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SPECTRA FOR GELFAND PAIRS
ASSOCIATED WITH THE HEISENBERG GROUP

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Let K be a closed Lie subgroup of the unitary group $U(n)$ acting by automorphisms on the $(2n+1)$-dimensional Heisenberg group H_n. We say that (K,H_n) is a Gelfand pair when the set $L^1_K(H_n)$ of integrable K-invariant functions on H_n is an abelian convolution algebra. In this case, the Gelfand space (or spectrum) for $L^1_K(H_n)$ can be identified with the set $\Delta(K,H_n)$ of bounded K-spherical functions on H_n. In this paper, we study the natural topology on $\Delta(K,H_n)$ given by uniform convergence on compact subsets in H_n. We show that $\Delta(K,H_n)$ is a complete metric space and that the "type 1" K-spherical functions are dense in $\Delta(K,H_n)$. Our main result shows that one can embed $\Delta(K,H_n)$ quite explicitly in a Euclidean space by mapping a spherical function to its eigenvalues with respect to a certain finite set of $(K\ltimes H_n)$-invariant differential operators on H_n. This viewpoint on the spectrum for $\Delta(K,H_n)$ was previously known for $K=U(n)$ and is referred to as "the Heisenberg fan".

1. Introduction. Given a locally compact group G and compact subgroup $K\subset G$, the pair (G,K) is called a *Gelfand pair* if $L^1(G//K)$, the space of integrable, K-bi-invariant functions on G, is commutative. Perhaps the best known examples are those defining symmetric spaces, that is, when G is a connected semisimple Lie group with finite center, and K is a maximal compact subgroup. The analysis associated with such pairs plays an important role in the representation theory of semisimple Lie groups and has been extensively developed in the last four decades (cf. e.g. [9], [11]). In sharp contrast to this case, one might begin by assuming that G is a solvable Lie group. But then, if G is simply connected for example, there may be no non-trivial compact subgroups. One can, however, consider pairs of the

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form $(K \ltimes G, K)$ where K is a compact subgroup of $\mathrm{Aut}(G)$, the group of automorphisms of G.

The Heisenberg group H_n of dimension $2n+1$ is defined in Section 2. The unitary group $U(n)$ is a maximal compact connected subgroup of $\mathrm{Aut}(H_n)$. Throughout this paper, K will denote a compact Lie subgroup of $U(n)$ for which $(K \ltimes H_n, K)$ is a Gelfand pair. We say, by abuse of terminology, that "(K, H_n) is a Gelfand pair". In this setting, $L^1(K \ltimes H_n//K)$ can be identified with the algebra $L^1_K(H_n)$ of integrable functions on H_n that are invariant under the action of K. In [1], it is shown that the Gelfand pairs of this sort play a central role in the study of Gelfand pairs associated with more general solvable Lie groups.

Under the conditions described above, the set $\mathbb{D}_K(H_n)$ of differential operators on H_n which are simultaneously left-H_n-invariant and K-invariant forms an abelian algebra. In Section 2 we construct a canonical set of generators $\{L_{\gamma_1}, \ldots, L_{\gamma_d}, T\}$ for this algebra. Here T is the derivative in the central direction of H_n, and each L_{γ_j} is a polynomial coefficient differential operator of degree at least 2 involving derivatives in non-central directions. The operators L_{γ_j} arise from the invariant theory for the action of K on the ring $\mathbb{C}[z_1, \ldots, z_n]$.

The Gelfand space (or spectrum) of the commutative Banach $*$-algebra $L^1_K(H_n)$ is the set of continuous non-zero algebra homomorphisms from $L^1_K(H_n)$ to $\mathbb{C}$. This can be identified, via integration, with the set $\Delta(K, H_n)$ of *bounded K-spherical functions* on H_n. These are the smooth bounded K-invariant functions $\psi : H_n \to \mathbb{C}$ which are joint eigenfunctions for the operators $D \in \mathbb{D}_K(H_n)$ and are normalized to take the value 1 at the identity. A description of the bounded K-spherical functions can be found in [2] and is summarized below in Section 2. These functions are of two types. The K-spherical functions of *type* 1 yield non-trivial characters when restricted to the center of H_n, whereas the K-spherical functions of *type* 2 are constant on the center. The latter can be regarded as functions defined on the quotient of H_n by its center and reflect the abelian component of analysis on H_n. For $K = U(n)$, these can be expressed in terms of Bessel functions (see equation (3.4)). The K-spherical functions of type 1 reflect the non-abelian component of analysis on H_n. For the case $K = U(n)$, these can be expressed in terms of Laguerre polynomials (see equation (3.3)).

Our focus here is on the topology of the Gelfand space, where the usual weak*-topology coincides with the compact-open topology on $\Delta(K, H_n)$. Our main result is stated as Theorem 4.1. It asserts that a sequence $(\psi_N)_{N=1}^{\infty}$ in $\Delta(K, H_n)$ converges to $\psi \in \Delta(K, H_n)$ if and only if the sequences of eigenvalues for ψ_N with respect to each generator for $\mathbb{D}_K(H_n)$ converge to the corresponding eigenvalue for ψ. We require a careful analysis of the behavior of such eigenvalues, and these results are described in Section 3.

As a corollary to Theorem 4.1, we find that the map which sends $\psi \in \Delta(K, H_n)$ to its eigenvalues with respect to $(L_{\gamma_1}, \ldots, L_{\gamma_d}, T)$ is a homeomorphism of the space $\Delta(K, H_n)$ onto its image in $\mathbb{C}^{d+1}$. In Section 3 we show that the eigenvalues for the L_{γ_j}'s are real numbers with constant sign and that the T-eigenvalues are pure imaginary numbers. Thus we obtain a map to $(\mathbb{R}^+)^d \times \mathbb{R}$ with image $\mathfrak{F}(K, H_n)$ homeomorphic to $\Delta(K, H_n)$. We call $\mathfrak{F}(K, H_n)$ the *Heisenberg fan* for the Gelfand pair (K, H_n) and describe this explicitly as the union of a d-dimensional algebraic set in $(\mathbb{R}^+)^d \times \{0\}$ and a countable family of 1-dimensional polynomial curves. This picture is derived from the invariant theory for the action of K on the ring of polynomials $\mathbb{C}[z_1, \ldots, z_n]$.

We refine our description of the topology on the Gelfand space by proving two final results. Proposition 4.5 asserts that $\Delta(K, H_n)$ is complete. That is, if a sequence of bounded K-spherical functions converges to some function in the compact-open topology, then the limit is necessarily a bounded K-spherical function. Proposition 4.4 asserts that the K-spherical functions of type 1 are dense in $\Delta(K, H_n)$. Thus, every bounded K-spherical function is a limit in the compact-open topology of some sequence of K-spherical functions of type 1. Our proof of Proposition 4.4 uses the fact that the K-spherical functions of type 2 form a set of measure zero with respect to the Godement–Plancherel measure on $\Delta(K, H_n)$. Section 5 contains a description of the Godement–Plancherel measure and the K-spherical transform.

Propositions 4.4 and 4.5 are clear for the case $K = U(n)$. Theorem 4.1 is not obvious, but was also known in this case (cf. [4]). We have adopted the term "Heisenberg fan" from [7] and [17], which discuss the case when $K = U(n)$. Our proof of Theorem 4.1 in Section 4 is, however, self-contained. The main interest in the current paper is that our results encompass Gelfand pairs (K, H_n) for proper subgroups K of $U(n)$.

2. Notation and preliminaries. We need to establish notation and recall some results concerning Gelfand pairs and spherical functions associated with Heisenberg groups. For details concerning this preliminary material, we refer the reader to two earlier papers, [1] and [2], by the first three authors.

2.1. *Heisenberg group***.** Given a complex vector space V of dimension n with Hermitian inner product $\langle \cdot, \cdot \rangle$, one forms the Heisenberg group $H_n = V \times \mathbb{R}$ with group law

$$(z, t)(z', t') = \big(z + z', t + t' - \tfrac{1}{2} \operatorname{Im}\langle z, z' \rangle\big).$$

At times, it will be convenient to work in coordinates on V. One can use an orthonormal basis to identify V with $\mathbb{C}^n$ so that $\langle z, z' \rangle = z \cdot \overline{z'}$ for $z, z' \in \mathbb{C}^n$. The left-invariant vector fields generated by the one-parameter subgroups through $((0, \dots, 0, 1 \pm i, 0, \dots, 0), 0)$ are written explicitly as

$$Z_j = 2\frac{\partial}{\partial \overline{z}_j} + i\frac{z_j}{2}\frac{\partial}{\partial t}, \qquad \overline{Z}_j = 2\frac{\partial}{\partial z_j} - i\frac{\overline{z}_j}{2}\frac{\partial}{\partial t}.$$

In addition let

$$T := \frac{\partial}{\partial t},$$

so that $\{Z_1, \dots, Z_n, \overline{Z}_1, \dots, \overline{Z}_n, T\}$ is a basis for the Lie algebra $\mathfrak{h}_n$ of H_n. With these conventions one has $[Z_j, \overline{Z}_j] = -2iT$.

2.2. *Gelfand pairs***.** The group $U(n)$ of unitary transformation of $(V, \langle \cdot, \cdot \rangle)$ acts by automorphisms on H_n via

$$k \cdot (z, t) := (kz, t) \quad \text{ for } k \in U(n) \text{ and } (z, t) \in H_n.$$

This yields a maximal compact connected subgroup of $\mathrm{Aut}(H_n)$. If K is a compact Lie subgroup of $U(n)$ then we say that (K, H_n) is a *Gelfand pair* when the algebra $L^1_K(H_n) := \{f \in L^1(H_n) \mid f(kz, t) = f(z, t)\}$ of integrable K-invariant functions on H_n is commutative under convolution. The group $U(n)$, and many proper subgroups K of $U(n)$, yield Gelfand pairs. One can find a complete classification of all such subgroups in [14] and [3]. Under our identification of V with $\mathbb{C}^n$, $U(n)$ can be regarded as the group of $n \times n$ unitary matrices.

