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PEŁCZYŃSKI'S PROPERTY (V) *ON SPACES OF VECTOR-VALUED FUNCTIONS*

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1. Introduction. Let E and F be Banach spaces and suppose $T : E \to F$ is a bounded linear operator. The operator T is said to be *unconditionally converging* if T does not fix any copy of c_0. A Banach space E is said to have *Pełczyński's property* (V) if every unconditionally converging operator with domain E is weakly compact. In a fundamental paper [15], Pełczyński showed that if Ω is a compact Hausdorff space then the space $C(\Omega)$, of all continuous scalar-valued functions on Ω, has property (V); and he asked ([15], Remark 1, p. 645; see also [9], p. 183) whether for a Banach space E the abstract continuous function space $C(\Omega, E)$ has property (V) whenever E does. This question has been considered by several authors. Perhaps the sharpest result in this direction is in the paper of Cembranos, Kalton, E. Saab and P. Saab [5], where they proved that if E has property (U) and contains no copy of ℓ^1 then $C(\Omega, E)$ has property (V). Property (U) and noncontainment of ℓ^1 was recently proved to be equivalent to hereditarily (V) by Rosenthal in [18]. There are, however, many known examples of Banach spaces that have property (V) but fail to satisfy the above conditions. For instance, Kisliakov in [13] (also Delbaen [7] independently) showed that the disk algebra has property (V); Bourgain did the same for ball algebras and polydisk algebras in [4] and H^∞ in [3]; recently Pfitzner proved that C^*-algebras have property (V) (see [16]). For more detailed discussion and examples of spaces with property (V), we refer to [11] and [20].

In this note, we obtain a positive answer to the above question for the separable case; namely we prove that if E is a separable Banach space then $C(\Omega, E)$ has property (V) if and only if E does. We also present some applications of the main theorem to Banach spaces of compact operators as well as to Bochner function spaces.

Our Banach space notation and terminology are standard, as may be found in the books [8] and [9].

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2. Definitions and some preliminary results

Definition 1. Let E be a Banach space. A series $\sum_{n=1}^{\infty} x_n$ in E is said to be *weakly unconditionally Cauchy* (*WUC*) if for every x^* in E^*, the series $\sum_{n=1}^{\infty} |x^*(x_n)|$ is convergent.

There are many criteria for a series to be a WUC series (see for instance [8]).

The following proposition was proved by Pełczyński in [15].

Proposition 1. *For a Banach space E, the following assertions are equivalent*:

(i) *A subset $H \subset E^*$ is relatively weakly compact whenever*

$$\lim_{n\to\infty} \sup_{x^*\in H} |x^*(x_n)| = 0$$

for every WUC series $\sum_{n=1}^{\infty} x_n$ in E;

(ii) *For any Banach space F, every bounded operator $T : E \to F$ that is unconditionally converging is weakly compact.*

Definition 2. A subset $H \subset E^*$ is called a (V)-*subset* if

$$\lim_{n\to\infty} \sup_{x^*\in H} |x^*(x_n)| = 0$$

for every WUC series $\sum_{n=1}^{\infty} x_n$ in E.

So a Banach space E has property (V) if (and only if) every (V)-subset of E^* is relatively weakly compact. This motivates us to study (V)-subsets of the dual of $C(\Omega, E)$ for a given Banach space E and a compact Hausdorff space Ω.

Recall that the space $C(\Omega, E)^*$ is isometrically isomorphic to the Banach space $M(\Omega, E^*)$ of all weak*-regular E^*-valued measures of bounded variation defined on the σ-field Σ of Borel subsets of Ω and equipped with the norm $\|m\| = |m|(\Omega)$, where $|m|$ is the variation of m. In this section we study different structures of subsets of $M(\Omega, E^*)$.

Let us begin by recalling some classical facts: Fix a probability measure λ on Σ and let $m \in M(\Omega, E^*)$ with $|m| \le \lambda$ and ϱ be a lifting of $L^\infty(\lambda)$ (see [10] and [12]). For $x \in E$, the scalar measure $x \circ m$ has density $d(x\circ m)/d\lambda \in L^\infty(\lambda)$. We define $\varrho(m)(\omega)(x) = \varrho(d(x \circ m)/d\lambda)(\omega)$. It is well known that

$$x(m(A)) = \int_A \langle \varrho(m)(\omega), x\rangle \, d\lambda(\omega) \quad \text{and} \quad |m|(A) = \int_A \|\varrho(m)(\omega)\| \, d\lambda(\omega)$$

for every measurable subset A of Ω. Note also that $\omega \mapsto \varrho(m)(\omega)$ $(\Omega \to E^*)$ is weak*-scalarly measurable.

The following proposition can be deduced from [2] but we will present a direct proof for the sake of completeness.

PROPOSITION 2. *Let H be a bounded subset of $M(\Omega, E^*)$. If H is a* (V)*-subset then $V(H) = \{|m| : m \in H\}$ is relatively weakly compact in $M(\Omega)$.*

Proof. Assume that $V(H)$ is not relatively weakly compact. Since the space $C(\Omega)$ has property (V), there exists a WUC series $\sum_{n=1}^{\infty} e_n$ in $C(\Omega)$ with $\sup_{n\in\mathbb{N}} \|e_n\| \le 1$, a sequence $(m_n)_n$ in H and $\varepsilon > 0$ so that $\langle e_n, |m_n|\rangle \ge \varepsilon$ for each $n \in \mathbb{N}$. Let $\lambda = \sum_{n=1}^{\infty} 2^{-n}|m_n|$. Since $|m_n| \le 2^n\lambda$, for a lifting ϱ of $L^\infty(\lambda)$ we can define $\varrho(m_n) : \Omega \to E^*$.

