oplus \bigoplus_{i=1}^{s} \overline{V}_i \oplus \bigoplus_{j=1}^{r} \overline{W}_j$$

is the wing decomposition of $\overline{T}$. We call $\overline{T}$ the *normalized form* or the *normalization* of T. As in [8, Lemma 2.5] we have

LEMMA 4.3. *Let T be a tilting sheaf with wing decomposition*

$$T = T_P \oplus \bigoplus_{i=1}^{s} T(V_i) \oplus \bigoplus_{j=1}^{r} T(W_j)$$

and let $\overline{T}$ be the normalization of T.

(a) *Assume that $F \in \mathrm{coh}(\mathbb{X})$ is not contained in the wings $\mathcal{W}(\tau_{\mathbb{X}} V_i)$ and $\mathcal{W}(\tau_{\mathbb{X}} W_j)$ for all i, j. Then $F \in \mathrm{coh}_{\geq}(T)$ if and only if $F \in \mathrm{coh}_{\geq}(\overline{T})$.*

(b) *Assume that $F \in \mathrm{coh}(\mathbb{X})$ is not contained in the wings $\mathcal{W}(V_i)$ and $\mathcal{W}(W_j)$ for all i, j. Then $F \in \mathrm{coh}_{\leq}(T)$ if and only if $F \in \mathrm{coh}_{\leq}(\overline{T})$.*

4.4. Furthermore we will frequently use the following information about wings proved in the situation of modules in [8] (see also [12]) and easily seen to be valid in our situation.

LEMMA 4.4. *Let U be indecomposable in* $\mathrm{vect}(\mathbb{X})$ *with quasi-length r and quasi-top X. Then*:

(a) *For an indecomposable vector bundle Y in* $\mathrm{vect}(\mathbb{X})$ *which is not in* $\mathrm{add}(\mathcal{W}(U))$ *the following conditions are equivalent*:

(1) $\mathrm{Hom}(Y, U) = 0$,
(2) $\mathrm{Hom}(Y, \tau_{\mathbb{X}}^{i} X) = 0$ *for* $i = 0, \ldots, r-1$,
(3) $\mathrm{Hom}(Y, W) = 0$ *for all* $W \in \mathrm{add}(\mathcal{W}(U))$.

(b) *For an indecomposable vector bundle Z in* $\mathrm{vect}(\mathbb{X})$ *which is not in* $\mathrm{add}(\mathcal{W}(U))$ *the following conditions are equivalent*:

(1) $\mathrm{Hom}(U, Z) = 0$,
(2) $\mathrm{Hom}(\tau_{\mathbb{X}}^{i} X, Z) = 0$ *for* $i = 0, \ldots, r-1$,
(3) $\mathrm{Hom}(W, Z) = 0$ *for all* $W \in \mathrm{add}(\mathcal{W}(U))$.

LEMMA 4.5. *Let W be an indecomposable vector bundle in* $\mathrm{vect}(\mathbb{X})$ *with quasi-length m and quasi-top X. Then the following conditions are equivalent*:

(a) *$X, \tau_{\mathbb{X}}X, \ldots, \tau_{\mathbb{X}}^{m-1}X$ are pairwise orthogonal.*
(b) *If $Z, Y \in \mathrm{add}(\mathcal{W}(W))$, then $\mathrm{rad}^{\infty}(Z, Y) = 0$.*

Here rad denotes the Jacobson radical of the category $\mathrm{coh}(\mathbb{X})$ and the infinite radical rad^{∞} is the intersection of all powers rad^{i}, $i \geq 1$, of rad. If one of the two conditions of the Lemma above is satisfied we call $\mathcal{W}(W)$ a *standard wing*.

LEMMA 4.6. *Let W be an indecomposable vector bundle in* $\mathrm{vect}(\mathbb{X})$ *with quasi-length m and let R be the indecomposable in* $\mathrm{vect}(\mathbb{X})$ *such that there is an irreducible epimorphism from R to W. Then $\mathcal{W}(R)$ is a standard wing if and only if W is exceptional.*

Recall that an indecomposable object X is *exceptional* if $\mathrm{Ext}^1_{\mathbb{X}}(X, X) = 0$.

5. Nonregular components for concealed-canonical algebras

THEOREM 5.1. *Let T be a tilting bundle on a wild weighted projective line $\mathbb{X}$ with wing decomposition*

$$T = T_P \oplus \bigoplus_{i=1}^{s} T(V_i) \oplus \bigoplus_{j=1}^{r} T(W_j).$$

Denote by X_j the quasi-socle of W_j and let $R_j \to W_j$ be an irreducible epimorphism for $j = 1, \ldots, r$. Then

(a) *$R_j \in \mathrm{coh}_{\geq}(T)$ for $j = 1, \ldots, r$.*
(b) *Let l be such that $\mathrm{rk}(X_l)$ is minimal and $\mu(X_l)$ is maximal among the X_j's with minimal rank. Then*

(i) *$\tau_{\mathbb{X}}^2 X_l \in \mathrm{coh}_{\geq}(T)$.*
(ii) *The $\tau_{\mathbb{X}}$-cone $(\to \tau_{\mathbb{X}}^2 X_l)$ is contained in $\mathrm{coh}_{\geq}(T)$ and is a full subquiver of the nonregular component in $\mathrm{mod}(\Sigma)$ containing W_l.*

Proof. By Lemma 4.3 we can assume that T is normalized. Therefore let $T = T_P \oplus \bigoplus_{i=1}^{s} \overline{V}_i \oplus \bigoplus_{j=1}^{r} \overline{W}_j$ using the notation of 4.3.