2.3. *Multiplicity free decomposition***.** An important ingredient in the theory of Gelfand pairs (K, H_n) is the representation of K on the space of polynomials $\mathbb{C}[V]$ given by

$$k \cdot p(z) := p(k^{-1}z).$$

One has the following result:

THEOREM 2.1 (cf. [5], [1]). *(K, H_n) is a Gelfand pair if and only if the representation of K on $\mathbb{C}[V]$ is multiplicity free.*

Throughout this paper, K will denote a closed Lie subgroup of $U(n)$ for which (K, H_n) is a Gelfand pair. We decompose $\mathbb{C}[V]$ into K-irreducible subspaces P_α,

$$\mathbb{C}[V] = \sum_{\alpha \in \Lambda} P_\alpha, \tag{2.1}$$

where Λ is some countably infinite index set. In view of Theorem 2.1, this is a canonical decomposition. Since the representation of K on $\mathbb{C}[V]$ preserves the space $\mathcal{P}_m(V)$ of homogeneous polynomials of degree m, each P_α is a subspace of some $\mathcal{P}_m(V)$. For $\alpha \in \Lambda$, we write $|\alpha|$ for the degree of homogeneity of the polynomials in P_α. Thus we have $P_\alpha \subset \mathcal{P}_{|\alpha|}(V)$ and

$$\mathcal{P}_m(V) = \sum_{|\alpha|=m} P_\alpha.$$

2.4. *Fock space***.** Fock space $\mathcal{F}$ consists of entire functions $f : V \to \mathbb{C}$ which are square integrable with respect to $e^{-|z|^2/2}dz$ with Hilbert space structure

$$\langle f, g\rangle_{\mathcal{F}} = \left(\frac{1}{2\pi}\right)^n \int_V f(z)\overline{g(z)}e^{-|z|^2/2}\,dz.$$

Here "dz" denotes Lebesgue measure on the underlying real space $V_{\mathbb{R}} \cong \mathbb{R}^{2n}$ for $V \cong \mathbb{C}^n$. The holomorphic polynomials $\mathbb{C}[V]$ form a dense subspace in $\mathcal{F}$ and one has the formula

$$\langle p, q\rangle_{\mathcal{F}} = p\left(2\frac{\partial}{\partial z}\right)\overline{q}(z)\Big|_{z=0} \qquad \text{for } p, q \in \mathbb{C}[V],$$

where $\overline{q}(z)$ denotes the holomorphic polynomial obtained in coordinates by conjugating the coefficients of q. The monomials $z^{\mathbf{a}} = z_1^{a_1}\dots z_n^{a_n}$ with $|\mathbf{a}| = a_1 + \dots + a_n = m$ form an orthogonal basis for $\mathcal{P}_m(V)$ in $\mathcal{F}$. One has $\|z^{\mathbf{a}}\|_{\mathcal{F}} = \sqrt{2^{|\mathbf{a}|}\mathbf{a}!}$, where as usual $\mathbf{a}! := a_1!\dots a_n!$.

2.5. *Invariant polynomials***.** Since the trivial representation of K occurs in $\mathbb{C}[V]$ as $\mathcal{P}_0(V)$, and $\mathbb{C}[V]$ is K-multiplicity free, there can be no non-constant K-invariant holomorphic polynomials. One does, however, have invariant polynomials on the underlying real vector space $V_{\mathbb{R}}$ for V. We denote the set of these by $\mathbb{C}[V_{\mathbb{R}}]^K$. More explicitly, given $\alpha \in \Lambda$, let $\{v_1, \dots, v_{\dim(P_\alpha)}\}$ be any orthonormal basis for P_α and define

$$p_\alpha(z) := \sum_{j=1}^{\dim(P_\alpha)} v_j(z)\overline{v}_j(\overline{z}). \tag{2.2}$$

This definition of p_α is independent of the basis chosen for P_α. Further, p_α is a K-invariant polynomial on $V_{\mathbb{R}}$ homogeneous of degree $2|\alpha|$, and $\{p_\alpha \mid \alpha \in \Lambda\}$ is a vector space basis for $\mathbb{C}[V_{\mathbb{R}}]^K$. Note that p_α takes values in the set $\mathbb{R}^+$ of non-negative reals.

A result due to Howe and Umeda (cf. [12]) shows that $\mathbb{C}[V_{\mathbb{R}}]^K$ is freely generated as an algebra. So there are polynomials $\gamma_1, \dots, \gamma_d \in \mathbb{C}[V_{\mathbb{R}}]^K$ so that

$$\mathbb{C}[V_{\mathbb{R}}]^K = \mathbb{C}[\gamma_1, \dots, \gamma_d].$$

We call $\gamma_1, \dots, \gamma_d$ the *fundamental invariants*. In fact, one can take, for each γ_j, a p_α for which P_α contains a primitive highest weight vector. We let $\delta_1, \dots, \delta_d \in \Lambda$ be the indices for which

$$\gamma_j = p_{\delta_j}.$$

One computes that

$$p_m(z) := \sum_{|\alpha|=m} p_\alpha(z) = \frac{1}{2^m m!}|z|^{2m}.$$

Letting $\gamma_0(z)$ be defined as

$$\gamma_0(z) = |z|^2,$$

this becomes $p_m = \gamma_0^m/(2^m m!)$. When K acts irreducibly on V, γ_0 will be one of the fundamental invariants. In general, one will have $\gamma_0 = \sum_{|\alpha_j|=1} \gamma_j$.

2.6. *The value space***.** The invariants for a smooth linear action of a compact Lie group K on a real vector space separate K-orbits (cf. for example page 133 in [16]). Thus, the map

$$\gamma := \gamma_1 \times \ldots \times \gamma_d : V \to (\mathbb{R}^+)^d$$

yields a bijection between the set V/K of K-orbits in V and the subset

$$\varGamma_K^+ := \gamma(V)$$

in $(\mathbb{R}^+)^d$. We call $\varGamma_K^+$ the *value space* for K.

2.7. *Invariant differential operators***.** The algebra $\mathbb{D}(H_n)$ of left-invariant differential operators on H_n is generated by $\{Z_1, \ldots, Z_n, \overline{Z}_1, \ldots, \overline{Z}_n, T\}$. We denote the subalgebra of K-invariant differential operators by

$$\mathbb{D}_K(H_n) := \{D \in \mathbb{D}(H_n) \mid D(f \circ k) = D(f) \circ k \text{ for } k \in K,\ f \in C^\infty(H_n)\}.$$

Since (K, H_n) is a Gelfand pair, $\mathbb{D}_K(H_n)$ is an abelian algebra. A result of Thomas (cf. [18]) shows that the converse is also true, at least when K is connected.

Each K-invariant polynomial $p \in \mathbb{C}[V_\mathbb{R}]^K$ gives rise to a differential operator

$$L_p := \widetilde{\mathcal{S}}(p) \in \mathbb{D}_K(H_n).$$

Here $\widetilde{\mathcal{S}} = \mathcal{S} \circ j \circ \varOmega$, where $\varOmega : \mathbb{C}[V_\mathbb{R}] \to \mathbb{C}[V_\mathbb{R}^*]$ is the algebra isomorphism induced by the symplectic pairing $-\operatorname{Im}\langle\cdot,\cdot\rangle$ on $V_\mathbb{R}$, and $j : \mathbb{C}[V_\mathbb{R}^*] \hookrightarrow \mathbb{C}[\mathfrak{h}_n^*]$ is inclusion and $\mathcal{S} : \mathbb{C}[\mathfrak{h}_n^*] \to \mathbb{D}(H_n)$ is the symmetrization map. We have

$$(L_p f)(z,t) = p\left(2\frac{\partial}{\partial\overline{\zeta}}, 2\frac{\partial}{\partial\zeta}\right) f\left(z+\zeta, t - \frac{1}{2}\operatorname{Im}\langle z,\zeta\rangle\right)\bigg|_{\zeta=0} \tag{2.3}$$

for $f \in C^\infty(H_n)$.

The operators L_p can be written explicitly using coordinates as follows: Let $p(z) = \sum c_{\mathbf{a},\mathbf{b}} z^{\mathbf{a}}\overline{z}^{\mathbf{b}}$, where $\mathbf{a}$, $\mathbf{b}$ are multi-indices and $z^{\mathbf{a}} = z_1^{a_1} \ldots z_n^{a_n}$, $\overline{z}^{\mathbf{b}} = \overline{z}_1^{b_1} \ldots \overline{z}_n^{b_n}$. Since p is K-invariant, it is easy to see that the operator

$p(Z,\overline{Z})$ defined by

$$p(Z,\overline{Z}) := \sum c_{\mathbf{a},\mathbf{b}} Z^{\mathbf{a}}\overline{Z}^{\mathbf{b}} = \sum c_{\mathbf{a},\mathbf{b}} Z_1^{a_1}\dots Z_n^{a_n}\overline{Z}_1^{b_1}\dots\overline{Z}_n^{b_n}$$

belongs to $\mathbb{D}_K(H_n)$. Note that the operator L_p is intrinsically defined, whereas $p(Z,\overline{Z})$ depends on the basis used to identify V with $\mathbb{C}^n$. One has

$$L_p = \mathrm{Sym}(p(Z,\overline{Z})),$$

where Sym is the linear map characterized by

$$\mathrm{Sym}(Z^{\mathbf{a}}\overline{Z}^{\mathbf{b}}) = \frac{1}{(|\mathbf{a}|+|\mathbf{b}|)!}\sum_{\sigma\in S_{|\mathbf{a}|+|\mathbf{b}|}}\sigma(Z^{\mathbf{a}}\overline{Z}^{\mathbf{b}}).$$

Here, as usual, $|\mathbf{a}| = a_1+\dots+a_n$ and $\sigma(Z^{\mathbf{a}}\overline{Z}^{\mathbf{b}})$ denotes the result of applying the permutation σ to the $|\mathbf{a}|+|\mathbf{b}|$ terms in $Z^{\mathbf{a}}\overline{Z}^{\mathbf{b}}$.