Now since $C(\Omega, E)$ is norming for $M(\Omega, E^*)$, there exists $\theta_n \in C(\Omega, E)$ with $\|\theta_n\| = 1$ and such that $\langle \theta_n, m_n\rangle \ge \|m_n\| - \varepsilon/2$ for each $n \in \mathbb{N}$, and that is equivalent to

$$\int \theta_n(\omega)\, dm_n(\omega) \ge \int \|\varrho(m_n)(\omega)\|\, d\lambda(\omega) - \frac{\varepsilon}{2}$$

or

$$\int \langle \theta_n(\omega), \varrho(m_n)(\omega)\rangle\, d\lambda(\omega) \ge \int \|\varrho(m_n)(\omega)\|\, d\lambda(\omega) - \frac{\varepsilon}{2}.$$

Notice also that since $\|\theta_n(\omega)\| \le 1$, $\langle \theta_n(\omega), \varrho(m_n)(\omega)\rangle \le \|\varrho(m_n)(\omega)\|$ and we get

$$\begin{aligned}
&\Big|\int e_n(\omega)\theta_n(\omega)\, dm_n(\omega) - \int e_n(\omega)\, d|m_n|(\omega)\Big| \\
&\qquad = \Big|\int e_n(\omega)(\langle \theta_n(\omega), \varrho(m_n)(\omega)\rangle - \|\varrho(m_n)(\omega)\|)\, d\lambda(\omega)\Big| \\
&\qquad \le \|e_n\| \int \Big|\|\varrho(m_n)(\omega)\| - \langle \theta_n(\omega), \varrho(m_n)(\omega)\rangle\Big|\, d\lambda(\omega) \\
&\qquad \le \int \|\varrho(m_n)(\omega)\|\, d\lambda(\omega) - \int \langle \theta_n(\omega), \varrho(m_n)(\omega)\rangle\, d\lambda(\omega) \\
&\qquad \le \int \|\varrho(m_n)(\omega)\|\, d\lambda(\omega) - \Big(\int \|\varrho(m_n)(\omega)\|\, d\lambda(\omega) - \frac{\varepsilon}{2}\Big) = \frac{\varepsilon}{2}.
\end{aligned}$$

So for each $n \in \mathbb{N}$,

$$\Big|\int e_n(\omega)\theta_n(\omega)\, dm_n\Big| > \frac{\varepsilon}{2}.$$

Fix $\psi_n = e_n(\cdot)\theta_n(\cdot)$ for each $n \in \mathbb{N}$; the function ψ_n belongs to $C(\Omega, E)$ and we claim that $\sum_{n=1}^{\infty} \psi_n$ is a WUC series in $C(\Omega, E)$. Indeed, there is a constant $C > 0$ such that $\sup_{n\in\mathbb{N}} \|\sum_{k=1}^{n} t_k e_k\| \le C \sup_{n\in\mathbb{N}} |t_n|$ for any $(t_n)_n \in \ell^\infty$ (see [8], p. 44). Now for any finite subset σ of $\mathbb{N}$ and $\omega \in \Omega$, we get

$$\begin{aligned}
\Big\|\sum_{n\in\sigma} \psi_n(\omega)\Big\|_E &= \Big\|\sum_{n\in\sigma} e_n(\omega)\theta_n(\omega)\Big\|_E = \sup_{\|x^*\|\le 1} \Big|\sum_{n\in\sigma} e_n(\omega)\langle \theta_n(\omega), x^*\rangle\Big| \\
&\le \sup_{\|x^*\|\le 1} C \cdot \sup_{n\in\mathbb{N}} |\langle \theta_n(\omega), x^*\rangle| \le C \cdot \sup_{n\in\mathbb{N}} \|\theta_n\| \le C,
\end{aligned}$$

which implies that for any finite subset σ of $\mathbb{N}$,

$$\Big\| \sum_{n\in\sigma} \psi_n \Big\| \le C.$$

This shows that $\sum_{n=1}^{\infty} \psi_n$ is a WUC series. Now $\langle \psi_n, m_n\rangle \ge \varepsilon/2$ for all $n \in \mathbb{N}$, a contradiction with the assumption that H is a (V)-subset. ∎

For the next proposition, we will use the following notation: for a given measure $m \in M(\Omega, E^*)$ and $A \in \Sigma$, $m\chi_A$ denotes the measure $(\Sigma \to E^*)$ given by $m\chi_A(B) = m(A \cap B)$ for any $B \in \Sigma$.

PROPOSITION 3. *Let H be a* (V)*-subset of $M(\Omega, E^*)$ and $(A_m)_{m\in H}$ a collection of measurable subsets of Ω. Then the subset $\{m\chi_{A_m} : m \in H\}$ is a* (V)*-subset of $M(\Omega, E^*)$.*