(a) First, W_l is exceptional, thus by Lemma 4.6, $\mathcal{W}(W_l)$ is a standard wing and by Lemma 4.5, $\mathrm{Hom}(R_j, \tau_{\mathbb{X}}W_j) = 0$. Moreover, Lemma 4.4 implies that $\mathrm{Hom}(R_j, \tau_{\mathbb{X}}\overline{W}_j) = 0$. Now, let $T = \overline{W}_j \oplus T'(W_j)$ and consider the exact sequence

$$0 \to \tau_{\mathbb{X}}W_j \to R_j \to Z_j \to 0.$$

Then $\mathrm{Hom}(Z_j, \tau_{\mathbb{X}} T'(W_j)) = 0$ because otherwise $0 \neq \mathrm{Hom}(W_j, \tau_{\mathbb{X}} T'(W_j)) \simeq \mathrm{DExt}^1(T'(W_j), W_j)$, which is impossible. Furthermore, from the wing decomposition of T we obtain $\mathrm{Hom}(W_j, T'(W_j))$=0, hence $\mathrm{Hom}(R_j, \tau_{\mathbb{X}} T'(W_j)) = 0$. Consequently, $\mathrm{Hom}(R_j, \tau_{\mathbb{X}} T) = 0$, which implies that $\mathrm{Ext}^1(T, R_j) = 0$.

(b) We know from (a) that $R_l \in \mathrm{coh}_{\geq}(T)$, thus by Proposition 3.1, $\tau_{\Sigma} R_l \simeq (\tau_{\mathbb{X}} R_l)_+$ and we have an exact sequence

$$0 \to \tau_{\Sigma} R_l \to \tau_{\mathbb{X}} R_l \to (\tau_{\mathbb{X}} R_l)_- \to 0.$$

Define $Q = (\tau_{\mathbb{X}} R_l)_-$. In the same way as in [8, Lemma 2.3] we conclude that $Q \in \mathrm{add}(\tau_{\mathbb{X}}(\bigoplus_{i=1}^s V_i \oplus \bigoplus_{j=1}^r W_j))$.

We claim that $\tau_{\mathbb{X}}^2 X_l \in \mathrm{coh}_{\geq}(T)$. Assume first that Q is of the form $(\tau_{\mathbb{X}} W_l)^{\oplus m}$ for some m. Since $\mathcal{W}(R_l)$ is a standard wing, $\mathrm{Hom}(\tau_{\mathbb{X}} R_l, \tau_{\mathbb{X}} W_l) = k$ and therefore applying the functor $\mathrm{Hom}(-, W_l)$ to the exact sequence (3) we obtain $m = 1$. Hence we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \tau_{\Sigma} R_1 \arrow[r] \arrow[d, dashed] & \tau_{\mathbb{X}} R_1 \arrow[r] \arrow[d, "\simeq"] & \tau_{\mathbb{X}} W_1 \arrow[r] \arrow[d, "\simeq"] & 0 \\
0 \arrow[r] & \tau_{\mathbb{X}}^2 X_1 \arrow[r] & \tau_{\mathbb{X}} R_1 \arrow[r] & \tau_{\mathbb{X}} W_1 \arrow[r] & 0
\end{tikzcd}$$

The induced morphism is an isomorphism, in particular we infer that $\tau_{\mathbb{X}}^2 X_l \simeq \tau_{\Sigma} R_l \in \mathrm{coh}_{\geq}(T)$.

Assume now that Q contains an indecomposable direct summand $\tau_{\mathbb{X}} Y \in \mathrm{add}(\tau_{\mathbb{X}}(\bigoplus_{i=1}^s V_i \oplus \bigoplus_{j \neq l} W_j))$. Then there is an epimorphism $\tau_{\mathbb{X}} R_l \to \tau_{\mathbb{X}} Y$ and therefore an epimorphism $R_l \to Y$. Moreover, applying the functor $\mathrm{Hom}(-, Y)$ to the exact sequence

$$0 \to \tau_{\mathbb{X}} X_l \to R_l \to W_l \to 0$$

we obtain a nonzero map $f : \tau_{\mathbb{X}} X_l \to Y$. Now,

$$\mathrm{Ext}^1(Y, \tau_{\mathbb{X}} X_l) \simeq \mathrm{DHom}(\tau_{\mathbb{X}} X_l, \tau_{\mathbb{X}} Y_l) \simeq \mathrm{DHom}(X_l, Y) = 0,$$

therefore it follows from [4] that f is an epimorphism or a monomorphism. Clearly f is not an isomorphism, because $Y \in \mathrm{coh}_{\geq}(T)$ but $\tau_{\mathbb{X}} X_l \in \mathrm{coh}_{\leq}(T)$.

Assume first that f is an epimorphism. If $Y = W_j$ for some j then $\mathrm{rk}(X_l) > \mathrm{rk}(W_j) \geq \mathrm{rk}(X_j)$, contrary to the assumption on l. If $Y = V_i$ for some i then using Lemma 4.2 we can compose f with an epimorphism $V_i \to W_j$ and again $\mathrm{rk}(X_l) > \mathrm{rk}(W_j) \geq \mathrm{rk}(X_j)$ gives a contradiction.

In case f is a monomorphism we also have a monomorphism $\tau_{\mathbb{X}}^2 X_l \hookrightarrow \tau_{\mathbb{X}} Y$. Applying the functor $\mathrm{Hom}(T, -)$ we see that $\mathrm{Hom}(T, \tau_{\mathbb{X}}^2 X_l) = 0$. Now, applying the functor $\mathrm{Hom}(T, -)$ to the exact sequence

$$0 \to \tau_{\mathbb{X}}^2 X_l \to \tau_{\mathbb{X}} R_l \to \tau_{\mathbb{X}} W_l \to 0$$

we conclude $\mathrm{Hom}(T, \tau_{\mathbb{X}} R_l) = 0$, which means that $\tau_{\mathbb{X}} R_l \in \mathrm{coh}_{\leq}(T)$. Therefore $\tau_{\Sigma} R_l = 0$ and R_l is projective in $\mathrm{mod}(\Sigma)$, which is impossible. This finishes the proof that $\tau_{\mathbb{X}}^2 X_l \in \mathrm{coh}_{\geq}(T)$.