For a d-multi-index $\mathbf{a}$, let $\gamma^{\mathbf{a}} := \gamma_1^{a_1}\dots\gamma_d^{a_d}$, $\|\mathbf{a}\| := a_1|\delta_1|+\dots+a_d|\delta_d|$ (the homogeneous degree of $\gamma^{\mathbf{a}}$), and $L_\gamma^{\mathbf{a}} := L_{\gamma_1}^{a_1}\dots L_{\gamma_d}^{a_d}$. Using the definition of the map $\widetilde{\mathcal{S}}$, together with the fact that $[Z_j,\overline{Z}_j] = -2iT$, one sees that

$$L_{\gamma^{\mathbf{a}}} = L_\gamma^{\mathbf{a}} + \sum_{\|\mathbf{b}\|<\|\mathbf{a}\|} c_{\mathbf{a},\mathbf{b}} L_\gamma^{\mathbf{b}} T^{\|\mathbf{a}\|-\|\mathbf{b}\|} \tag{2.4}$$

for some coefficients $c_{\mathbf{a},\mathbf{b}}\in\mathbb{C}$. Since $\gamma_1,\dots,\gamma_d$ generate $\mathbb{C}[V_\mathbb{R}]$, it follows easily that $\{L_{\gamma_1},\dots,L_{\gamma_d},T\}$ generates the algebra $\mathbb{D}_K(H_n)$.

2.8. *Spherical functions.* A smooth function $\psi : H_n \to \mathbb{C}$ is called *K-spherical* if

(1) ψ is K-invariant,
(2) ψ is an eigenfunction for every $D\in\mathbb{D}_K(H_n)$, and
(3) $\psi(0,0)=1$.

Since $L_{\gamma_1},\dots,L_{\gamma_d},T$ generate $\mathbb{D}_K(H_n)$, (2) holds if and only if ψ is an eigenfunction for each of $L_{\gamma_1},\dots,L_{\gamma_d},T$. We write $\widehat{D}(\psi)$ for the eigenvalue of $D\in\mathbb{D}_K(H_n)$ on a K-spherical function ψ, that is, $D(\psi)=\widehat{D}(\psi)\psi$. Note that since $\psi(0,0)=1$, one has

$$\widehat{D}(\psi) = D(\psi)(0,0). \tag{2.5}$$

We denote the set of positive definite K-spherical functions on H_n by $\Delta(K,H_n)$. In [1], it is shown that every bounded K-spherical function is positive definite, so $\Delta(K,H_n)$ is also the set of bounded K-spherical functions. We remark that this result is not true for more general Gelfand pairs (G,K) and contrasts with the situation for symmetric spaces. The positive definite K-spherical functions ψ are the matrix coefficients obtained using unit K-fixed vectors in the representation spaces for the K-spherical representations of the semidirect product group $K\ltimes H_n$. In [2], it is shown that these functions can be described directly in terms of the representation theory for H_n and the action of K on V.

One has an irreducible unitary representation π of H_n on $\mathcal{F}$ defined as

$$(\pi(z,t)f)(w) = e^{it-\langle w,z\rangle/2-|z|^2/4} f(w+z).$$

For $\alpha \in \Lambda$ let

$$\phi_\alpha(z,t) := \frac{1}{\dim(P_\alpha)} \sum_{j=1}^{\dim(P_\alpha)} \langle \pi(z,t)v_j, v_j\rangle_{\mathcal{F}}, \tag{2.6}$$

where $\{v_1, \ldots, v_{\dim(P_\alpha)}\}$ is an orthonormal basis for P_α. This description of ϕ_α does not depend on our choice of basis $\{v_j\}$. Define $\phi_{\lambda,\alpha}$ for $\lambda \in \mathbb{R}^\times$ and $\alpha \in \Lambda$ by

$$\phi_{\lambda,\alpha}(z,t) := \phi_\alpha\big(\sqrt{|\lambda|}z, \lambda t\big), \tag{2.7}$$

so that $\phi_\alpha = \phi_{1,\alpha}$. The $\phi_{\lambda,\alpha}$'s are distinct bounded K-spherical functions. We refer to these elements of $\Delta(K,H_n)$ as the spherical functions of *type* 1. From equation (2.6) one can show that ϕ_α has the general form

$$\phi_\alpha(z,t) = e^{it} q_\alpha(z) e^{-|z|^2/4},$$

where q_α is a K-invariant polynomial on $V_{\mathbb{R}}$ with homogeneous component of highest degree given by $(-1)^{|\alpha|} p_\alpha / \dim(P_\alpha)$.

In addition to the K-spherical functions of type 1, there are K-spherical functions which arise from the one-dimensional representations of H_n. For $w \in V$, let

$$\eta_w(z,t) := \int_K e^{i\,\mathrm{Re}\langle w,kz\rangle}\, dk = \int_K e^{i\,\mathrm{Re}\langle z,kw\rangle}\, dk, \tag{2.8}$$

where "dk" denotes normalized Haar measure on K. The η_w are the bounded K-spherical functions of *type* 2. Note that η_0 is the constant function 1 and $\eta_w = \eta_{w'}$ if and only if $Kw = Kw'$. We have one K-spherical function for each K-orbit in V and sometimes write "η_{Kw}" in place of "η_w".

It is shown in [2] that every bounded K-spherical function is of type 1 or type 2. Thus we have:

THEOREM 2.2. *The bounded K-spherical functions on H_n are parameterized by the set* $(\mathbb{R}^\times \times \Lambda) \cup (V/K)$ *via*

$$\Delta(K,H_n) = \{\phi_{\lambda,\alpha} \mid \lambda \in \mathbb{R}^\times,\ \alpha \in \Lambda\} \cup \{\eta_{Kw} \mid w \in V\}.$$

Note that, for $\psi \in \Delta(K,H_n)$, one has

$$\psi(z,t) = e^{i\lambda t}\psi(z,0), \tag{2.9}$$

where $\lambda = -i\widehat{T}(\psi) \in \mathbb{R}$. We observe that ψ is of type 2 if and only if $\lambda = 0$.

3. Eigenvalues $\widehat{D}(\psi)$. The eigenvalue $\widehat{T}(\psi)$ of T on $\psi \in \Delta(K, H_n)$ is easily understood via equation (2.9). Eigenvalues of the form $\widehat{L}_p(\psi)$ are much more subtle. In this section, we present a number of results concerning these eigenvalues. The first of these shows that $\widehat{L}_p(\psi)$ can be computed by replacing L_p with its "highest order terms".

LEMMA 3.1. *For $p \in \mathbb{C}[V_{\mathbb{R}}]^K$ and $\psi \in \Delta(K, H_n)$, one has*

$$\widehat{L}_p(\psi) = \partial_p(\psi)(0,0),$$

where $\partial_p := p\big(2\frac{\partial}{\partial \overline{z}}, 2\frac{\partial}{\partial z}\big)$. That is, ∂_p is the operator obtained by replacing each occurrence of z_j in p by $2\frac{\partial}{\partial \overline{z}_j}$ and each $\overline{z}_j$ by $2\frac{\partial}{\partial z_j}$.

Proof. Using equations (2.5) and (2.3) we see that

$$\widehat{L}_p(\psi) = L_p(\psi)(0,0) = p\bigg(2\frac{\partial}{\partial \overline{\zeta}}, 2\frac{\partial}{\partial \zeta}\bigg)\psi\bigg(0+\zeta, 0 - \frac{1}{2}\operatorname{Im}\langle 0, \zeta\rangle\bigg)\bigg|_{\zeta=0}$$

$$= \bigg(p\bigg(2\frac{\partial}{\partial \overline{z}}, 2\frac{\partial}{\partial z}\bigg)\psi\bigg)(0,0). \ \blacksquare$$

The eigenvalue for an operator of the form L_{p_α} on a K-spherical function of type 2 can be computed as follows. Recall that $P_\alpha \subset \mathcal{P}_{|\alpha|}$, so that p_α is homogeneous of degree $2|\alpha|$.

LEMMA 3.2. *$\widehat{L}_{p_\alpha}(\eta_w) = (-1)^{|\alpha|} p_\alpha(w)$.*

Proof. Using equation (2.8) together with Lemma 3.1 yields

$$\widehat{L}_{p_\alpha}(\eta_w) = \partial_{p_\alpha}(\eta_w)(0,0) = \partial_{p_\alpha}\Big[\int_K e^{i\operatorname{Re}\langle z, kw\rangle}\, dk\Big]\Big|_{z=0}.$$

One computes

$$\partial_{p_\alpha}(e^{i\operatorname{Re}\langle z,kw\rangle})(z) = i^{2|\alpha|} p_\alpha(kw) e^{i\operatorname{Re}\langle z,kw\rangle} = (-1)^{|\alpha|} p_\alpha(w) e^{i\operatorname{Re}\langle z,kw\rangle}.$$

Thus, $\widehat{L}_{p_\alpha}(\eta_w) = (-1)^{|\alpha|} p_\alpha(w)\eta_w(0) = (-1)^{|\alpha|} p_\alpha(w)$. $\blacksquare$

Lemmas 3.3, 3.4 and 3.5 concern eigenvalues on K-spherical functions of type 1.