Proof. Assume that H is a (V)-subset of $M(\Omega, E^*)$. By Proposition 2, the set $V(H)$ is relatively weakly compact in $M(\Omega)$. Let λ be a control measure for $V(H)$. Fix a sequence $(m_n\chi_{A_{m_n}})_{n\in\mathbb{N}}$ in $\{m\chi_{A_m} : m \in H\}$. We need to show that the countable subset $\{m_n\chi_{A_{m_n}} : n \in \mathbb{N}\}$ is a (V)-subset of $M(\Omega, E^*)$. Let $\sum_{n=1}^{\infty} f_n$ be a WUC series in $C(\Omega, E)$ with $\sup_n \|f_n\| \le 1$. For $\varepsilon > 0$ (fixed), there exists $\delta > 0$ such that if $A \in \Sigma$ with $\lambda(A) < \delta$ then $|m|(A) < \varepsilon/2$, for all $m \in H$. For each $n \in \mathbb{N}$, choose a compact set C_n and an open set O_n such that $C_n \subset A_{m_n} \subset O_n$ and $\lambda(O_n \setminus C_n) < \delta$. Fix a continuous function $g_n : \Omega \to [0, 1]$ with $g_n(\omega) = 1$ for $\omega \in C_n$ and $g_n(\omega) = 0$ for $\omega \in \Omega \setminus O_n$. Let $\phi_n = g_n f_n$; we claim that $\sum_{n=1}^{\infty} \phi_n$ is a WUC series in $C(\Omega, E)$. For that, recall that $\sum_{n=1}^{\infty} f_n$ is a WUC series in $C(\Omega, E)$ and since $(g_n)_n$ is bounded in $C(\Omega)$, one can use similar argument to that in the proof of Proposition 2 to conclude that $\sum_{n=1}^{\infty} \phi_n$ is a WUC series. Consequently, $\lim_{n\to\infty} \langle m_n, \phi_n\rangle = 0$. Now we have the following estimate:

$$\begin{aligned}
|\langle m_n\chi_{A_{m_n}}, f_n\rangle| &= \Big| \int f_n \chi_{A_{m_n}}\, dm_n \Big| = \Big| \int_{A_{m_n}} f_n\, dm_n \Big| \\
&\le \Big| \int_{C_n} f_n\, dm_n \Big| + \Big| \int_{A_{m_n}\setminus C_n} f_n\, dm_n \Big| \\
&\le \Big| \int_{\Omega} \phi_n\, dm_n \Big| + \Big| \int_{O_n\setminus C_n} f_n\, dm_n \Big| + \Big| \int_{A_{m_n}\setminus C_n} f_n\, dm_n \Big| \\
&\le |\langle \phi_n, m_n\rangle| + |m_n|(O_n \setminus C_n) + |m_n|(A_{m_n} \setminus C_n) \\
&\le |\langle \phi_n, m_n\rangle| + 2|m_n|(O_n \setminus C_n) \le |\langle \phi_n, m_n\rangle| + \varepsilon.
\end{aligned}$$

This implies that $\limsup_{n\to\infty} |\langle m_n\chi_{A_{m_n}}, f_n\rangle| \le \varepsilon$ and since ε is arbitrary, we conclude that $\lim_{n\to\infty} |\langle m_n\chi_{A_{m_n}}, f_n\rangle| = 0$. This shows that $\{m_n\chi_{A_{m_n}} : n \in \mathbb{N}\}$ is a (V)-subset. ∎

If we denote by $M^\infty(\lambda, E^*)$ the set $\{m \in M(\Omega, E^*) : |m| \le \lambda\}$ then we obtain the following corollary.

Corollary 1. *Let H be a* (V)-*subset of $M(\Omega, E^*)$ and consider λ the control measure of $V(H)$. For $\varepsilon > 0$ fixed, there exist $N \in \mathbb{N}$ and H_ε a* (V)-*subset of $M(\Omega, E^*)$ with $H_\varepsilon \subset NM^\infty(\lambda, E^*)$ so that $H \subseteq H_\varepsilon + \varepsilon B$, where B denotes the closed unit ball of $M(\Omega, E^*)$.*

Proof. Let $g_m : \Omega \to \mathbb{R}_+$ be the density of $|m|$ with respect to λ. Then

$$\lim_{N\to\infty} \int_{\{\omega: g_m(\omega) > N\}} g_m(\omega)\, d\lambda(\omega) = 0 \qquad \text{uniformly on } H.$$

For $\varepsilon > 0$, choose $N \in \mathbb{N}$ so that

$$\int_{\{\omega: g_m(\omega) > N\}} g_m(\omega)\, d\lambda(\omega) < \varepsilon$$

and let $A_m = \{\omega : g_m(\omega) \le N\}$. It is clear that $H_\varepsilon = \{m\chi_{A_m} : m \in H\}$ is a subset of $NM^\infty(\lambda, E^*)$ and is a (V)-subset by Proposition 3. Also each measure m in H satisfies $m = m\chi_{A_m} + m\chi_{A_m^c}$ with $\|m\chi_{A_m^c}\| < \varepsilon$. ∎

Our next proposition can be viewed as a generalization of Theorem 1 of [17] for sequences of weak*-scalarly measurable maps. We denote by (e_n) the unit vector basis of c_0, by $(\Omega, \Sigma, \lambda)$ a probability space and, for any Banach space F, by F_1 the closed unit ball of F.

