COLLOQUIUM MATHEMATICUM

VOL. 71 1996 NO. 2

THE MINIMAL EXTENSION OF SEQUENCES III. ON PROBLEM 16 OF GRÄTZER AND KISIELEWICZ

BY

J. DUDEK (WROCŁAW)

The main result of this paper is a description of totally commutative idempotent groupoids. In particular, we show that if an idempotent groupoid $(G,\cdot)$ has precisely $m\ge 2$ distinct essentially binary polynomials and they are all commutative, then G contains a subgroupoid isomorphic to the groupoid N_m described below. In [2], this fact was proved for $m=2$.

1. Given an algebra A, we denote by $p_n=p_n(A)$ the number of essentially n-ary polynomials over A. For definitions we refer the reader to [5] and [6]. We say that an infinite (or finite) sequence $(a_0,\ldots)$ of cardinals is *representable* if there exists an algebra A_0 such that $p_n(A_0)=a_n$ for all n. If additionally A_0 is taken from a given class of algebras, then we say that the sequence is representable in that class. A sequence $a^*=(a_0,\ldots,a_m,\ldots)$ of nonnegative integers (cardinals) is a *minimal extension* of the sequence $a=(a_0,\ldots,a_m)$ (in a given class K of algebras) if a^* is representable (in the class K) and for every algebra A ($\in K$) which represents $a=(a_0,\ldots,a_m)$ we have $p_n(A)\ge a_n$ for all n.

Problems concerning minimal extensions of sequences were raised by G. Grätzer in [4]. He also initiated (together with R. Padmanabhan and J. Płonka) the problems of characterization of algebras (varieties) by means of the number p_n (see Problem 42, p. 195 of [5]).

In this paper we deal with Problem 16 of G. Grätzer and A. Kisielewicz [6]: Does the sequence $(0,1,n)$ have a minimal extension, for any $n\ge 0$?

The main result was presented during the Conference on Logic and Algebra dedicated to Roberto Magari on his 60th birthday, Pontignano (Siena), 26–30 April 1994 (see Theorem 2 of [3]).

2. A groupoid $(G,\cdot)$ is said to be *totally commutative* if all of its essentially binary polynomials are commutative operations. As usual, we use the notation xy^n defined by induction as follows: $xy^1=xy$ and $xy^{n+1}=(xy^n)y$.

1991 *Mathematics Subject Classification*: Primary 08B05.

By $\mathbf{N}_n$ ($n = 1, 2, \ldots$) we denote the variety of all commutative idempotent groupoids $(G, \cdot)$ satisfying $xy^2 = yx^2$ and $xy^n = xy^{n+1}$. This family of varieties was introduced in [2] together with the family of groupoids $N_n = (\{-1, 0, 1, \ldots, n\}, \cdot)$, where the fundamental operation $\cdot$ is defined as follows:

$$xy = \begin{cases} x & \text{if } x = y, \\ 1 + \max(x, y) & \text{if } x \neq y \text{ and } x, y \leq n - 1, \\ n & \text{otherwise.} \end{cases}$$

By Theorem 4 of [1], if a commutative idempotent groupoid $(G, \cdot)$ satisfies $xy^2 = yx^2$, then it is totally commutative (and moreover, each essentially binary polynomial is of the form $xy^k = yx^k$). Hence, every variety $\mathbf{N}_n$ consists entirely of totally commutative groupoids.

Moreover, it is not difficult to see that $N_n \in \mathbf{N}_n$ for every $n \geq 1$. In fact, it follows from the proof of the theorem below that N_n is isomorphic to the free groupoid with two free generators in the variety $\mathbf{N}_n$.

From the proof we will also see that $p_2(N_n) = n$. (Incidentally, it seems that this very family should have been mentioned in [6], p. 77, as one showing that all sequences $(0, 1, n)$ are representable. Otherwise, the example given in [6] is not quite correct; for instance, the case $n = 3$ does not have the property claimed.)

In [2] we have proved that if a commutative idempotent groupoid $(G, \cdot)$ satisfies $p_2(G, \cdot) = 2$ (and hence is totally commutative!), then $(G, \cdot)$ contains a subgroupoid isomorphic to N_2. As a corollary, the sequence $(0, 1, 2, 10, \ldots \ldots, p_n(N_2), \ldots)$ is the minimal extension of the sequence $(0, 1, 2)$ in the class of all commutative groupoids.

Here we generalize this result as follows:

THEOREM. *Let $(G, \cdot)$ be a totally commutative idempotent groupoid. Then the following conditions are equivalent for every positive integer $m \geq 2$:*

(i) $p_2(G, \cdot) = m$;

(ii) $(G, \cdot) \in \mathbf{N}_m$ *and* $(G, \cdot) \notin \mathbf{N}_k$ *for any* $k < m$;

(iii) $(G, \cdot) \in \mathbf{N}_m$ *and* $(G, \cdot)$ *contains a subgroupoid isomorphic to* N_m;

(iv) *there exists a positive integer $n > m$ such that $xy^n = xy^m$ and m is minimal with this property.*

As an immediate consequence we have

COROLLARY. *The sequence $(0, 1, m, \ldots, p_n(N_m), p_{n+1}(N_m), \ldots)$ is the minimal extension of the sequence $(0, 1, m)$ in the class of all totally commutative groupoids.*

Proof of the Theorem. Suppose that $p_2(G) = m > 1$. Then, since G is totally commutative, xy^2 is essentially binary and $xy^2 = yx^2$. By Theorem 4 of [1], every essentially binary polynomial of G is of the form

$f(x,y)=xy^k$; from the proof of this theorem it also follows that

$$(*)\qquad xy^k = yx^k \quad \text{for all } k \geq 1.$$

If all the polynomials xy^k, $k>1$, are different, then $p_2(G)$ is infinite. Otherwise, $xy^n = xy^k$ for some $1<n<k$. Suppose that $n>1$ is minimal with this property. Then using $(*)$ we have

$$(**)\qquad \begin{aligned} xy^n &= (xy^n)(xy^n) = (xy^k)(xy^n) = ((xy^n)y^{k-n})(xy^n) \\ &= (y(xy^n)^{k-n})(xy^n) = y(xy^n)^{k-n+1} = (xy^n)y^{k-n+1} = xy^{k+1}. \end{aligned}$$

It follows that $xy^n = xy^{n+1}$.