LEMMA 3.3. *For $D \in \mathbb{D}_K(H_n)$, $\alpha \in \Lambda$ and all $v \in P_\alpha$ one has*

$$\pi(D)v = \widehat{D}(\phi_\alpha)v.$$

Proof. This is Proposition 3.20 in [2]. For the reader's convenience, we outline the proof here. One observes that the operator $\pi(D)$ on $\mathcal{F}$ preserves $\mathbb{C}[V]$ and commutes with the action of K. Since the decomposition given by equation (2.1) is multiplicity free, $\pi(D)$ must preserve each P_α, and by Schur's Lemma, $\pi(D)|_{P_\alpha}$ is a scalar operator, $c_\alpha I_{P_\alpha}$ say. Equations (2.5) and (2.6) now show that

$$\widehat{D}(\phi_\alpha) = D(\phi_\alpha)(0,0) = \frac{1}{\dim(P_\alpha)} \sum_{j=1}^{\dim(P_\alpha)} \langle \pi(0,0)\pi(D)v_j, v_j\rangle_{\mathcal{F}} = c_\alpha. \ \blacksquare$$

LEMMA 3.4. *$\widehat{L}_{p_\alpha}(\phi_{\lambda,\beta}) = |\lambda|^{|\alpha|}\widehat{L}_{p_\alpha}(\phi_\beta)$ for $\alpha, \beta \in \Lambda$, $\lambda \in \mathbb{R}^\times$.*

Proof. Using Lemma 3.1 together with equation (2.7), we compute

$$\begin{aligned}\widehat{L}_{p_\alpha}(\phi_{\lambda,\beta}) = \partial_{p_\alpha}(\phi_{\lambda,\beta})(0,0) &= p_\alpha\left(2\frac{\partial}{\partial \overline{z}}, 2\frac{\partial}{\partial z}\right)\left[\phi_\beta\big(\sqrt{|\lambda|}z, 0\big)\right]\Big|_{z=0} \\ &= \big(\sqrt{|\lambda|}\big)^{2|\alpha|} p_\alpha\left(2\frac{\partial}{\partial \overline{z}}, 2\frac{\partial}{\partial z}\right)[\phi_\beta(z,0)]\Big|_{z=0},\end{aligned}$$

since p_α is homogeneous of degree $2|\alpha|$. Thus, $\widehat{L}_{p_\alpha}(\phi_{\lambda,\beta}) = |\lambda|^{|\alpha|}\partial_{p_\alpha}(\phi_\beta)(0,0) = |\lambda|^{|\alpha|}\widehat{L}_{p_\alpha}(\phi_\beta)$ as desired. $\blacksquare$

LEMMA 3.5. *$\widehat{L}_{p_\alpha}(\psi)$ is a real number with sign $(-1)^{|\alpha|}$ for all $\alpha \in \Lambda$ and $\psi \in \Delta(K, H_n)$.*

Proof. In view of Lemmas 3.2 and 3.4, it suffices to show that $\widehat{L}_{p_\alpha}(\phi_\beta)$ is a real number with sign $(-1)^{|\alpha|}$ for $\alpha, \beta \in \Lambda$. We have

$$L_{p_\alpha} = \mathrm{Sym}(p_\alpha(Z, \overline{Z})) = \frac{1}{(2|\alpha|)!}\Big[p_\alpha(Z,\overline{Z}) + \sum_{|\delta|<|\alpha|} c_{\alpha,\delta}p_\delta(Z,\overline{Z})(2iT)^{|\alpha|-|\delta|}\Big],$$

for some coefficients $c_{\alpha,\delta}$. The value of $c_{\alpha,\delta}$ is obtained by collecting terms after reordering the monomials arising from symmetrization, so that the Z_j's precede the $\overline{Z}_j$'s. Since $\overline{Z}_jZ_j = Z_j\overline{Z}_j + 2iT$, we see that $c_{\alpha,\delta} \geq 0$. Thus we have

$$\begin{aligned}(3.1)\qquad &\widehat{L}_{p_\alpha}(\phi_\beta) \\ &= \frac{1}{(2|\alpha|)!}\Big[p_\alpha(Z,\overline{Z})^\wedge(\phi_\beta) + \sum_{|\delta|<|\alpha|} c_{\alpha,\delta}p_\delta(Z,\overline{Z})^\wedge(\phi_\beta)(-2)^{|\alpha|-|\delta|}\Big].\end{aligned}$$

Let $\{v_1, \ldots, v_{\dim(P_\delta)}\}$ be an orthonormal basis for P_δ and let u_β be a unit vector in P_β. Using Lemma 3.3 and equation (2.2) we see that

$$\begin{aligned}p_\delta(Z,\overline{Z})^\wedge(\phi_\beta) = \langle \pi(p_\delta(Z,\overline{Z}))u_\beta, u_\beta\rangle_{\mathcal{F}} &= \sum_j \langle \pi(v_j(Z))\pi(\overline{v}_j(\overline{Z}))u_\beta, u_\beta\rangle_{\mathcal{F}} \\ &= \sum_j \langle \pi(\overline{v}_j(\overline{Z}))u_\beta, \pi(\overline{v}_j(-\overline{Z}))u_\beta\rangle_{\mathcal{F}} \\ &= (-1)^{|\delta|}\sum_j \|\pi(\overline{v}_j(\overline{Z}))u_\beta)\|_{\mathcal{F}}^2.\end{aligned}$$

Here we have used the fact that $\pi(-\overline{Z}_j)$ is the adjoint operator for $\pi(Z_j)$ on $\mathcal{F}$. This is equation (4.13) in [2] and follows easily from the formulas

$$\pi(Z_j)f(z) = -z_j f(z), \quad \pi(\overline{Z}_j)f(z) = 2\frac{\partial f}{\partial z_j}(z) \quad \text{for } f \in \mathcal{F}. \tag{3.2}$$

Thus, in equation (3.1), $p_\alpha(Z,\overline{Z})^\wedge(\phi_\beta)$ is a real number with sign $(-1)^{|\alpha|}$ and each of the $p_\delta(Z,\overline{Z})^\wedge(\phi_\beta)$'s are real numbers with sign $(-1)^{|\delta|}$. As the $c_{\alpha,\delta}$'s are non-negative real numbers, we conclude that $\widehat{L}_{p_\alpha}(\phi_\beta)$ is a real number with sign $(-1)^{|\alpha|}$. ■

The Gelfand pair $(U(n), H_n)$ plays a special role in our proof of Theorem 4.1. The $U(n)$-spherical functions on H_n are well known and have been computed independently by many authors. We refer the reader to [2] for one treatment and additional references. The $U(n)$-spherical functions of type 1 associated with $\mathcal{P}_r(V)$ $(r = 0, 1, \ldots)$ can be written as

$$\phi_{\lambda,r}(z,t) = e^{i\lambda t} L_r^{(n-1)}\left(\frac{|\lambda|}{2}|z|^2\right) e^{-|\lambda||z|^2/4}, \tag{3.3}$$

where $L_r^{(n-1)}$ is the Laguerre polynomial of order $n-1$ and degree r normalized to have value 1 at $z = 0$. The $U(n)$-spherical functions of type 2 are $\eta_0(z,t) \equiv 1$ and

$$\eta_w(z,t) = \frac{2^{n-1}(n-1)!}{(|w||z|)^{n-1}} J_{n-1}(|w||z|), \tag{3.4}$$

for $w \neq 0$. Here J_{n-1} is the Bessel function of order $n-1$ of the first kind.

We have $L_{\gamma_0} = \frac{1}{2}\sum_j (Z_j\overline{Z}_j + \overline{Z}_j Z_j)$, and using equations (3.2) one computes that $\pi(L_{\gamma_0}) = -2E - n$, where $E = \sum_j z_j \frac{\partial}{\partial z_j}$ is the degree operator. Thus, $\pi(L_{\gamma_0})|_{\mathcal{P}_r(V)}$ is the scalar operator $-(2r+n)$. From Lemmas 3.3 and 3.4 we obtain the following result, which is, in any case, very well known.

LEMMA 3.6. *The eigenvalues for L_{γ_0} on the $U(n)$-spherical functions of type 1 are $\widehat{L}_{\gamma_0}(\phi_{\lambda,r}) = -|\lambda|(2r+n)$.*

The eigenvalues $\widehat{L}_p(\phi_{\lambda,r})$ for operators L_p arising from polynomials $p \in \mathbb{C}[V_\mathbb{R}]^{U(n)}$ $(= \mathbb{C}[\gamma_0])$ of degree greater than 2 are more difficult to compute. We provide the following estimate for $\widehat{L}_{p_m}(\phi_r)$. As explained in Section 2.5, $p_m = \gamma_0^m/(2^m m!)$ is the $U(n)$-invariant polynomial obtained from $\mathcal{P}_m(V)$ via equation (2.2).

LEMMA 3.7. $|\widehat{L}_{p_m}(\phi_r)| \leq \binom{n+r+m-1}{m}$.

Proof. Using Lemma 3.3, we have $\widehat{L}_{\gamma_0^m}(\phi_r) = \langle \pi(L_{\gamma_0^m})u_r, u_r\rangle_{\mathcal{F}}$, where u_r is any unit vector in $\mathcal{P}_r(V)$. We take $u_r(z) := z_1^r/(2^r r!)$. One has

$$L_{\gamma_0^m} = \sum_{|\mathbf{a}|=m} \frac{m!}{\mathbf{a}!}\,\mathrm{Sym}(Z^{\mathbf{a}}\overline{Z}^{\mathbf{a}}).$$

Applying π and using equations (3.2) we obtain

$$\begin{aligned}\widehat{L}_{\gamma_0^m}(\phi_r) &= \sum_{|\mathbf{a}|=m} \frac{m!}{\mathbf{a}!}\left\langle \mathrm{Sym}\left((-z)^{\mathbf{a}}\left(2\frac{\partial}{\partial z}\right)^{\mathbf{a}}\right)u_r, u_r\right\rangle_{\mathcal{F}} \\ &= \sum_{|\mathbf{a}|=m} \frac{(-2)^m m!}{\mathbf{a}!}\left\langle \mathrm{Sym}\left(z^{\mathbf{a}}\left(\frac{\partial}{\partial z}\right)^{\mathbf{a}}\right)u_r, u_r\right\rangle_{\mathcal{F}}.\end{aligned}$$

One sees that

$$\begin{aligned}\left\langle \mathrm{Sym}\left(z^{\mathbf{a}}\left(\frac{\partial}{\partial z}\right)^{\mathbf{a}}\right)u_r, u_r\right\rangle_{\mathcal{F}} &\le \left\langle \left(\frac{\partial}{\partial z}\right)^{\mathbf{a}} z^{\mathbf{a}}u_r, u_r\right\rangle_{\mathcal{F}} \\ &= \frac{(a_1+r)!}{r!}a_2!\ldots a_n!,\end{aligned}$$

where the quantity on the left is a positive real number. We obtain

$$\begin{aligned}|\widehat{L}_{\gamma_0^m}(\phi_r)| &\le \sum_{|\mathbf{a}|=m} \frac{2^m m!}{\mathbf{a}!}\frac{(a_1+r)!}{r!}a_2!\ldots a_n! \\ &= m!2^m \sum_{|\mathbf{a}|=m}\binom{a_1+r}{r} \\ &= m!2^m \sum_{a_1=0}^{m}\binom{a_1+r}{r}\binom{m-a_1+n-2}{n-2},\end{aligned}$$

where the second binomial coefficient in the last expression counts the number of terms $(a_2,\ldots,a_n)$ with $a_2+\ldots+a_n = m-a_1$.