Proposition 4. *Let Z be a separable subspace of a real Banach space E and $(f_n)_n$ be a sequence of maps from Ω to E^* that are weak*-scalarly measurable with $\sup_n \|f_n\|_\infty \le 1$. Let a, b be real numbers with $a < b$. Then there exist a sequence $g_n \in \operatorname{conv}\{f_n, f_{n+1}, \ldots\}$ and measurable subsets C and L of Ω with $\lambda(C \cup L) = 1$ such that*

(i) *if $\omega \in C$ and $T \in \mathcal{L}(c_0, Z)_1$ then either*

$$\limsup_{n\to\infty} \langle g_n(\omega), Te_n\rangle \le b \quad \textit{or} \quad \liminf_{n\to\infty} \langle g_n(\omega), Te_n\rangle \ge a;$$

(ii) *if $\omega \in L$ then there exists $k \in \mathbb{N}$ so that for each infinite sequence σ of zeros and ones, there exists $T \in \mathcal{L}(c_0, Z)_1$ such that for $n \ge k$,*

$$\sigma_n = 1 \Rightarrow \langle g_n(\omega), Te_n\rangle \ge b \quad \textit{and} \quad \sigma_n = 0 \Rightarrow \langle g_n(\omega), Te_n\rangle \le a.$$

Proof. The proof is a further refinement of the techniques used in [22] and [17] so we recommend that the reader should get familiar with the proof of Theorem 1 of [17] before reading our extension.

We begin by introducing some notations, part of which were already used in [22] and [17].

Let $f_n : \Omega \to E^*$ be a sequence as in the statement of the proposition. We write $u \ll f$ (or $(u_n) \ll (f_n)$) if there exist $k \in \mathbb{N}$ and $p_1 < q_1 < p_2 <$

$q_2 < \ldots < p_n < q_n < \ldots$ so that for $n \geq k$,

$$u_n = \sum_{i=p_n}^{q_n} \lambda_i f_i \quad \text{with } \lambda_i \in [0,1] \text{ and } \sum_{i=p_n}^{q_n} \lambda_i = 1.$$

Consider $\mathcal{L}(c_0, Z)_1$, the closed unit ball of $\mathcal{L}(c_0, Z)$ with the strong operator topology. It is not difficult to see (using the fact that Z is separable) that $\mathcal{L}(c_0, Z)_1$ is a Polish space; in particular, it has a countable basis $(O_n)_n$. Since $\mathcal{L}(c_0, Z)_1$ is a metric space, we can assume that the O_n's are open balls.

The letter $\mathcal{K}$ will stand for the set of all (strongly) closed subsets of $\mathcal{L}(c_0, Z)_1$. We will say that $\omega \mapsto K(\omega)$ ($\Omega \to \mathcal{K}$) is *measurable* if the set $\{\omega : K(\omega) \cap O_n \neq \emptyset\}$ is a measurable subset of Ω for every $n \in \mathbb{N}$.

Let $h_n = \sum_{i=p_n}^{q_n} \lambda_i f_i$ with $\sum_{i=p_n}^{q_n} \lambda_i = 1$, $\lambda_i \geq 0$ and $p_1 < q_1 < p_2 < q_2 < \ldots$; let V be an open subset of $\mathcal{L}(c_0, Z)_1$ and $\omega \mapsto K(\omega)$ be a fixed measurable map. We set

$$\overline{h}_n(\omega) = \sup_{k \geq q_n} \sup\{\langle h_n(\omega), Te_k\rangle : T \in V \cap K(\omega)\}, \tag{1}$$

$$\theta(h)(\omega) = \limsup_{n\to\infty} \overline{h}_n(\omega). \tag{2}$$

Notice that the definition of $\overline{h}_n$ depends on the representation of h_n as a block convex combination of f_n's. Note also that since V is fixed, it does not appear in the notation. The measurability of $\overline{h}_n$ can be deduced with similar argument to that in [22] (see also [17]) and it is clear that $\|\overline{h}_n\|_\infty \leq 1$. Similarly we set

$$\widetilde{h}_n(\omega) = \inf_{k \geq q_n} \inf\{\langle h_n(\omega), Te_k\rangle : T \in V \cap K(\omega)\}, \tag{3}$$

$$\varphi(h)(\omega) = \liminf_{n\to\infty} \widetilde{h}_n(\omega). \tag{4}$$

The proof of the following lemma is just a notational adjustment of the proof of Lemma 2 of [17].

LEMMA 1. *There exists* $(g_n) \ll (f_n)$ *such that if* $(h_n) \ll (g_n)$ *then*

$$\lim_{n\to\infty} \|\theta(g) - \overline{h}_n\|_1 = 0 \quad \textit{and} \quad \lim_{n\to\infty} \|\varphi(g) - \widetilde{h}_n\|_1 = 0.$$

MAIN CONSTRUCTION. Fix $a < b$ and let τ be the first uncountable ordinal. Set $h_n^0 = f_n$ and $K_0(\omega) = \mathcal{L}(c_0, Z)_1$ for every $\omega \in \Omega$. We construct inductively, as in [17], for $\alpha < \tau$, sequences $h^\alpha = (h_n^\alpha)_n$ and measurable maps $K_\alpha : \Omega \to \mathcal{K}$ with the following property:

$$\text{for } \beta < \alpha < \tau, \quad h^\alpha \ll h^\beta. \tag{5}$$

For $\alpha < \tau$ and $h \ll f$ with $h_n = \sum_{j=p_n}^{q_n} \lambda_i f_i$, if we define

$$\begin{aligned}\overline{h}_{n,l,\alpha}(\omega) &= \sup_{k\ge q_n} \sup\{\langle h_n(\omega), Te_k\rangle : T \in O_l \cap K_\alpha(\omega)\},\\ \theta_{l,\alpha}(h)(\omega) &= \limsup_{n\to\infty} \overline{h}_{n,l,\alpha}(\omega),\\ \widetilde{h}_{n,l,\alpha}(\omega) &= \inf_{k\ge q_n} \inf\{\langle h_n(\omega), Te_k\rangle : T \in O_l \cap K_\alpha(\omega)\},\\ \varphi_{l,\alpha}(h)(\omega) &= \liminf_{n\to\infty} \widetilde{h}_{n,l,\alpha}(\omega),\end{aligned} \tag{6}$$

then for each α of the form $\beta+1$, each $l \ge 1$ and each $h \ll h^\alpha$, we have

$$\lim_{n\to\infty} \|\theta_{l,\beta}(h^\alpha) - \overline{h}_{n,l,\beta}\|_1 = 0 \quad \text{and} \quad \lim_{n\to\infty} \|\varphi_{l,\beta}(h^\alpha) - \widetilde{h}_{n,l,\beta}\|_1 = 0.$$