Now we show by induction on n that $\tau_{\mathbb{X}}^n X_l \in \mathrm{coh}_{\geq}(T)$. From the induction hypothesis and Proposition 3.1 we see that $\tau_{\Sigma}(\tau_{\mathbb{X}}^{n-1} X_l) \simeq (\tau_{\mathbb{X}}^n X_l)_+$, thus there is an exact sequence

$$0 \to \tau_{\Sigma}(\tau_{\mathbb{X}}^{n-1} X_l) \to \tau_{\mathbb{X}}^n X_l \to Q \to 0.$$

Again by [8, Lemma 2.3] we have $Q \in \mathrm{add}(\tau_{\mathbb{X}}(\bigoplus_{i=1}^s V_i \oplus \bigoplus_{j=1}^r W_j))$. Assume that $Q \neq 0$. Then there is an epimorphism $f : \tau_{\mathbb{X}}^{n-1} X_l \to Y$ where Y is some W_j or some V_i. Using Lemma 4.2 the second case can be reduced to the first one. Now, if $f : \tau_{\mathbb{X}}^{n-1} X_l \to W_j$ is an epimorphism but not an isomorphism then $\mathrm{rk}(X_l) > \mathrm{rk}(W_j) \geq \mathrm{rk}(X_j)$, which contradicts the choice of l. On the other hand, if f is an isomorphism then $W_j = X_j$ and therefore $\mathrm{rk}(X_j) = \mathrm{rk}(X_l)$ but $\mu(X_j) = \mu(X_l((n-1)\vec{\omega})) > \mu(X_l)$, again a contradiction to the assumption on l. Hence $Q = 0$ and consequently $\tau_{\mathbb{X}}^n X_l \simeq \tau_{\Sigma}(\tau_{\mathbb{X}}^{n-1} X_l)$, in particular $\tau_{\mathbb{X}}^n X_l \in \mathrm{coh}_{\geq}(T)$.

Finally, as in the proof of Theorem 3.4, one shows that all Auslander–Reiten sequences in the $\tau_{\mathbb{X}}$-cone $(\to \tau_{\mathbb{X}}^2 X_l)$ in $\mathrm{vect}(\mathbb{X})$ are also Auslander–Reiten sequences in $\mathrm{mod}(\Sigma)$. Using $\tau_{\Sigma} R_l \simeq \tau_{\mathbb{X}}^2 X_l$ and the existence of an irreducible morphism from R_l to W_l we see that the cone $(\to \tau_{\mathbb{X}}^2 X_l)$ and W_l are in the same component in $\mathrm{mod}(\Sigma)$.

Definition 5.2. Let T be a tilting bundle. An indecomposable direct summand $S \in \mathrm{add}(T)$ is called a *special summand* of T if S is a sink summand of T and T has no $\overline{S}^{\perp}$-preinjective direct summands. Recall that $\overline{S}$ denotes the direct sum of all indecomposable projectives in the wing $\mathcal{W}(S)$.

This definition is the analogue of [14, Def. 7.3] and makes sense because by [6], $\overline{S}^{\perp}$ is equivalent to a module category. In our situation we can characterize special summands using the rank and the degree functions.

Theorem 5.3. *Let T be a tilting bundle on a wild weighted projective line $\mathbb{X}$ with wing decomposition*

$$T = T_P \oplus \bigoplus_{i=1}^{s} T(V_i) \oplus \bigoplus_{j=1}^{r} T(W_j).$$

Denote by X_j the quasi-socle of W_j. Let l be such that $\mathrm{rk}(X_l)$ is minimal and $\mu(X_l)$ is maximal among the X_j's with minimal rank. Then W_l is a special summand.

Proof. We can assume that T is normalized. Indeed, we have $T(W_l)^{\perp} = \overline{W_l}^{\perp}$ [14]. Furthermore, the whole wings $\mathcal{W}(W_j)$ and $\mathcal{W}(V_i)$ are contained in

$\overline{W}_l^{\perp}$ and the irreducible maps between the projectives of those wings remain irreducible in $\overline{W}_l^{\perp}$.

We suppose therefore that $T = T_P \oplus \bigoplus_{i=1}^s \overline{V}_i \oplus \bigoplus_{j=1}^r \overline{W}_j$ and consider the tilting bundle $T' = T'(W_l) = T_P \oplus \bigoplus_{i=1}^s \overline{V}_i \oplus \bigoplus_{j \neq l} \overline{W}_j$ in $\overline{W}_l^{\perp}$. Denote $\Sigma' = \mathrm{End}(T')$. Then an indecomposable direct summand from T_P is preprojective in $\mathrm{mod}(\Sigma)$, thus preprojective in $\mathrm{mod}(\Sigma')$ and consequently not preinjective in $\overline{W}_l^{\perp}$.

Next we show that no X_j, $j \neq l$, is preinjective in $\overline{W}_l^{\perp}$. Fix such an X_j. By [11, Theorem 2.7] there exists an N such that $\mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}}^{-N} X_j, X_j) \neq 0$. Now, consider the chain of irreducible maps in $\mathrm{coh}(\mathbb{X})$:

$$(*) \qquad X_j \overset{\mu_0}{\hookrightarrow} Y_j \overset{\varepsilon_0}{\twoheadrightarrow} \tau_{\mathbb{X}}^{-} X_j \overset{\mu_1}{\hookrightarrow} \tau_{\mathbb{X}}^{-} Y_j \overset{\varepsilon_1}{\twoheadrightarrow} \ldots \tau_{\mathbb{X}}^{-n} X_j \overset{\mu_n}{\hookrightarrow} \tau_{\mathbb{X}}^{-n} Y_j \overset{\varepsilon_n}{\twoheadrightarrow} \tau_{\mathbb{X}}^{-n-1} X_j$$
$$\overset{\mu_{n+1}}{\hookrightarrow} \tau_{\mathbb{X}}^{-n-1} Y_j \overset{\varepsilon_{n+1}}{\twoheadrightarrow} \ldots \overset{\mu_{N-1}}{\hookrightarrow} \tau_{\mathbb{X}}^{-N+1} Y_j \overset{\varepsilon_{N-1}}{\twoheadrightarrow} \tau_{\mathbb{X}}^{-N} X_j,$$

where all μ_n are monomorphisms and all ε_n are epimorphisms.