We have a sum of the general form $\sum_{a_1=0}^{m} A_{a_1}B_{m-a_1}$, which is the coefficient of the mth term in the product of the following two series:

$$\begin{aligned}\sum_{j=0}^{\infty} A_j x^j &= \sum_{j=0}^{\infty}\binom{r+j}{j}x^j = \left(\frac{1}{1-x}\right)^{r+1}, \\ \sum_{j=0}^{\infty} B_j x^j &= \sum_{j=0}^{\infty}\binom{j+n-2}{j}x^j = \left(\frac{1}{1-x}\right)^{n-1}.\end{aligned}$$

This product is

$$\left(\frac{1}{1-x}\right)^{n+r} = \sum_{j=0}^{\infty}\binom{n+r+j-1}{j}x^j$$

and the mth term has coefficient $\binom{n+r+m-1}{m}$. Thus we have shown that

$$|\widehat{L}_{\gamma_0^m}(\phi_r)| \le m!2^m \binom{n+r+m-1}{m}.$$

This completes the proof, since $p_m = \gamma_0^m/(m!2^m)$. ∎

The following lemma shows that the eigenvalues $\widehat{L}_{p_\alpha}(\phi_{\lambda,\beta})$ for a Gelfand pair (K, H_n) are controlled by eigenvalues for the pair $(U(n), H_n)$.

LEMMA 3.8. *For $\alpha, \beta \in \Lambda$ one has*

$$|\widehat{L}_{p_\alpha}(\phi_\beta)| \le |\widehat{L}_{p_{|\alpha|}}(\phi_{|\beta|})|.$$

Here the eigenvalue on the right hand side of this inequality is for the pair $(U(n), H_n)$.

Proof. Letting $m := |\alpha|$, we have $p_m = \sum_{|\delta|=m} p_\delta$ and thus

$$\widehat{L}_{p_m}(\phi_\beta) = \sum_{|\delta|=m} \widehat{L}_{p_\delta}(\phi_\beta).$$

Lemma 3.5 shows that the $\widehat{L}_{p_\delta}(\phi_\beta)$'s are real numbers with common sign. We conclude that

$$|\widehat{L}_{p_\alpha}(\phi_\beta)| \le |\widehat{L}_{p_m}(\phi_\beta)|.$$

Since $P_\beta \subset \mathcal{P}_{|\beta|}(V)$, Lemma 3.3 shows that both $\widehat{L}_{p_m}(\phi_\beta)$ and $\widehat{L}_{p_m}(\phi_{|\beta|})$ are equal to $\langle \pi(L_{p_m})u_\beta, u_\beta\rangle_{\mathcal{F}}$, where u_β is any unit vector in P_β. Thus,

$$|\widehat{L}_{p_\alpha}(\phi_\beta)| \le |\widehat{L}_{p_m}(\phi_\beta)| = |\widehat{L}_{p_m}(\phi_{|\beta|})|. \ ∎$$

The eigenvalues $\widehat{L}_{p_\alpha}(\psi)$ for a K-spherical function ψ are related to the coefficients in a Taylor series expansion for ψ. One can find results closely related to Proposition 3.9 in [19].

PROPOSITION 3.9. *For $\psi \in \Delta(K, H_n)$, one has*

$$\psi(z,0) = \sum_{\delta\in\Lambda} \frac{\widehat{L}_{p_\delta}(\psi)}{\dim(P_\delta)} p_\delta(z),$$

where the series converges absolutely and uniformly on compact subsets in V. Thus we have the following series expansions for the K-spherical functions of types 1 and 2 respectively:

$$\phi_{\lambda,\alpha}(z,t) = e^{i\lambda t} \sum_{\delta\in\Lambda} \frac{|\lambda|^{|\delta|}\widehat{L}_{p_\delta}(\phi_\delta)}{\dim(P_\delta)} p_\delta(z),$$

$$\eta_w(z,t) = \sum_{\delta\in\Lambda} \frac{(-1)^{|\delta|} p_\delta(w)}{\dim(P_\delta)} p_\delta(z).$$

Here, convergence is absolute and uniform on compact subsets in H_n.

P r o o f. The expansions for $\phi_{\lambda,\alpha}(z,t)$ and $\eta_w(z,t)$ follow immediately from that for $\psi(z,0)$ together with Lemmas 3.4 and 3.2. It is a general fact that the spherical functions for a Gelfand pair (G,K) are real analytic (cf. Proposition 1.5.15 in [9]). For pairs of the form (K,H_n), one can see this directly from the functional forms of the two types of K-spherical functions. Write the Taylor series expansion of $\psi(z,0)$ centered at $z=0$ as $\psi(z,0)=\sum_{m=0}^{\infty}h_m(z)$, where $h_m(z)$ is a homogeneous polynomial of degree m on $V_{\mathbb{R}}$ (i.e. in the variables $(z,\overline{z})$). Since ψ is K-invariant, one sees by K-averaging this expression that each h_m is K-invariant. As $\{p_\delta \mid \delta\in\Lambda\}$ is a basis for $\mathbb{C}[V_{\mathbb{R}}]^K$, we can rewrite the Taylor series as

$$\psi(z,0)=\sum_{\delta\in\Lambda}c_\delta p_\delta(z)$$

for some coefficients c_δ. Note that since Taylor series converge absolutely, it is not necessary to specify an ordering on the set Λ of indices for this sum. We use Lemma 3.1 and perform term-wise differentiation of this Taylor series to obtain

$$\widehat{L}_{p_\alpha}(\psi)=\partial_{p_\alpha}(\psi)(0,0)=\sum_{\delta\in\Lambda}c_\delta\partial_{p_\alpha}(p_\delta)(0).$$

Let $\{v_1,\dots,v_{\dim(P_\alpha)}\}$ be an orthonormal basis for P_α and $\{u_1,\dots,u_{\dim(P_\delta)}\}$ be an orthonormal basis for P_δ. Thus $p_\alpha=\sum v_j(z)\overline{v}_j(\overline{z})$, $p_\delta=\sum u_i(z)\overline{u}_i(\overline{z})$ and

$$\begin{aligned}\partial_{p_\alpha}(p_\delta)(0)&=\sum_{i,j}v_j\left(2\frac{\partial}{\partial\overline{z}}\right)\overline{u}_i(\overline{z})\overline{v}_j\left(2\frac{\partial}{\partial z}\right)u_i(z)\bigg|_{z=0}\\&=\sum_{i,j}|\langle v_j,u_i\rangle_{\mathcal{F}}|^2=\begin{cases}0 & \text{if } \delta\neq\alpha,\\ \dim(P_\alpha) & \text{if } \delta=\alpha.\end{cases}\end{aligned}$$

Hence $\widehat{L}_{p_\alpha}(\psi)=c_\alpha\dim(P_\alpha)$ and $c_\delta=\widehat{L}_{p_\delta}(\psi)/\dim(P_\delta)$. Since this is a Taylor series, the convergence is absolute and uniform on compact sets. ∎

It is well known that a spherical function is completely determined by its spectrum of eigenvalues. Proposition 3.9 shows explicitly how $\psi\in\Delta(K,H_n)$ is determined by $\{\widehat{D}(\psi)\mid D\in\mathbb{D}_K(H_n)\}$. Since $L_{\gamma_1},\dots,L_{\gamma_d},T$ generate $\mathbb{D}_K(H_n)$, it follows that ψ is in fact determined by $(\widehat{L}_{\gamma_1}(\psi),\dots,\widehat{L}_{\gamma_d}(\psi),$ $\widehat{T}(\psi))$. Indeed, equation (2.4) shows that eigenvalues of the form $\widehat{L}_{\gamma^{\mathbf{a}}}(\psi)$ are determined by the eigenvalues of the sort $(L_\gamma^{\mathbf{b}})^\wedge(\psi)=\widehat{L}_{\gamma_1}(\psi)^{b_1}\dots\widehat{L}_{\gamma_d}(\psi)^{b_d}$ together with $\widehat{T}(\psi)$. Since each p_δ is a linear combination of $\gamma^{\mathbf{a}}$'s, we obtain the following corollary from Proposition 3.9.

COROLLARY 3.10. *A K-spherical function $\psi\in\Delta(K,H_n)$ is completely determined by the $d+1$ eigenvalues*

$$(\widehat{L}_{\gamma_1}(\psi),\dots,\widehat{L}_{\gamma_d}(\psi),\widehat{T}(\psi)).$$

4. The topology on $\Delta(K,H_n)$. Given a Gelfand pair (K,H_n), one can consider three topological spaces:

- The set $\Delta(K,H_n)$ of bounded K-spherical functions on H_n endowed with the compact-open topology. That is, we give $\Delta(K,H_n)$ the topology of uniform convergence on compact sets.
- The *Gelfand space* (or *spectrum*) $\Delta(L^1_K(H_n))$ of the commutative Banach algebra $L^1_K(H_n)$. This is the set of continuous non-zero algebra homomorphisms from $L^1_K(H_n)$ to $\mathbb{C}$ regarded as a subspace of $L^1_K(H_n)^*$ with the weak*-topology.
- The set $\widehat{G}_K$ of K-spherical representations of $G := K \ltimes H_n$. This is the set of irreducible unitary represention of G that possess cyclic K-fixed vectors. We give $\widehat{G}_K$ the Fell topology.

Since bounded K-spherical functions are necessarily positive definite, we have a natural bijection between $\widehat{G}_K$ and $\Delta(K,H_n)$. Moreover, integration produces a bijection between $\Delta(K,H_n)$ and $\Delta(L^1_K(H_n))$. That is, $\psi \in \Delta(K,H_n)$ yields a continuous algebra homomorphism $L^1_K(H_n) \to \mathbb{C}$ via

$$f \mapsto \int\limits_{H_n} f(z,t)\psi(z,t)\,dz\,dt.$$

It is standard that these set bijections are homeomorphisms when $\Delta(K,H_n)$, $\Delta(L^1_K(H_n))$ and $\widehat{G}_K$ are endowed with the natural topologies described above. Moreover, it is known that these spaces are locally compact, second countable and metrizable. The study of these topologies amounts to the following analytic question:

Under what conditions does a given sequence $(\psi_N)_{N=1}^\infty$ of K-spherical functions converge uniformly on compact sets to a given K-spherical function ψ?