If α is limit, we set

$$K_\alpha(\omega) = \bigcap_{\beta<\alpha} K_\beta(\omega). \tag{7}$$

If $\alpha = \beta + 1$, then

$$K_\alpha(\omega) = \{T \in K_\beta(\omega) : T \in O_\ell \Rightarrow \theta_{l,\beta}(h^\alpha)(\omega) \ge b,\ \varphi_{l,\beta}(h^\alpha)(\omega) \le a\}. \tag{8}$$

The construction is done in the same manner as in [17] and is a direct application of Lemma 1 of the present paper, so we will not present the details.

As in [17], one can fix an ordinal $\alpha < \tau$ such that for a.e. $\omega \in \Omega$, $K_\alpha(\omega) = K_{\alpha+1}(\omega)$.

Let $h = h^{\alpha+1}$,

$$C = \{\omega : K_\alpha(\omega) = \emptyset\} \quad \text{and} \quad M = \{\omega : K_\alpha(\omega) = K_{\alpha+1}(\omega) \ne \emptyset\}.$$

Clearly C and M are measurable and $\lambda(C \cup M) = 1$.

The next lemma is the analogue of Lemma 4 of [17].

LEMMA 2. *Let $\omega \in C$ and $T \in \mathcal{L}(c_0, Z)_1$. If $u \ll h$ then either*

$$\limsup_{n\to\infty} \langle u_n(\omega), Te_n\rangle \le b \quad \textit{or} \quad \liminf_{n\to\infty} \langle u_n(\omega), Te_n\rangle \ge a.$$

Proof. Let $\omega \in C$, $T \in \mathcal{L}(c_0, Z)_1$ and fix $u \ll h \ll f$ (say $u = \sum_{j=a_n}^{b_n} \alpha_j f_j$); let $S : c_0 \to c_0$ be an operator defined as follows: $Se_{b_n} = e_n$ and $Se_j = 0$ if $j \ne b_n$, $n \in \mathbb{N}$. The operator S is obviously bounded linear with $\|S\| = 1$. So $T \circ S \in \mathcal{L}(c_0, Z)_1 = K_0(\omega)$. Since $T \circ S \notin K_\alpha(\omega)$, there exists a least ordinal β for which $T \circ S \notin K_\beta(\omega)$. The ordinal β cannot be limit, so $\beta = \gamma + 1$ and $T \circ S \in K_\gamma(\omega)$. By the definition of $K_\beta(\cdot)$, there exists $l \in \mathbb{N}$ with $T \circ S \in O_l$ but either $\theta_{l,\gamma}(h^\beta)(\omega) \le b$ or $\varphi_{l,\gamma}(h^\beta)(\omega) \ge a$.

Now since $u \ll h^\beta$, we get either

$$\limsup_{n\to\infty}\langle u_n(\omega), Te_n\rangle = \limsup_{n\to\infty}\langle u_n(\omega), T\circ Se_{q_n}\rangle \le \theta_{l,\gamma}(u)(\omega) \le \theta_{l,\gamma}(h^\beta)(\omega) \le b$$

or

$$\liminf_{n\to\infty}\langle u_n(\omega), Te_n\rangle \ge \varphi_{l,\gamma}(u)(\omega) \ge \theta_{l,\gamma}(h^\beta)(\omega) \ge a.$$

Lemma 2 is proved. ■

The following property of the measurable subset M is somewhat stronger than that obtained in Lemma 5 of [17] and is the main adjustment of the entire proof.

LEMMA 3. *There exists a subsequence $(n(i))$ of integers such that for a.e. $\omega \in M$, if σ is an infinite sequence of zeros and ones then there exists an operator $T \in \mathcal{L}(c_0, Z)_1$ (which may depend on ω and σ) such that:*

$$\sigma_i = 1 \Rightarrow \langle h_{n(i)}(\omega), Te_i\rangle \ge b \quad \textit{and} \quad \sigma_i = 0 \Rightarrow \langle h_{n(i)}(\omega), Te_i\rangle \le a.$$

Proof. Let us denote by F the set of finite sequences of zeros and ones and by F^∞ the set of infinite sequences of zeros and ones. For $s \in F$, $|s|$ will denote the length of s. Let $s = (s_1, \ldots, s_n)$ and $r = (r_1, \ldots, r_m)$ with $n \le m$. We say that $s < r$ if $s_i = r_i$ for $i \le n$. Let us fix a representation of (h_n) as a block convex combination of (f_n):

$$h_n = \sum_{i=p_n}^{q_n} \lambda_i f_i.$$

We will construct sequences of integers $n(i)$ and $m(i)$, measurable sets $B_i \subset M$ and measurable maps $Q(s,\cdot) : M \to \mathbb{N}$ (for $s \in F$) such that:

(9) $q_{n(1)} < m(1) < q_{n(2)} < m(2) < \ldots < q_{n(i)} < m(i) < \ldots$;

(10) for all $s \in F$, $\sup\{Q(s,\omega) : \omega \in M\} < \infty$;