In case all $\tau_{\mathbb{X}}^{-n} X_j$ and all $\tau_{\mathbb{X}}^{-n} Y_j$ appearing in $(*)$ belong to $\overline{W}_l^{\perp}$ we obtain a cycle in $\overline{W}_l^{\perp}$ and then X_j is regular. Thus we can assume that one $\tau_{\mathbb{X}}^{-n} Z_j$ with $Z = X$ or $Z = Y$ is not contained in $\overline{W}_l^{\perp}$.

We claim that $\mathrm{Hom}(\overline{W}_l, \tau_{\mathbb{X}}^{-n} X_j) = 0$ for $n = 0, \ldots, N$. First, as a consequence of Theorem 5.1, we have $0 = \mathrm{Ext}^1(T, \tau_{\mathbb{X}}^{n} X_l) \simeq \mathrm{DHom}(\tau_{\mathbb{X}}^{n} X_l, \tau_{\mathbb{X}} T)$ for $n \geq 2$. Moreover, $\mathrm{Hom}(W_l, X_j) = 0$, which implies by Lemma 4.4, $\mathrm{Hom}(\tau_{\mathbb{X}}^{-m} X_l, X_j) = 0$ for $m = 0, 1, \ldots, t$ where $t+1$ is the quasi-length of W_l. Therefore $\mathrm{Hom}(\tau_{\mathbb{X}}^{-m} X_l, \tau_{\mathbb{X}}^{-n} X_j) = 0$ for $m = 0, 1, \ldots, t$ and $n = 0, 1, \ldots, N$. Observe that $\tau_{\mathbb{X}}^{-n} X_j \notin \mathrm{add}(\mathcal{W}(W_l))$. Indeed, otherwise X_l is in the τ-orbit of X_j, which implies that X_l and X_j have equal rank. Since the wings $\mathcal{W}(W_l)$ and $\mathcal{W}(W_j)$ are disjoint we then have $X_l = \tau_{\mathbb{X}}^{-m} X_j$ for some $m \geq 0$, hence $\mu(X_l) < \mu(X_j)$, which contradicts the assumption on l. Therefore $\tau_{\mathbb{X}}^{-n} X_j \notin \mathrm{add}(\mathcal{W}(W_l))$ and consequently $\mathrm{Hom}(\overline{W}_l, \tau_{\mathbb{X}}^{-n} X_j) = 0$ by Lemma 4.4.

It follows that in our case some $\mathrm{Ext}^1(\overline{W}_l, \tau_{\mathbb{X}}^{-i} Z_j) \neq 0$ for some $\tau_{\mathbb{X}}^{-i} Z_j$, $Z = X$ or $Z = Y$. Because the ε_i are epimorphisms, the first sheaf of $(*)$ which is not contained in $\overline{W}_l^{\perp}$ is some $\tau_{\mathbb{X}}^{-n} Y_j$. For this n we have $\mathrm{Ext}^1(W_l, \tau_{\mathbb{X}}^{-n} Y_j) \neq 0$. Now, by [6] the embedding $W_l^{\perp} \hookrightarrow \mathrm{coh}(\mathbb{X})$ admits a left adjoint functor $l : \mathrm{coh}(\mathbb{X}) \to W_l^{\perp}$. Then we can proceed as in [14, Lemma 7.2]. The object $l(\tau_{\mathbb{X}}^{-n} Y_j)$ is indecomposable by [14, 2.2] using $\mathrm{Hom}(\tau_{\mathbb{X}}^{-n} Y_j, W_l) \simeq \mathrm{DExt}^1(W_l, \tau_{\mathbb{X}}^{-n+1} Y_j) = 0$ by the choice of n. Moreover, the map $l(\mu_n) : l(\tau_{\mathbb{X}}^{-n} X_j) \to l(\tau_{\mathbb{X}}^{-n} Y_j)$ is nonzero. Now, $W_l^{\perp}$ is the coproduct of $\overline{W}_l^{\perp}$ and the category of a finite wing by [14, Theorem 3.5] and we conclude that $l(\tau_{\mathbb{X}}^{-n} Y_j) \in \overline{W}_l^{\perp}$. It follows that $\mathrm{Hom}(l(\tau_{\mathbb{X}}^{-n} Y_j), R_l) \simeq \mathrm{Hom}(l(\tau_{\mathbb{X}}^{-n} Y_j), W_l)$, where R_l is defined as in Theorem 5.1. By the construction of the functor l (see [2]) the last term is nonzero. Thus we have a chain of nonzero maps in $\overline{W}_l^{\perp}$:

$$X_j \to Y_j \to \tau_{\mathbb{X}}^{-} X_j \to \tau_{\mathbb{X}}^{-} Y_j \to \ldots \to \tau_{\mathbb{X}}^{-n} X_j = l(\tau_{\mathbb{X}}^{-n} X_j) \to \tau_{\mathbb{X}}^{-n} Y_j \to R_l.$$

By [14], R_l is regular in $\overline{W}_l^{\perp}$. Therefore X_j and consequently no direct summand of $\overline{W}_j$ is preinjective in $\overline{W}_l^{\perp}$.

In order to finish the proof it remains to show that no V_i is preinjective in $\overline{W}_l^{\perp}$. By Lemma 4.2, for each V_i there is some epimorphism $f : V_i \twoheadrightarrow W_j$. If $j \neq l$ we conclude from the fact that W_j is not preinjective that V_i is not preinjective. If $j = l$, then $f : V_i \to W_l$ factors through the middle term of the Auslander–Reiten sequence ending in W_l, and since $\mathrm{Hom}(V_i, \tau_{\mathbb{X}} \overline{W}_l) \simeq \mathrm{DExt}^1(\overline{W}_l, V_i) = 0$ it factors through R_l. Now the fact that R_l is regular in $\overline{W}_l^{\perp}$ implies that V_i is not preinjective in $\overline{W}_l^{\perp}$.