The following theorem answers this question and is our main result.

THEOREM 4.1. *Let $(\psi_N)_{N=1}^\infty$ be a sequence of K-spherical functions and $\psi \in \Delta(K,H_n)$. Then ψ_N converges to ψ in the topology of $\Delta(K,H_n)$ (i.e. uniformly on compact sets) if and only if $\widehat{L}_{\gamma_j}(\psi_N) \to \widehat{L}_{\gamma_j}(\psi)$ for $j=1,\dots,d$, and $\widehat{T}(\psi_N) \to \widehat{T}(\psi)$.*

Proof. Suppose that $(\psi_N)_{N=1}^\infty$ converges uniformly to ψ on compact subsets of H_n. Since $\psi_N(0,t) = e^{\widehat{T}(\psi_N)t}$ and $\psi(0,t) = e^{\widehat{T}(\psi)t}$, we must have $\widehat{T}(\psi_N) \to \widehat{T}(\psi)$. The series expansions for $\psi_N(z,0)$ and $\psi(z,0)$ given in Proposition 3.9 ensure that $\widehat{L}_{p_\delta}(\psi_N) \to \widehat{L}_{p_\delta}(\psi)$ for each $\delta \in \Lambda$. Since $\{\gamma_1,\dots,\gamma_d\} \subset \{p_\delta \mid \delta \in \Lambda\}$, this shows in particular that $\widehat{L}_{\gamma_j}(\psi_N) \to \widehat{L}_{\gamma_j}(\psi)$ for $j=1,\dots,d$.

Conversely, suppose that $\widehat{L}_{\gamma_j}(\psi_N) \to \widehat{L}_{\gamma_j}(\psi)$ for $j = 1, \dots, d$ and $\widehat{T}(\psi_N) \to \widehat{T}(\psi)$. It follows that $\widehat{L}_p(\psi_N) \to \widehat{L}_p(\psi)$ for every $p \in \mathbb{C}[V_\mathbb{R}]^K$. Indeed, each $p \in \mathbb{C}[V_\mathbb{R}]^K$ is a linear combination of monomials $\gamma^{\mathbf{a}}$ in the fundamental invariants, and from equation (2.4) one obtains

$$\begin{aligned}\lim_{N\to\infty} (L_{\gamma^{\mathbf{a}}})^\wedge(\psi_N) &= \lim_{N\to\infty} \Big[\widehat{L}^{\mathbf{a}}_\gamma(\psi_N) + \sum_{\|\mathbf{b}\|<\|\mathbf{a}\|} c_{\mathbf{a},\mathbf{b}} \widehat{L}^{\mathbf{b}}_\gamma(\psi_N)\widehat{T}(\psi_N)^{\|\mathbf{a}\|-\|\mathbf{b}\|}\Big] \\ &= \widehat{L}^{\mathbf{a}}_\gamma(\psi) + \sum_{\|\mathbf{b}\|<\|\mathbf{a}\|} c_{\mathbf{a},\mathbf{b}} \widehat{L}^{\mathbf{b}}_\gamma(\psi)\widehat{T}(\psi)^{\|\mathbf{a}\|-\|\mathbf{b}\|} = (L_{\gamma^{\mathbf{a}}})^\wedge(\psi).\end{aligned}$$

It suffices to consider two cases:

(1) Each ψ_N is a K-spherical function of type 1.
(2) Each ψ_N is a K-spherical function of type 2.

Indeed, if $(\psi_N)_{N=1}^\infty$ contains infinitely many terms of both types, then one reduces the situation to these cases by forming the subsequences consisting of the terms of each type.

We begin with the easier case (2) and write $\psi_N = \eta_{w_N}$. Since $\widehat{T}(\psi) = \lim \widehat{T}(\psi_N) = 0$, ψ must be a K-spherical function of type 2, $\psi = \eta_w$ say. Lemma 3.2 shows that $\widehat{L}_{\gamma_j}(\eta_{w_N}) = (-1)^{|\delta_j|}\gamma_j(w_N)$ and $\widehat{L}_{\gamma_j}(\eta_w) = (-1)^{|\delta_j|}\gamma_j(w)$. We conclude that $\gamma(w_N) \to \gamma(w)$ in the value space Γ_K^+ and hence $Kw_N \to Kw$ in V/K. Equation (2.8) shows that η_{w_N} and η_w can be obtained via integration over the K-orbits Kw_N and Kw. It follows easily that η_{w_N} converges to η_w uniformly on compact sets.

Next consider case (1). Let $\psi_N = \phi_{\lambda_N,\alpha_N}$ and let S be a given compact subset of H_n. In case (1), ψ can be a K-spherical function of either type. Suppose first that ψ is of type 2 and write $\psi = \eta_w$. (Later we will also treat the situation where ψ is a K-spherical function of type 1.)

Choose a constant c_1 large enough so that $\gamma_0(z) = |z|^2 \le c_1$ for all $(z,t) \in S$. We write r_N for $|\alpha_N|$ and consider the tail of the series expansion for $\phi_{\lambda_N,\alpha_N}$ consisting of the terms with indices δ satisfying $|\delta| \ge M$. For $M \ge 2$ and all $(z,t) \in S$, we obtain the following bound:

$$\begin{aligned}\sum_{\substack{\delta\in\Lambda\\|\delta|\ge M}} \frac{|\widehat{L}_{p_\delta}(\phi_{\alpha_N})|}{\dim(P_\delta)} p_\delta(z)|\lambda_N|^{|\delta|} &\le \sum_{m=M}^\infty |\widehat{L}_{p_m}(\phi_{r_N})| \sum_{|\delta|=m} p_\delta(z)|\lambda_N|^m \\ &\le \sum_{m=M}^\infty \binom{m+n+r_N-1}{m} \frac{1}{2^m m!}\gamma_0^m(z)|\lambda_N|^m \\ &\le \frac{1}{2^M M!} \sum_{m=M}^\infty \binom{m+n+r_N-1}{m} |c_1\lambda_N|^m\end{aligned}$$

$$\le \frac{1}{2^M M!} \sum_{m=2}^{\infty} \binom{m+n+r_N-1}{m} |c_1\lambda_N|^m$$
$$= \frac{R_2(|c_1\lambda_N|)}{2^M M!} \quad \text{say.}$$

We have used Lemmas 3.7 and 3.8 here, together with the identity $\sum_{|\delta|=m} p_\delta = \gamma_0^m/(2^m m!)$.

We now regard $R_2(x)$ as a remainder term for the Taylor series for $(1-x)^{-(n+r_N)}$. Computing the integral form for this remainder term yields

$$R_2(x) = \int_0^x \frac{(n+r_N)(n+r_N+1)(x-t)}{(1+t)^{n+r_N+1}}\, dt \le (n+r_N)(n+r_N+1)x^2,$$

provided $0 < x < 1$. Since $\widehat{T}(\phi_{\lambda_N,\alpha_N}) = i\lambda_N$ converges to $\widehat{T}(\eta_w) = 0$, we have $\lambda_N \to 0$. Thus for N sufficiently large, $N \ge N_0$ say, we have $0 < |c_1\lambda_N| < 1$ and hence

$$R_2(|c_1\lambda_N|) \le (n+r_N)(n+r_N+1)|c_1\lambda_N|^2.$$

Moreover, we have $\widehat{L}_{\gamma_0}(\phi_{\lambda_N,\alpha_N}) = \widehat{L}_{\gamma_0}(\phi_{\lambda_N,r_N}) = -|\lambda_N|(2r_N+n)$ by Lemma 3.6 and since $\widehat{L}_{\gamma_0}(\phi_{\lambda_N,\alpha_N})$ converges to $\widehat{L}_{\gamma_0}(\eta_w)$, we conclude that $(|\lambda_N|r_N)$ is convergent. We now see, in particular, that both (λ_N) and $(|\lambda_N|r_N)$ are bounded sequences and hence

$$(n+r_N)(n+r_N+1)|c_1\lambda_N|^2 \le c_2$$

for some constant c_2 and all N. Thus, for all $N \ge N_0$ and all $(z,t) \in S$ we have

$$\sum_{\substack{\delta\in\Lambda\\ |\delta|\ge M}} \frac{|\widehat{L}_{p_\delta}(\phi_{\lambda_N,\alpha_N})|}{\dim(P_\delta)} p_\delta(z) \le \frac{c_2}{2^M M!}.$$

This bound ensures that the series converges uniformly to η_w on compact subsets of H_n.