(11) $\lambda(M \setminus B_i) \le 2^{-i}$;

(12) for all $s \in F$, $\operatorname{diam}(O_{Q(s,\omega)}) \le 1/|s|$;

(13) for $s, r \in F$, $s < r$, and $\omega \in \bigcap_{|s|\le i\le |r|} B_i$, $O_{Q(r,\omega)} \subset O_{Q(s,\omega)}$;

(14) for all $\omega \in M$ and $s \in F$, $K_\alpha(\omega) \cap O_{Q(s,\omega)} \ne \emptyset$;

(15) for all $s \in F$, for all $i \le p = |s|$ and all $\omega \in \bigcap_{i\le j\le |s|} B_j$,

$$s_i = 1 \Rightarrow \text{for all } T \in O_{Q(s,\omega)},\ \sup_{q_{n(i)}\le k\le m(i)} \langle h_{n(i)}(\omega), Te_k\rangle \ge b,$$

$$s_i = 0 \Rightarrow \text{for all } T \in O_{Q(s,\omega)},\ \inf_{q_{n(i)}\le k\le m(i)} \langle h_{n(i)}(\omega), Te_k\rangle \le a.$$

The construction is done in a similar fashion to that in [17]; the only difference is in the selection of the measurable map $Q(s,\cdot):\Omega\to\mathbb{N}$ so that (12) is satisfied. For that we consider, instead of $\mathbb{N}$, the subset $\mathcal{M}\subset\mathbb{N}$ defined by

$$\mathcal{M}=\left\{k\in\mathbb{N}:\operatorname{diam}O_k\le\frac{1}{|s|}\right\},$$

and since $\mathbb{N}$ and $\mathcal{M}$ are equipped with the discrete topology, we can replace $\mathbb{N}$ by $\mathcal{M}$ and use the same argument to get $Q(s,\cdot):\Omega\to\mathcal{M}$.

To complete the proof, let $L=\bigcup_k\bigcap_{i\ge k}B_i$. It is clear that $\lambda(M\setminus L)=0$. Fix $\omega\in\bigcap_{i\ge k}B_i$ and $\sigma\in F^\infty$ with $\sigma=(\sigma_i)_{i\in\mathbb{N}}$. Let

$$\sigma^{(m)}=(\sigma_1,\ldots,\sigma_m)\in F\quad\text{ for all } m\in\mathbb{N}.$$

By (15) we deduce that for $m\in\mathbb{N}$ and all $i\le m$ and $\omega\in\bigcap_{i\le j\le m}B_j$,

$$\sigma_i=1\Rightarrow\text{for all } T\in O_{Q(\sigma^{(m)},\omega)},\ \sup_{q_{n(i)}\le k\le m(i)}\langle h_{n(i)}(\omega),Te_k\rangle\ge b,$$

$$\sigma_i=0\Rightarrow\text{for all } T\in O_{Q(\sigma^{(m)},\omega)},\ \inf_{q_{n(i)}\le k\le m(i)}\langle h_{n(i)}(\omega,Te_k\rangle\le a.$$

It is easy to check that the same conclusion holds for $T\in\overline{O}_{Q(\sigma^{(m)},\omega)}$ (the closure of $O_{Q(\sigma^{(m)},\omega)}$ for the strong operator topology). So if we let $\mathcal{A}=\bigcap_{m\in\mathbb{N}}\overline{O}_{Q(\sigma^{(m)},\omega)}$, then $\mathcal{A}\ne\emptyset$. In fact, $(\overline{O}_{Q(\sigma^{(m)},\omega)})_{m\in\mathbb{N}}$ is a nested sequence of nonempty closed sets (by (13)) of a complete metric space and such that $\operatorname{diam}(\overline{O}_{Q(\sigma^{(m)},\omega)})\to 0$ (as $m\to\infty$) (by (12)), so $\mathcal{A}\ne\emptyset$ (see for instance [14], p. 270).

It is now clear that if $\omega\in\bigcap_{i\ge k}B_i$ and $A\in\mathcal{A}$, then for $i\ge k$,

$$\sigma_i=1\Rightarrow\sup_{q_{n(i)}\le k\le m(i)}\langle h_{n(i)}(\omega),Ae_k\rangle\ge b,$$

$$\sigma_i=0\Rightarrow\sup_{q_{n(i)}\le k\le m(i)}\langle h_{n(i)}(\omega),Ae_k\rangle\le a.$$

We complete the proof as in [17]: choose $k(i)\in[q_{n(i)},m(i)]$ such that

$$\sup_{q_{n(i)}\le k\le m(i)}\langle h_{n(i)}(\omega),Ae_k\rangle=\langle h_{n(i)}(\omega),Ae_{k(i)}\rangle$$

for $\sigma_i=1$ and

$$\inf_{q_{n(i)}\le k\le m(i)}\langle h_{n(i)}(\omega),Ae_k\rangle=\langle h_{n(i)}(\omega),Ae_{k(i)}\rangle$$

for $\sigma_i=0$. The sequence $(k(i))$ is an increasing sequence by (9) so one can construct an operator $S:c_0\to c_0$ with $Se_i=e_{k(i)}$ for all $i\in\mathbb{N}$ and it is now clear that

$$\sigma_i=1\Rightarrow\langle h_{n(i)}(\omega),A\circ Se_i\rangle\ge b\quad\text{and}\quad\sigma_i=0\Rightarrow\langle h_{n(i)}(\omega),A\circ Se_i\rangle\le a.$$

The operator $T = A \circ S$ satisfies the required property. The proof of Lemma 3 is complete. ■

To finish the proof of Proposition 4, we take $g_i = h_{n(i)}$ for $i \in \mathbb{N}$. ■

3. Main theorem

THEOREM 1. *Let E be a separable Banach space and Ω be a compact Hausdorff space. Then the space $C(\Omega, E)$ has property* (V) *if and only if E has property* (V).