THEOREM 5.4. *Let T be a tilting bundle on a wild weighted projective line $\mathbb{X}$ with wing decomposition*

$$T = T_P \oplus \bigoplus_{i=1}^{s} T(V_i) \oplus \bigoplus_{j=1}^{r} T(W_j).$$

Define $\Sigma_P = \mathrm{End}(T_P)$. Let $\mathcal{C}$ be a component in $\mathrm{mod}_+(\Sigma)$. Then there exists an indecomposable $Z \in \mathcal{C}$ such that the τ_{Σ}^{-}-cone $(Z \to)$ is a full subquiver of a component in $\mathrm{mod}(\Sigma_P)$.

Proof. The proof of this theorem is similar to the proof of [8, Theorem 1].

We can assume that $\mathcal{C}$ is not the preprojective component and furthermore that $T = T_P \oplus \bigoplus_{i=1}^{s} \overline{V}_i \oplus \bigoplus_{j=1}^{r} \overline{W}_j$ is normalized. Define $T'(W_j) = T_P \oplus \bigoplus_{i=1}^{s} \overline{V}_i \oplus \bigoplus_{t \neq j} \overline{W}_t$.

Choose l such that $\mathrm{rk}(X_l)$ is minimal and $\mu(X_l)$ is maximal among the X_j's with minimal rank. To simplify notation we write $W = W_l$ and $X = X_l$, where as before X_l is the quasi-socle of W_l. Let $Z \in \mathcal{C}$. We will first show that for some $N \geq 0$,

$$\mathrm{Hom}_{\mathbb{X}}(W, \tau_{\Sigma}^{-t} Z) = 0 \quad \text{for } t \geq N. \tag{1}$$

By [9, 2.9] there is an integer M such that

$$\mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}}^{i} X, Z) = 0 \quad \text{for } i \geq M. \tag{2}$$

Let $i \geq 2$. We know from Theorem 5.1 that $\tau_{\mathbb{X}}^{i} X \in \mathrm{coh}_{\geq}(T)$ and for these objects the application of τ_{Σ}^{-} and $\tau_{\mathbb{X}}^{-}$ coincides. Therefore the application of τ_{Σ}^{-} gives an isomorphism

$$\overline{\mathrm{Hom}}_{\Sigma}(\tau_{\mathbb{X}}^{i+1} X, \tau_{\Sigma}^{-t} Z) \simeq \underline{\mathrm{Hom}}_{\Sigma}(\tau_{\mathbb{X}}^{i} X, \tau_{\Sigma}^{-t-1} Z). \tag{3}$$

The first term of (3) equals $\mathrm{Hom}_{\Sigma}(\tau_{\mathbb{X}}^{i+1} X, \tau_{\Sigma}^{-t} Z)$ because $\mathrm{mod}_+(\Sigma)$ contains no nonzero injective Σ-modules and the second term of (3) equals $\mathrm{Hom}_{\Sigma}(\tau_{\mathbb{X}}^{i} X, \tau_{\Sigma}^{-t-1} Z)$ because for $i \geq 2$, $\mathrm{Hom}_{\Sigma}(\tau_{\mathbb{X}}^{i} X, T) = \mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}}^{i} X, T) \simeq$

$\mathrm{DExt}^1(T, \tau_{\mathbb{X}}^{i+1}) = 0$ and therefore a nontrivial factorization through a projective module is not possible. Iterating the arguments above $t+1$ times we obtain $\mathrm{Hom}_{\Sigma}(\tau_{\mathbb{X}}^{i} X, \tau_{\Sigma}^{-t-1} Z) \simeq \mathrm{Hom}_{\Sigma}(\tau_{\mathbb{X}}^{i+t+1} X, Z)$, which vanishes by (2) for $t \geq M - i - 1$.

Thus we have shown that there exists an $N \in \mathbb{N}$ such that

$$\mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}}^{i} X, \tau_{\Sigma}^{-t} Z) = 0 \quad \text{for } i \geq 2 \text{ and } t \geq N. \tag{4}$$

Now we show that

$$\mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}}^{i} X, \tau_{\Sigma}^{-t} Z) = 0 \quad \text{for } i \geq 0 \text{ and } t \geq N+2. \tag{5}$$

Consider the exact sequence

$$0 \to \tau_{\Sigma}^{-t} Z \to \tau_{\mathbb{X}}(\tau_{\Sigma}^{-t-1} Z) \xrightarrow{p} Q_t \to 0 \tag{$*$}$$

where $Q_t = (\tau_{\mathbb{X}}(\tau_{\Sigma}^{-t-1} Z))_{-}$. By [8, Lemma 2.3], $Q_t \in \mathrm{add}(\tau_{\mathbb{X}}(\bigoplus_{i=1}^{s} V_i \oplus \bigoplus_{j=1}^{r} W_j))$. Now, for $t \geq N$, $\mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}}^{2} X, \tau_{\Sigma}^{-t} Z) = 0$ and $\mathrm{Hom}(\tau_{\mathbb{X}}^{2} X, Q_t) = 0$ because there are no nonzero morphisms from $\mathrm{coh}_{\geq}(T)$ to $\mathrm{coh}_{\leq}(T)$ and consequently $0 = \mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}}^{2} X, \tau_{\mathbb{X}}(\tau_{\Sigma}^{-t-1} Z)) \simeq \mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}} X, \tau_{\Sigma}^{-t-1} Z)$.

Assume now that there is a nonzero morphism

$$f \in \mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}} X, \tau_{\mathbb{X}}(\tau_{\Sigma}^{-t-1} Z)) \simeq \mathrm{Hom}_{\mathbb{X}}(X, \tau_{\Sigma}^{-t-1} Z) \quad \text{for } t \geq N+1.$$

Applying the functor $\mathrm{Hom}(\tau_{\mathbb{X}} X, -)$ to $(*)$ we see from $\mathrm{Hom}(\tau_{\mathbb{X}} X, \tau_{\Sigma}^{-t} Z) = 0$ that the composition $p \circ f : \tau_{\mathbb{X}} X \to Q_t$ is nonzero. Since it factorizes over $\tau_{\mathbb{X}}(\tau_{\Sigma}^{-t-1} Z)$ it is in $\mathrm{rad}^{\infty}(\tau_{\mathbb{X}} X, Q_t)$.