Consequently, every essentially binary polynomial of G is of the form $f(x,y)=xy^k$ with $k\leq n$. Therefore $n=m$, $G\in \mathbf{N}_m$, and there are $a,b\in G$ such that $ab^{m-1}\neq ab^m$ (note that, since G is idempotent, $a\neq b$).

Let $G(a,b)$ be the subgroupoid of G generated by a,b. From what we have proved so far it follows that $G(a,b)=\{a,b,ab,\dots,ab^m\}$. We show that all these elements are different.

Indeed, first suppose that $ab^k=ab^n$ for some $0\leq k<n\leq m$. Multiplying this identity by b, $m-1-k\geq 0$ times, we get $ab^{m-1}=ab^{m+(n-1-k)}=ab^m$, a contradiction. Suppose now that $b=ab^k$ for some $k\geq 1$. Then, by $(*)$, $b=ba^k$, and in consequence, $ab=ba=ba^{k+1}=ab^{k+1}$, which has already been shown to be impossible.

It remains to show that $G(a,b)$ is isomorphic to $\mathbf{N}_m$. The one-to-one correspondence is defined by $b\to -1$ and $ab^k\to k$. For $k>n\geq 0$, using $(**)$, we get $(ab^k)(ab^n)=ab^{k+1}$. Also, $(ab^k)b=ab^{k+1}$. In view of the identity $xy^m=xy^{m+1}$ the isomorphism is clear, which completes the proof.

3. The theorem above characterizes totally commutative idempotent groupoids $(G,\cdot)$ with $p_2(G,\cdot)\geq 2$ finite. We close the paper with some remarks concerning the remaining totally commutative idempotent groupoids.

If $p_2(G,\cdot)=1$ then, according to Lemma 1 of [2], $(G,\cdot)$ is either a nontrivial near-semilattice (a member of the variety $\mathbf{N}_1$) or a nontrivial Steiner quasigroup (an idempotent commutative groupoid satisfying $xy^2=x$).

If $p_2(G,\cdot)=0$ then $(G,\cdot)$ is a left or right zero-semigroup (i.e., one satisfying $xy=y$ or $xy=x$).

If $p_2(G,\cdot)$ is infinite, then (as in the proof of the theorem) $(G,\cdot)$ is a commutative groupoid satisfying $xy^2=yx^2$, but not belonging to any variety $\mathbf{N}_n,\ n=1,2,\dots$

An infinite analogue of N_n is $N_\omega=(\{-1,0,1,\dots\},\cdot)$, where xy is defined to be idempotent and $xy=1+\max(x,y)$ for $x\neq y$. Here, $p_2(N_\omega)$ is infinite (the polynomials xy^k are pairwise distinct), the set $\{-1,0\}$ is the set of generators, and moreover N_ω is isomorphic to the free groupoid with two free generators in the variety $\mathbf{N}_\omega$ of all commutative idempotent groupoids $(G,\cdot)$ satisfying $xy^2 = yx^2$.

Another member of $\mathbf{N}_\omega$ is $\widetilde{N}_n = (\{-1,0,\ldots,n\}\cup\{n+1,\ldots\},\cdot)$, where the operation $\cdot$ is defined by

$$xy = \begin{cases} x & \text{if } x = y,\\ 1+\max(x,y) & \text{if } x\ne y \text{ and } n\notin\{x,y\},\\ n & \text{otherwise,}\end{cases}$$

and $n\ge 1$ is a fixed integer.

It is not difficult to see that $p_2(\widetilde{N}_n)$ is infinite and $\widetilde{N}_n$ contains no isomorphic copy of N_ω. (Note that for every n, N_n is embeddable in $\widetilde{N}_n$.) Thus, the condition (iii) of the theorem does not extend to the case of $p_2(G,\cdot)$ infinite.

REFERENCES

[1] J. Dudek, *Variety of idempotent commutative groupoids*, Fund. Math. 120 (1984), 193–204.

[2] —, *On the minimal extension of sequences*, Algebra Universalis 23 (1986), 308–312.

[3] —, *p_n-sequences. The minimal extension of sequences* (*Abstract*), presented at the Conference on Logic and Algebra dedicated to Roberto Magari on his 60th birthday, Pontignano (Siena), 26–30 April 1994, 1–6.

[4] G. Grätzer, *Composition of functions*, in: Proc. Conference on Universal Algebra, Kingston, 1969, Queen's Univ., Kingston, Ont., 1970, 1–106.

[5] —, *Universal Algebra*, 2nd ed., Springer, New York, 1979.

[6] G. Grätzer and A. Kisielewicz, *A survey of some open problems on p_n-sequences and free spectra of algebras and varieties*, in: Universal Algebra and Quasigroup Theory, A. Romanowska and J. D. H. Smith (eds.), Heldermann, Berlin, 1992, 57–88.

Mathematical Institute
University of Wrocław
Pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: dudek@math.uni.wroc.pl

Received 2 June 1195;
revised 11 October 1995