Proof. If $C(\Omega, E)$ has property (V), then the space E has property (V) since E is isomorphic to a complemented subspace of $C(\Omega, E)$.

For the converse, we will present the case where Ω is compact metrizable; the reduction of the general case to the metrizable case was already done in the proof of Theorem 3 (case 2) of [5], so we will not present the details.

Assume that E has property (V) and Ω is a compact metric space. Let H be a (V)-subset of $M(\Omega, E^*)$. Our goal is to show that H is relatively weakly compact. Using Corollary 1, we can assume without loss of generality that there exists a probability measure λ on Σ such that $|m| \le \lambda$ for each $m \in H$. Observe that if E has property (V), then E^* is weakly sequentially complete (Corollary 5 of [15]) and thus $M(\Omega, E^*)$ is weakly sequentially complete, as shown in [22] (Theorem 17). If H is not relatively weakly compact, then it contains a sequence $(m_n)_n$ that is equivalent to the ℓ^1-basis. By Theorem 14 of [22], there exist $m'_n \in \operatorname{conv}\{m_n, m_{n+1}, \ldots\}$ and $\Omega' \subset \Omega$, $\lambda(\Omega') > 0$, so that for $\omega \in \Omega'$, there exists $l \in \mathbb{N}$ such that $(\varrho(m'_n)(\omega))_{n \ge l}$ is equivalent to the ℓ^1-basis in E^*. Let

$$f_n(\omega) = \varrho(m'_n)(\omega)\chi_{\Omega'}(\omega), \qquad n \in \mathbb{N}.$$

$(f_n)_n$ is a sequence of weak*-scalarly measurable maps and $\sup_n \|f_n\|_\infty < \infty$.

PROPOSITION 5. *There exist a sequence $g_n \in \operatorname{conv}\{f_n, f_{n+1}, \ldots\}$, a positive number δ and a strongly measurable map $T : \Omega \to \mathcal{L}(c_0, E)_1$ such that*

$$\liminf_{n\to\infty} \left| \int \langle g_n(\omega), T(\omega)e_n \rangle \, d\lambda(\omega) \right| \ge \delta.$$

Proof. Let $(a(k), b(k))_{k\in\mathbb{N}}$ be an enumeration of all pairs of rationals with $a < b$. By induction, we construct sequences (g^k) and measurable sets C_k, L_k of Ω satisfying the following:

(i) $g^{k+1} \ll g^k$ for each $k \in \mathbb{N}$;
(ii) $C_{k+1} \subset C_k$, $L_k \subset L_{k+1}$, $\lambda(C_k \cup L_k) = 1$;
(iii) for all $\omega \in C_k$ and all $j \ge k$, if $T \in \mathcal{L}(c_0, E)_1$, then either

$$\limsup_{n\to\infty} \langle g^j_n(\omega), Te_n \rangle \le b(k) \quad \text{or} \quad \liminf_{n\to\infty} \langle g^j_n(\omega), Te_n \rangle \ge a(k);$$

(iv) for all $\omega \in L_k$, there exists $j \in \mathbb{N}$ such that for each infinite sequence σ of zeros and ones, there exists $T \in \mathcal{L}(c_0, E)_1$ such that if $n \ge j$, then

$$\sigma_n = 1 \Rightarrow \langle g_n^k(\omega), Te_n \rangle \ge b(k) \quad \text{and} \quad \sigma_n = 0 \Rightarrow \langle g_n^k(\omega), Te_n \rangle \le a(k).$$

This is just an application of Proposition 4 inductively, starting from $g^0 = f$.

Let $\mathcal{P} = \{k \in \mathbb{N} : b(k) > 0\}$ and $\mathcal{N} = \{k \in \mathbb{N} : a(k) < 0\}$. It is clear that $\mathbb{N} = \mathcal{N} \cup \mathcal{P}$. Consider $C = \bigcap_k C_k$ and $L = \bigcup_k L_k$; we have $\lambda(C \cup L) = 1$.

Case 1: $\lambda(L) > 0$. Since $L = \bigcup_k L_k$, there exists $k \in \mathbb{N}$ such that $\lambda(L_k) > 0$. Let $(g_n) = (g_n^k)$. We claim that (g_n) satisfies the requirements of Proposition 5. For that let us assume first that $k \in \mathcal{P}$ (i.e. $b(k) > 0$). Fix $\sigma = (1, 1, 1, \ldots)$. For each $\omega \in L_k$, there exists $T \in \mathcal{L}(c_0, E)_1$ such that $\langle g_n(\omega), Te_n \rangle \ge b(k)$ for all $n \ge j$ for some $j \in \mathbb{N}$. We can choose the above operator measurably using the following lemma:

LEMMA 4. *There exists a strongly measurable map* $T : \Omega \to \mathcal{L}(c_0, E)_1$ *such that*:

(α) $T(\omega) = 0$ *for all* $\omega \notin L_k$;

(β) *For* $\omega \in L_k$, *there exists* $j \in \mathbb{N}$ *such that if* $n \ge j$, *then* $\langle g_n(\omega), T(\omega)e_n \rangle \ge b(k)$.