Let $Q = E_1 \oplus E_2$ where $E_1 \in \mathrm{add}(\tau_{\mathbb{X}} W)$ and E_2 is without direct summand isomorphic to $\tau_{\mathbb{X}} W$ and decompose $p = \binom{p_1}{p_2}$, $p_i : \tau_{\mathbb{X}}(\tau_{\Sigma}^{-t-1} Z) \to E_i$. Then $p_2 \circ f = 0$ by the wing decomposition of T. It follows that $0 \neq p_1 \circ f \in \mathrm{rad}^{\infty}(\tau_{\mathbb{X}} X, (\tau_{\mathbb{X}} W)^m)$, which gives a contradiction to the fact that $\mathcal{W}(W)$ is a standard wing. Thus formula (5) holds.

Let q be the quasi-length of W and denote by $X(j)$ the indecomposable vector bundle with quasi-length j and quasi-socle X. We show by induction on u that for $1 \leq u \leq q$ there exists an $N' \in \mathbb{N}$ such that

$$\mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}}^{i} X(j), \tau_{\Sigma}^{-t} Z) = 0 \tag{6}$$

for all $i \geq 0$, $j \leq u$, $t \geq N'$. The case $u = 1$ was already proved. For $u \geq 0$ we consider the Auslander–Reiten sequence

$$0 \to \tau_{\mathbb{X}}^{i} X(u-1) \to \tau_{\mathbb{X}}^{i} X(u) \oplus \tau_{\mathbb{X}}^{i-1} X(u-2) \to \tau_{\mathbb{X}} X(u-1) \to 0$$

(where for $u = 1$ the middle term consists only of the first summand). Applying the functor $\mathrm{Hom}_{\mathbb{X}}(-, \tau_{\Sigma}^{-t} Z)$ we obtain by induction $\mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}}^{i} X(u), \tau_{\Sigma}^{-t} Z) = 0$ for $t \geq N'$ and $i \geq 1$. In order to show that also $\mathrm{Hom}_{\mathbb{X}}(X(u), \tau_{\Sigma}^{-t} Z) = 0$ assume to the contrary that there is a nonzero map $f' \in \mathrm{Hom}_{\mathbb{X}}(X(u), \tau_{\Sigma}^{-t} Z)$. Then for the corresponding $f \in \mathrm{Hom}_{\mathbb{X}}(\tau_{\mathbb{X}} X(u), \tau_{\mathbb{X}}(\tau_{\Sigma}^{-t} Z))$ the composition $p \circ f$ is nonzero and is in $\mathrm{rad}^{\infty}(\tau_{\mathbb{X}} X(u), Q_t)$. Using again the decomposition $Q_t = E_1 \oplus E_2$, and $p = \binom{p_1}{p_2}$ we have $p_2 = 0$

and then $0 \neq p_1 \circ f \in \mathrm{rad}^\infty(\tau_{\mathbb{X}} X(u), (\tau_{\mathbb{X}} W)^m)$, a contradiction because $\mathcal{W}(\tau_{\mathbb{X}} W)$ is a standard wing. It follows that $\mathrm{Hom}_{\mathbb{X}}(X(u), \tau_\Sigma^{-t} Z) = 0$ for $t \geq N'$.

For $u = q$ we obtain $\mathrm{Hom}_{\mathbb{X}}(W, \tau_\Sigma^{-t} Z) = 0$, which proves formula (1). As a consequence for $Z' = \tau_\Sigma^{-N'} Z$ the τ_Σ^--cone $(Z' \rightarrow)$ consists of modules over $\mathrm{End}(T'(W))$. Now the perpendicular category $\overline{W}_l^\perp$ is equivalent to a module category over a hereditary algebra H. Under this equivalence $T'(W)$ corresponds to a tilting module in $\mathrm{mod}(H)$, which is, by Theorem 5.3, without a preinjective direct summand. The modules of $(Z' \rightarrow)$ are contained in the class of torsion-free modules $\mathcal{Y}_{\mathrm{mod}(H)}(T'(W))$ of the torsion pair in $\mathrm{mod}(H)$ defined by the tilting module $T'(W)$. Moreover, the Auslander–Reiten sequences of $\mathcal{C}$ which are in this cone are also Auslander–Reiten sequences in $\mathcal{Y}_{\mathrm{mod}(H)}(T'(W))$. This means that $(Z' \rightarrow)$ is part of a component in $\mathcal{Y}_{\mathrm{mod}(H)}(T'(W))$ and our result follows from [8, Theorem 1].

COROLLARY 5.5. *Let Σ be a wild concealed-canonical algebra and $\mathcal{C}$ a component in* $\mathrm{mod}_+(\Sigma)$ *different from the preprojective component. Then the stable part of $\mathcal{C}$ is of type $\mathbb{Z}A_\infty$.*

COROLLARY 5.6. *Let $\mathbb{X}$ be a wild weighted projective line*, *T a tilting bundle on $\mathbb{X}$ and $\Sigma = \mathrm{End}(T)$ the corresponding concealed-canonical algebra. Then T defines bijections between the following three sets*:

- $\Omega(\mathbb{X})$ *of components of* $\mathrm{vect}(\mathbb{X})$,
- $\Omega_+(\Sigma)$ *of components of* $\mathrm{mod}_+(\Sigma)$ *different from the preprojective component*,
- $\Omega(\Sigma_P)$ *of regular components of* $\mathrm{mod}(\Sigma_P)$.