Finally, suppose that $\psi_N = \phi_{\lambda_N,\alpha_N}$ and $\psi_N \to \psi$, where $\psi = \phi_{\lambda,\alpha}$ is a K-spherical function of type 1. Since $\widehat{T}(\phi_{\lambda_N,\alpha_N}) = i\lambda_N$ converges to $\widehat{T}(\phi_{\lambda,\alpha}) = i\lambda$, we have $\lambda_N \to \lambda$. Since $\widehat{L}_{\gamma_0}(\phi_{\lambda_N,\alpha_N}) = |\lambda_N|\widehat{L}_{\gamma_0}(\phi_{|\alpha_N|}) = |\lambda_N|(2|\alpha_N|+n)$ converges to $\widehat{L}_{\gamma_0}(\phi_{\lambda,\alpha}) = |\lambda|(2|\alpha|+n)$, also $r_N := |\alpha_N|$ must converge to $|\alpha|$. Thus, both (λ_N) and (r_N) are bounded sequences and we choose constants c_1, c_2 with

$$|\lambda| \le c_1, \quad 0 \le r_N \le c_2$$

for all N. Choose constants c_3 and c_4 with $\gamma_0(z) = |z|^2 \le c_3$ for all $(z,t) \in S$ and

$$\frac{\gamma_0(z)^m|\lambda_N|^m}{m!} \le \frac{(c_3c_1)^m}{m!} \le \frac{c_4}{2^m}$$

for all m, N and all $(z,t) \in S$. As before, we obtain, for $(z,t) \in S$,

$$\begin{aligned}
&\Bigg| e^{i\lambda_N t} \sum_{\substack{\delta\in\Lambda\\|\delta|\geq M}} \frac{\widehat{L}_{p_\delta}(\phi_{\lambda_N,\alpha_N})}{\dim(P_\delta)} p_\delta(z)\Bigg| \\
&\qquad\leq \sum_{m=M}^{\infty} \binom{m+n+r_N-1}{m} \frac{1}{2^m m!} \gamma_0^m(z)|\lambda_N|^m \\
&\qquad\leq \frac{c_4}{2^M} \sum_{m=M}^{\infty} \binom{m+n+r_N-1}{m} \left(\frac{1}{2}\right)^m \\
&\qquad\leq \frac{c_4}{2^M} \sum_{m=0}^{\infty} \binom{m+n+r_N-1}{m} \left(\frac{1}{2}\right)^m \\
&\qquad= \frac{c_4}{2^M} 2^{n+r_N} \leq \frac{c_5}{2^M},
\end{aligned}$$

where $c_5 := c_4 2^{n+c_2}$. From this, we conclude that $\phi_{\lambda_N,\alpha_N} \to \phi_{\lambda,\alpha}$ uniformly on S by reasoning as in the case where $\phi_N = \phi_{\lambda_N,\alpha_N}$ and $\psi = \eta_w$. ∎

For the Gelfand pair $(U(n), H_n)$, one has the single fundamental invariant $\gamma_0(z) = |z|^2$. Theorem 4.1, together with Lemma 3.6, shows that a sequence $(\phi_{\lambda_N,r_N})_{N=1}^{\infty}$ of $U(n)$-spherical functions of type 1 converges in $\Delta(U(n), H_n)$ to a $U(n)$-spherical function η_w of type 2 if and only if

$$\lambda_N \to 0 \quad \text{and} \quad |\lambda_N|(2r_N + n) \to |w|^2.$$

These spherical functions are given explicitly by equations (3.3) and (3.4). From this we obtain, for example, the following corollary to Theorem 4.1.

COROLLARY 4.2. *The sequence*

$$\frac{x^{n-1}}{2^{n-1}(n-1)!} L_m^{(n-1)}\left(\frac{x^2}{2(2m+n)}\right) e^{-\frac{x^2}{4(2m+n)}}$$

converges to $J_{n-1}(x)$ *uniformly on compact subsets of* $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$.

The reader will find classical viewpoints on Corollary 4.2 discussed on page 92 in [7].

One can obtain a clearer understanding of the topology on $\Delta(K, H_n)$ by recasting Theorem 4.1 as follows. Consider the map $E : \Delta(K, H_n) \to \mathbb{C}^{d+1}$ defined by

$$E(\psi) := ((-1)^{|\delta_1|}\widehat{L}_{\gamma_1}(\psi), \ldots, (-1)^{|\delta_d|}\widehat{L}_{\gamma_d}(\psi), -i\widehat{T}(\psi)).$$

Since $\widehat{L}_{\gamma_j}(\psi)$ is a real number with sign $(-1)^{|\delta_j|}$ and $\widehat{T}(\psi)$ is pure imaginary,

E takes values in $(\mathbb{R}^+)^d \times \mathbb{R}$. We denote the image of E by

$$\mathfrak{F}(K, H_n) := E(\Delta(K, H_n)) \subset (\mathbb{R}^+)^d \times \mathbb{R}$$

and give this the subspace topology from $(\mathbb{R}^+)^d \times \mathbb{R}$. We call $\mathfrak{F}(K, H_n)$ the *Heisenberg fan for* (K, H_n). In view of Corollary 3.10, the map E is a bijection between the sets $\Delta(K, H_n)$ and $\mathfrak{F}(K, H_n)$. Since both $\Delta(K, H_n)$ and $\mathfrak{F}(K, H_n)$ are metrizable spaces, Theorem 4.1 shows that both E and E^{-1} are continuous maps. Thus we have proved the following theorem.

THEOREM 4.3. *The map E is a homeomorphism between $\Delta(K, H_n)$ and the Heisenberg fan $\mathfrak{F}(K, H_n)$.*

Lemma 3.2 shows that the image under E of the K-spherical functions of type 2 is precisely the value space Γ_K^+ in $(\mathbb{R}^+)^d$:

$$E(\{\eta_w \mid w \in V\}) = \Gamma_K^+ \times \{0\}.$$

Lemma 3.4 shows that for fixed $\alpha \in \Lambda$ we have

$$E(\{\phi_{\lambda,\alpha} \mid \lambda \in \mathbb{R}^\times\}) = \{(\kappa_{\alpha,1}|\lambda|^{|\delta_1|}, \ldots, \kappa_{\alpha,d}|\lambda|^{|\delta_d|}, \lambda) \mid \lambda \in \mathbb{R}^\times\},$$

where $\kappa_{\alpha,j} := |\widehat{L}_{\gamma_j}(\phi_\alpha)|$. This is the disjoint union of two algebraic curves,

$$E(\{\phi_{\lambda,\alpha} \mid \lambda \in \mathbb{R}^\times\}) = \mathcal{L}_\alpha^+ \amalg \mathcal{L}_\alpha^-,$$

where

$$\mathcal{L}_\alpha^\pm = \{(\kappa_{\alpha,1}\lambda^{|\delta_1|}, \ldots, \kappa_{\alpha,d}\lambda^{|\delta_d|}, \pm\lambda) \mid \lambda > 0\}.$$

Thus we see that $\mathfrak{F}(K, H_n)$ is the algebraic set in $(\mathbb{R}^+)^d \times \mathbb{R}$ consisting of the value space Γ_K^+ together with a countable family of curves:

$$\mathfrak{F}(K, H_n) = (\Gamma_K^+ \times \{0\}) \amalg \Big(\coprod_{\alpha \in \Lambda} \mathcal{L}_\alpha^+\Big) \amalg \Big(\coprod_{\alpha \in \Lambda} \mathcal{L}_\alpha^-\Big).$$

$\mathfrak{F}(U(n), H_n)$ is depicted in Figure 1.

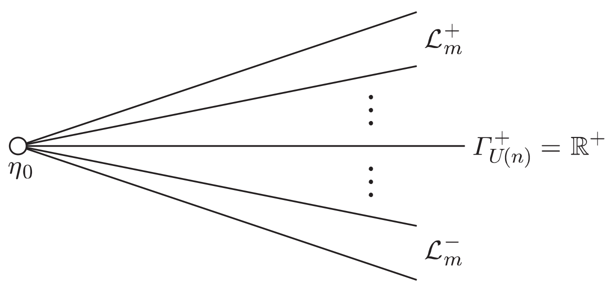

Fig. 1. The Heisenberg fan for $(U(n), H_n)$

One has

$$\mathcal{F}(U(n), H_n) = (\mathbb{R}^+ \times \{0\}) \amalg \Big(\coprod_{m=0}^{\infty} \mathcal{L}_m^+\Big) \amalg \Big(\coprod_{m=0}^{\infty} \mathcal{L}_m^-\Big),$$

where $\mathcal{L}_m^\pm$ are the lines

$$\mathcal{L}_m^\pm = \{((2m+n)\lambda, \pm\lambda) \mid \lambda > 0\}$$

of slope $\pm 1/(2m+n)$ in $\mathbb{R}^+ \times \mathbb{R}$. The Heisenberg fan for $(U(n), H_n)$ is described in [7] and in [17], although the latter paper does not explicitly discuss the topology on $\Delta(U(n), H_n)$. A proof that $E : \Delta(U(n), H_n) \to \mathfrak{F}(U(n), H_n)$ is a homeomorphism is implicit in Bougerol's paper [4], which includes a description of a system of open neighborhoods for $\eta_0 \equiv 1$ in $\Delta(U(n), H_n)$. Thus Theorems 4.1 and 4.3 are known results for the case $K = U(n)$.

The following result ensures that each K-spherical function η_w of type 2 is the limit in $\Delta(K, H_n)$ of some sequence $(\phi_{\lambda_N, \alpha_N})_{N=1}^\infty$ of K-spherical functions of type 1. Proposition 4.4 is clear for the case $K = U(n)$ but is rather surprising when $K \neq U(n)$. In these cases, Γ_K^+ has dimension $d > 1$ but the $\mathcal{L}_\alpha^\pm$'s are one-dimensional. The result reflects the manner in which the countable family of $\mathcal{L}_\alpha^\pm$'s sit over the value space in $\mathfrak{F}(K, H_n)$.

PROPOSITION 4.4. *The K-spherical functions of type* 1 *are dense in the space $\Delta(K, H_n)$.*

Our proof for Proposition 4.4 requires the spherical transform and will be given in Section 5. Proposition 4.5 complements Proposition 4.4. If a sequence in $\Delta(K, H_n)$ converges uniformly on compact subsets of H_n then the limit is necessarily a bounded K-spherical function.

PROPOSITION 4.5. *$\Delta(K, H_n)$ is a complete metric space. That is, if $(\psi_N)_{N=1}^\infty$ is a sequence of bounded K-spherical functions that converges uniformly to ψ on compact subsets in H_n then ψ is a bounded K-spherical function.*

Proof. It is clear that ψ is continuous, K-invariant and bounded and that $\psi(0,0) = 1$. Moreover, if $f, g \in L_K^1(H_n)$ have compact support then

$$\begin{aligned}
\int_{H_n} \psi(z,t)(f*g)(z,t)\,dz\,dt &= \lim_{N\to\infty} \int_{H_n} \psi_N(z,t)(f*g)(z,t)\,dz\,dt \\
&= \lim_{N\to\infty} \int_{H_n} \psi_N(z,t) f(z,t)\,dz\,dt \int_{H_n} \psi_N(z,t) g(z,t)\,dz\,dt \\
&= \int_{H_n} \psi(z,t) f(z,t)\,dz\,dt \int_{H_n} \psi(z,t) g(z,t)\,dz\,dt.
\end{aligned}$$

Thus $f \mapsto \int_{H_n} \psi(z,t) f(z,t)\, dz\, dt$ defines a continuous non-zero algebra homomorphism $L^1_K(H_n) \to \mathbb{C}$. It follows that $\psi \in \Delta(K, H_n)$. ■

5. The K-spherical transform. The K-spherical transform for $f \in L^1_K(H_n)$ is the function

$$\widehat{f} : \Delta(K, H_n) \to \mathbb{C}, \qquad \widehat{f}(\psi) := \int_{H_n} f(z,t)\psi(z,t)\, dz\, dt.$$

Here "$dzdt$" denotes Haar measure for the group H_n, which is simply Euclidean measure on $V_{\mathbb{R}} \times \mathbb{R}$. One has

$$(f * g)^\wedge(\psi) = \widehat{f}(\psi)\widehat{g}(\psi) \quad \text{and} \quad (f^*)^\wedge(\psi) = \overline{\widehat{f}(\psi)} \tag{5.1}$$

for $f, g \in L^1_K(H_n)$, $\psi \in \Delta(K, H_n)$, where $f^*(z,t) := \overline{f(-z,-t)}$.