Proof. Let $\mathcal{C}$ be a component of $\mathrm{mod}_+(\Sigma)$ different from the preprojective component. It follows from Theorems 3.4 and 5.4 that there exist a unique component $\mathcal{C}'$ in $\mathrm{vect}(\mathbb{X})$ and a unique regular component $\mathcal{C}''$ in $\mathrm{mod}(\Sigma_P)$ such that $\mathcal{C}$ and $\mathcal{C}'$ coincide on a τ-cone and $\mathcal{C}$ and $\mathcal{C}''$ coincide on a τ^--cone. Thus we obtain injective maps $\mu_1 : \Omega_+(\Sigma) \rightarrow \Omega(\mathbb{X})$ and $\mu_2 : \Omega_+(\Sigma) \rightarrow \Omega(\Sigma_P)$.

Let $\mathcal{D}$ be a component of $\mathrm{vect}(\mathbb{X})$ and X a quasi-simple vector bundle in $\mathcal{D}$. Then $\mathrm{DExt}^1_{\mathbb{X}}(T, \tau^n_{\mathbb{X}} X) = \mathrm{Hom}(\tau^n_{\mathbb{X}} X, \tau_{\mathbb{X}} T) = 0$ for all $n \geq n_0$ by [11, Theorem 2.9]. Therefore all objects in the τ-cone $(\rightarrow \tau^{n_0}_{\mathbb{X}} X)$ are in $\mathrm{mod}_+(\Sigma)$ and the Auslander–Reiten sequences of that cone are also Auslander–Reiten sequences in $\mathrm{mod}(\Sigma)$. Consequently, μ_1 is surjective.

In order to show that also μ_2 is surjective we proceed as in [8, Theorem 3]. Σ is an iterated branch-enlargement

$$\Sigma = C_0[Z_1, Q_1] \ldots [Z_m, Q_m],$$

where $C_0 = \Sigma_P$ and, for $j = 1, \ldots, m$, $C_j = C_0[Z_1, Q_1] \ldots [Z_j, Q_j]$ is obtained by a one-point extension of C_{j-1} by a quasi-simple C_{j-1}-module Z_j

and then rooting a linear quiver $Q_j = \circ \to \circ \to \ldots \to \circ$ at the module Z_j. Now, let $\mathcal{D}$ be a regular component in $\mathrm{mod}(\Sigma_P)$ and Y a quasi-simple object in $\mathcal{D}$. For $j = 0, \ldots, m-1$, C_j is a tilted algebra of some wild connected hereditary algebra with tilting module without preinjective direct summand (using Theorem 5.3). Therefore by [8, Corollary 3.2] there are $N_j \in \mathbb{N}$, $j = 0, \ldots, m-1$, such that $\mathrm{Hom}_{C_j}(Z_{j+1}, \tau_{C_j}^{-l} Y) = 0$ for $l \geq N_j$. It follows that the Auslander–Reiten sequences of a τ^--cone $(\tau_{\Sigma_P}^{-n} Y \to)$ are also Auslander–Reiten sequences in $\mathrm{mod}(\Sigma)$ and consequently μ_2 is surjective.

Using the duality $D : \mathrm{coh}_+(T) \to \mathrm{coh}_-(T)$, $F \mapsto F^{\check{}}(\vec{c} + \vec{\omega})$, we have for a tilting bundle the same results for $\mathrm{mod}_-(\Sigma)$.

6. Nonregular components for almost concealed-canonical algebras

6.1. In this section we assume that $T = T' \oplus T''$ is a tilting sheaf on a wild weighted projective line $\mathbb{X}$ where $T' \in \mathrm{vect}(\mathbb{X})$ and $T'' \in \mathrm{coh}_0(\mathbb{X})$. Because $\mathrm{mod}_+(\Sigma)$ coincides with $\mathrm{mod}(\Sigma')$, where $\Sigma' = \mathrm{End}(T')$ ([10]), the structure of the components of $\mathrm{mod}_+(\Sigma)$ follows from the description of the previous sections (see 3.6 and 5.4).

In order to describe the left hand side of a component of $\mathrm{mod}_-(\Sigma)$ we use the dual wing decomposition. The proofs of the following results are dual and therefore omitted.

THEOREM 6.1. *Let T be a tilting sheaf over a wild weighted projective line $\mathbb{X}$. Then there exists a decomposition*

$$T = T_2 \oplus T_I$$

which satisfies the following conditions:

(i) *The left perpendicular category ${}^{\perp}T_2$ is equivalent to the module category of a connected wild hereditary algebra.*

(ii) *T_I is ${}^{\perp}T_2$-preinjective.*

(iii) *The preinjective component of the algebra $\Sigma_I = \mathrm{End}(T_I)$ is a full component of the Auslander–Reiten quiver for Σ. Moreover, this is the only preinjective component for Σ.*

6.2. Dually to 4.2 we have a decomposition

$$T = \bigoplus_{j=1}^{a} T(W_j) \oplus \bigoplus_{i=1}^{b} T(V_i) \oplus T_I.$$

THEOREM 6.2. *Let T be a tilting sheaf on a wild weighted projective line $\mathbb{X}$ with a decomposition*

$$T = \bigoplus_{j=1}^{a} T(W_j) \oplus \bigoplus_{i=1}^{b} T(V_i) \oplus T_I.$$

Denote by Z_j the quasi-top of W_j and let $R_j \to W_j$ be an irreducible epimorphism for $j = 1, \ldots, a$. Then:

(a) $R_j \in \mathrm{coh}_{\leq}(T)$ *for* $j = 1, \ldots, a$.

(b) *Let l be such that* $\mathrm{rk}(Z_l)$ *is maximal and* $\mu(Z_l)$ *is minimal among the Z_j's with maximal rank. Then*

(i) $\tau_{\mathbb{X}}^{-} Z_l \in \mathrm{coh}_{\leq}(T)$.

(ii) *The $\tau_{\mathbb{X}}^{-}$-cone $(\tau_{\mathbb{X}}^{-} Z_l \to)$ is contained in* $\mathrm{coh}_{\leq}(T)$ *and* $(\tau_{\mathbb{X}}^{-} Z_l \to)[1]$ *is a full subquiver of the nonregular component in* $\mathrm{mod}(\Sigma)$ *containing $\tau_{\mathbb{X}} W_l$.*

Observe that this is the dual situation of Theorem 5.1 shifted by $\tau_{\mathbb{X}}$. Moreover, because T is not contained in $\mathrm{coh}_0(\mathbb{X})$, X_l is a vector bundle, and therefore we can apply dual arguments as in the proof of Theorem 5.1.