The compact-open topology on $\Delta(K, H_n)$ is the smallest topology that makes all of the maps $\{\widehat{f} \mid f \in L^1_K(H_n)\}$ continuous. Since $L^1_K(H_n)$ is a Banach $*$-algebra with respect to the involution $f \mapsto f^*$, it follows that $\widehat{f}$ belongs to the space $C_0(\Delta(K, H_n))$ of continuous functions on $\Delta(K, H_n)$ that vanish at infinity. Moreover, we have

$$\|\widehat{f}\|_\infty \leq \|f\|_1$$

for $f \in L^1_K(H_n)$. This follows immediately from the fact that for $\psi \in \Delta(K, H_n)$ one has $|\psi(z,t)| \leq \psi(0,0) = 1$, since ψ is positive definite.

Godement's Plancherel Theory for Gelfand pairs (G, K) (cf. [10], or Section 1.6 in [9]) ensures that there exists a unique positive Borel measure $d\mu$ on the space $\Delta(K, H_n)$ for which

$$\int_{H_n} |f(z,t)|^2\, dz\, dt = \int_{\Delta(K,H_n)} |\widehat{f}(\psi)|^2\, d\mu(\psi) \tag{5.2}$$

for all continuous functions $f \in L^1_K(H_n) \cap L^2_K(H_n)$. If $f \in L^1_K(H_n) \cap L^2_K(H_n)$ is continuous and $\widehat{f}$ is integrable with respect to $d\mu$ then one has the Inversion Formula

$$f(z,t) = \int_{\Delta(K,H_n)} \widehat{f}(\psi)\overline{\psi(z,t)}\, d\mu(\psi). \tag{5.3}$$

In particular, this formula holds when f is continuous, positive definite and K-invariant. Moreover, the spherical transform $f \mapsto \widehat{f}$ extends uniquely to an isomorphism between $L^2_K(H_n)$ and $L^2(\Delta(K, H_n), d\mu)$.

The following result makes the Godement–Plancherel measure on $\Delta(K, H_n)$ explicit. Given $F : \Delta(K, H_n) \to \mathbb{C}$, we write $F(w)$ and $F(\lambda, \alpha)$ in place of $F(\eta_w)$ and $F(\phi_{\lambda,\alpha})$ respectively. The reader can find another proof of Theorem 5.1 in [19]. The result for $K = U(n)$ is also discussed in [7] and [17].

THEOREM 5.1. *The Godement–Plancherel measure $d\mu$ on $\Delta(K, H_n)$ is given by*

$$\int\limits_{\Delta(K,H_n)} F(\psi)\, d\mu(\psi) = \int\limits_{\mathbb{R}^\times} \sum_{\alpha\in\Lambda} \dim(P_\alpha) F(\lambda,\alpha)|\lambda|^n \, d\lambda.$$

Proof. For $\lambda \in \mathbb{R}^\times$ let π_λ be the irreducible unitary representation of H_n on $\mathcal{F}$ defined by $\pi_\lambda(z,t) := \pi(\sqrt{|\lambda|}, \lambda t)$. One has the formula (cf. [2])

$$\phi_{\lambda,\alpha}(z,t) = \int\limits_K \langle \pi_\lambda(kz,t)u_\alpha, u_\alpha\rangle_{\mathcal{F}}\, dk,$$

where u_α is any unit vector in P_α.

Since existence and uniqueness of the Godement–Plancherel measure is guaranteed, we need only verify that equation (5.2) holds for $d\mu$ as in the statement of the theorem and all f in a dense self-adjoint subalgebra of $C(H_n) \cap L^1_K(H_n) \cap L^2_K(H_n)$. For f continuous, K-invariant and of compact suppport, we compute

$$\begin{aligned}
\langle \pi_\lambda(f * f^*)u_\alpha, u_\alpha\rangle_{\mathcal{F}} &= \int\limits_{H_n} (f*f^*)(z,t)\langle \pi_\lambda(z,t)u_\alpha, u_\alpha\rangle_{\mathcal{F}}\, dz\, dt \\
&= \int\limits_{H_n} \int\limits_K (f*f^*)(k^{-1}z,t)\langle \pi_\lambda(z,t)u_\alpha, u_\alpha\rangle_{\mathcal{F}}\, dk\, dz\, dt \\
&= \int\limits_{H_n} \int\limits_K (f*f^*)(z,t)\langle \pi_\lambda(kz,t)u_\alpha, u_\alpha\rangle_{\mathcal{F}}\, dk\, dz\, dt \\
&= \int\limits_{H_n} (f*f^*)(z,t)\phi_{\lambda,\alpha}(z,t)\, dz\, dt \\
&= (f*f^*)^\wedge(\lambda,\alpha) = |\widehat{f}(\lambda,\alpha)|^2.
\end{aligned}$$

Thus we see that $\operatorname{tr}(\pi_\lambda(f*f^*)) = \sum_{\alpha\in\Lambda} \dim(P_\alpha)|\widehat{f}(\lambda,\alpha)|^2$. The Plancherel formula for H_n (cf. pages 37–39 in [8]) now gives

$$\begin{aligned}
\int\limits_{H_n} |f(z,t)|^2\, dz\, dt &= \int\limits_{\mathbb{R}^\times} \operatorname{tr}(\pi_\lambda(f*f^*))|\lambda|^n\, d\lambda \\
&= \int\limits_{\mathbb{R}^\times} \sum_{\alpha\in\Lambda} \dim(P_\alpha)|\widehat{f}(\lambda,\alpha)|^2|\lambda|^n\, d\lambda. \ \blacksquare
\end{aligned}$$

Theorem 5.1 shows that the K-spherical functions of type 2 form a set of Godement–Plancherel measure zero in $\Delta(K, H_n)$. Our proof of Proposition 4.4, stated earlier, uses this fact together with the Inversion Formula (5.3).

Proof of Proposition 4.4. Take a point w in V, and suppose that η_w is not in the closure of $\{\phi_{\lambda,\alpha} \mid \lambda \in \mathbb{R}^\times,\ \alpha \in \Lambda\}$.

$\Delta(K, H_n)$ is metrizable, hence it is completely regular, so one can find a continuous function $J : \Delta(K, H_n) \to \mathbb{R}$ with $J(w) = 1$ and $J(\lambda, \alpha) = 0$ for all $\lambda \in \mathbb{R}^\times$, $\alpha \in \Lambda$. We can assume that J has compact support.

The second of equations (5.1) ensures that $L^1_K(H_n)$ is a *symmetric* Banach $*$-algebra. It follows that $\{\widehat{f} \mid f \in L^1_K(H_n)\}$ is dense in $(C_0(\Delta(K, H_n)), \|\cdot\|_\infty)$. (See, for example, §14 in Chapter III of [15].) Thus we can find a sequence (j_N) in $L^1_K(H_n)$ with $\widehat{j}_N \to J$ uniformly on $\Delta(K, H_n)$. We can assume that each j_N is continuous and compactly supported. Moreover, since J is real-valued, we can assume that $j_N^* = j_N$.

The proof of Proposition 3 in [13] shows that one can find an approximate identity $(a_s)_{s>0}$ in $L^1_K(H_n)$ with $\widehat{a}_s$ compactly supported in $\Delta(K, H_n)$ for all $s > 0$. For s sufficiently small, one has $\widehat{a}_s(w) > 3/4$. Moreover, for each s one sees that $(a_s * j_N)^\wedge = \widehat{a}_s \widehat{j}_N$ converges uniformly to $\widehat{a}_s J$ as $N \to \infty$. Thus we can choose s_0 sufficiently small and N_0 sufficiently large that $g := a_{s_0} * j_{N_0}$ satisfies

$$\widehat{g}(w) > \tfrac{1}{2} \quad \text{and} \quad |\widehat{g}(\lambda, \alpha)| < \tfrac{1}{4} \text{ for all } \lambda,\ \alpha.$$

Note that g is continuous, integrable, square-integrable and $g^* = g$.

Dixmier's functional calculus (cf. [6]) ensures that "sufficiently smooth functions operate on $L^1_K(H_n)$". Thus, if $\zeta : \mathbb{R} \to \mathbb{R}$ is sufficiently smooth with ζ and its derivatives integrable and $\zeta(0) = 0$, then there is a function $f := \zeta\{g\} \in L^1_K(H_n) \cap L^2_K(H_n) \cap C(H_n)$ with the property that $\widehat{f} = \zeta \circ \widehat{g}$. We choose such a ζ with $\zeta(t) = 1$ for $t > 1/2$ and $\zeta(t) = 0$ for $t < 1/4$. Then $F := \widehat{f} = (\zeta\{g\})^\wedge$ satisfies $F(w) = 1$ and $F(\lambda, \alpha) = 0$ for all λ, α.

Now Theorem 5.1 shows that $F = 0$ a.e. on $\Delta(K, H_n)$. In particular, F is integrable on $\Delta(K, H_n)$ and we can apply the Inversion Formula (5.3) to conclude that $f \equiv 0$ on H_n. This implies that $F = \widehat{f}$ is identically zero on $\Delta(K, H_n)$, which contradicts $F(w) = 1$. ■

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