THEOREM 6.3. *Let T be a tilting sheaf on a wild weighted projective line $\mathbb{X}$ with decomposition*

$$T = \bigoplus_{j=1}^{a} T(W_j) \oplus \bigoplus_{i=1}^{b} T(V_i) \oplus T_I.$$

Denote by Z_j the quasi-top of W_j. Let l be such that $\mathrm{rk}(Z_l)$ *is maximal and* $\mu(Z_l)$ *is minimal among the Z_j's with maximal rank. Then no direct summand of T is ${}^{\perp}\underline{W_l}$-preprojective where $\underline{W_l}$ denotes the direct sum of all injectives in the wing $\mathcal{W}(W_l)$.*

The theorem can be proved again by using dual arguments with simple modifications. It is essential for the induction step of the following result.

THEOREM 6.4. *Let T be a tilting sheaf on a wild weighted projective line $\mathbb{X}$ with decomposition*

$$T = \bigoplus_{j=1}^{a} T(W_j) \oplus \bigoplus_{i=1}^{b} T(V_i) \oplus T_I.$$

Define $\Sigma_I =$ $\mathrm{End}(T_I)$. *Let $\mathcal{C}$ be a component in* $\mathrm{mod}_{\leq}(\Sigma)$. *Then there exists an indecomposable $Z \in \mathcal{C}$ such that the τ_{Σ}-cone $(\to Z)$ is a full subquiver of a component in* $\mathrm{mod}(\Sigma_I)$.

COROLLARY 6.5. *Let T be a tilting sheaf on a wild weighted projective line $\mathbb{X}$ and $T = T' \oplus T''$ with $T' \in$* $\mathrm{vect}(\mathbb{X})$ *and $T'' \in$* $\mathrm{coh}_0(\mathbb{X})$. *Furthermore, let $T = T_2 \oplus T_I$ be the decomposition of Theorem* 6.1. *Then T'' is a direct summand of T_I.*

Proof. Let $\mathcal{C}$ be a component in $\mathrm{mod}_{\le}(\Sigma)$ different from the preinjective component containing some injective Σ-module of the form $Y = \tau_{\mathbb{X}} M_j[1]$ where $M = W$ or V and $M_j \in \mathrm{coh}_0(\mathbb{X})$. It follows from Theorems 3.4 and 6.4 that Y has infinitely many successors in the component $\mathcal{C}$. This is impossible, since the Σ-modules from $\mathrm{mod}_0^-(\Sigma)$ have only finitely many successors in $\mathrm{mod}(\Sigma)$.

COROLLARY 6.6. *Let $\mathbb{X}$ be a wild weighted projective line, T a tilting sheaf on $\mathbb{X}$ and $\Sigma = \mathrm{End}(T)$ the corresponding almost concealed-canonical algebra. Then T defines bijections between the following three sets*:

- $\Omega(\mathbb{X})$ *of components of* $\mathrm{vect}(\mathbb{X})$,
- $\Omega_-(\Sigma)$ *of components of* $\mathrm{mod}_-(\Sigma)$ *different from the preinjective component*,
- $\Omega(\Sigma_I)$ *of regular components of* $\mathrm{mod}(\Sigma_I)$.

Proof. A component of $\mathrm{mod}_-(\Sigma)$ coincides on a τ-cone with a component of $\mathrm{mod}(\Sigma_I)$ and on a τ^--cone with a component of $\mathrm{vect}(\mathbb{X})$. Now we can apply similar arguments as in the proof of Corollary 5.6. For the proof of the surjectivity of μ_1 we use the fact that there are no nonzero morphisms from T'' to vector bundles.

REFERENCES

[1] W. Geigle and H. Lenzing, *A class of weighted projective curves arising in representation theory of finite dimensional algebras*, in: Singularities, Representations of Algebras, and Vector Bundles, Lecture Notes in Math. 1273, Springer, 1987, 265–297.

[2] —, — *Perpendicular categories with applications to representations and sheaves*, J. Algebra 144 (1991), 273–343.

[3] D. Happel, I. Reiten and S. Smalø, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. 575 (1996).

[4] D. Happel and C. M. Ringel, *Tilted algebras*, Trans. Amer. Math. Soc. 274 (1982), 399–443.

[5] M. Hoshino, *On splitting torsion theories induced by tilting modules*, Comm. Algebra 11 (1983), 493–500.

[6] T. Hübner and H. Lenzing, *Categories perpendicular to exceptional bundles*, preprint, 1993.

[7] O. Kerner, *Tilting wild algebras*, J. London Math. Soc. 39 (1989), 29–47.

[8] —, *Stable components of wild tilted algebras*, J. Algebra 142 (1991), 37–57.

[9] H. Lenzing and H. Meltzer, *Sheaves on a weighted projective line of genus one and representations of a tubular algebra*, in: Representations of Algebras, Sixth International Conference, Ottawa 1992, CMS Conf. Proc. 14, 1993, 317–337.

[10] —, —, *Tilting sheaves and concealed-canonical algebras*, in: Proceedings of the Seventh International Conference on Representations of Algebras, Cocoyoc, Mexico, 1994, CMS Conf. Proc. 18, 1996, 455–473.

[11] H. Lenzing and J. A. de la Peña, *Wild canonical algebras*, Math. Z., to appear.

[12] C. M. Ringel, *Finite dimensional algebras of wild representation type*, Math. Z. 161 (1978), 235–255.

[13] —, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. 1099, Springer, 1984.

[14] H. Strauss, *On the perpendicular category of a partial tilting module*, J. Algebra 144 (1991), 43–66.

Technische Universität Chemnitz-Zwickau
Fakultät für Mathematik
09107 Chemnitz, Germany
E-mail: meltzer@mathematik.tu-chemnitz.de

Received 15 November 1995