Proof. Consider $\mathcal{L}(c_0, E)_1$ with the strong operator topology and E_1^* with the weak*-topology. The space E_1^* is a compact metric space and hence is a Polish space. The space $E_1^{*\mathbb{N}} \times \mathcal{L}(c_0, E)_1$ equipped with the product topology is a Polish space. Let $\mathcal{A}$ be the following subset of $E_1^{*\mathbb{N}} \times \mathcal{L}(c_0, E)_1$:

$$\{(x_n^*), T\} \in \mathcal{A} \Leftrightarrow \text{there is } j \in \mathbb{N} \text{ for which } \langle x_n^*, Te_n \rangle \ge b(k) \text{ for all } n \ge j.$$

The set $\mathcal{A}$ is clearly a Borel subset of $E^{*\mathbb{N}} \times \mathcal{L}(c_0, E)_1$ and if $\Pi : E_1^{*\mathbb{N}} \times \mathcal{L}(c_0, E)_1 \to E_1^{*\mathbb{N}}$ is the first projection, then $\Pi(\mathcal{A})$ is an analytic subset of $E_1^{*\mathbb{N}}$. By Theorem 8.5.3 of [6], there exists a universally measurable map $\Theta : \Pi(\mathcal{A}) \to \mathcal{L}(c_0, E)_1$ such that the graph of Θ is a subset of $\mathcal{A}$. Notice that if $\omega \in L_k$, then $(g_n(\omega))_{n \ge 1} \in \Pi(\mathcal{A})$. We define

$$T(\omega) = \begin{cases} \Theta((g_n(\omega))_{n\ge 1}) & \text{if } \omega \in L_k, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that T satisfies all the requirements of the lemma. Lemma 4 is proved. ∎

Back to the proof of the proposition, we have $\langle g_n(\omega), T(\omega)e_n \rangle \ge b(k)$ for all $\omega \in L_k$ and $n \ge j$ (for some $j \in \mathbb{N}$). So $\liminf_{n\to\infty} \langle g_n(\omega), T(\omega)e_n \rangle \ge b(k)$ for $\omega \in L_k$, and by Fatou's lemma,

$$\liminf_{n\to\infty} \int \langle g_n(\omega), T(\omega)e_n \rangle \, d\lambda(\omega) \ge b(k)\lambda(L_k).$$

The integrand in the above integral is clearly integrable and if we set $\delta = b(k)\lambda(L_k) > 0$, the proof is complete for $k \in \mathcal{P}$.

Now if $k \in \mathcal{N}$ (i.e. $a(k) < 0$), we consider $\sigma = (0, 0, \ldots)$ and choose a strongly measurable map $\omega \mapsto T(\omega)$ (using similar argument to that in the above lemma) with $T(\omega) = 0$ for $\omega \notin L_k$; for $\omega \in L_k$, there exists $j \in \mathbb{N}$ such that $\langle g_n(\omega), T(\omega)e_n \rangle \le a(k) < 0$ for $n \ge j$. So we get

$$\limsup_{n\to\infty} \langle g_n(\omega), T(\omega)e_n \rangle \le a(k)$$

for each $\omega \in L_k$ and hence

$$\limsup_{n\to\infty} \int \langle g_n(\omega), T(\omega)e_n \rangle \, d\lambda(\omega) \le a(k)\lambda(L_k) < 0,$$

which implies that

$$\liminf_{n\to\infty} \left| \int \langle g_n(\omega), T(\omega)e_n \rangle \, d\lambda(\omega) \right| \ge |a(k)|\lambda(L_k),$$

so the proof of the proposition is complete for the case $\lambda(L) > 0$.

Case 2: $\lambda(L) = 0$. Since $\lambda(C \cup L) = 1$, we have $\lambda(\Omega \setminus C) = 0$. Choose a sequence (g_n) so that $(g_n) \ll (g_n^k)$ for every $k \in \mathbb{N}$ (see Lemma 1 of [22]). By the definition of the C_k's and by (iii), we have either $\limsup_{n\to\infty} \langle g_n(\omega), Te_n \rangle \le b(k)$ or $\liminf_{n\to\infty} \langle g_n(\omega), Te_n \rangle \ge a(k)$ for all $k \in \mathbb{N}$, and therefore for each $\omega \in C$,

$(*)$ $\quad \lim_{n\to\infty} \langle g_n(\omega), Te_n \rangle$ exists for every $T \in \mathcal{L}(c_0, E)_1$.

But for $\omega \in \Omega'$, the sequence $(f_n(\omega))_n$ is equivalent to the ℓ^1-basis in E^*; and since $(g_n) \ll (f_n)$, the sequence $(g_n(\omega))_n$ is also equivalent to the ℓ^1-basis in E^*; and since E has property (V), the set $\{g_n(\omega) : n \ge 1\}$ cannot be a (V)-subset of E^*, i.e., there exists a WUC series $\sum_{n=1}^{\infty} x_n$ in E such that $\limsup_{n\to\infty} \langle g_n(\omega), x_n \rangle > 0$. Define $T : c_0 \to E$ by $T((t_n)_n) = \sum_{n=1}^{\infty} t_n x_n$ for every $(t_n)_n \in c_0$; T is well defined, linear and bounded (see for intance [8]). Clearly $Te_n = x_n$ for all $n \in \mathbb{N}$. Replacing T by $T/\|T\|$ (if necessary), we conclude that there exists an operator $T \in \mathcal{L}(c_0, E)_1$ such that $\limsup_{n\to\infty} \langle g_n(\omega), Te_n \rangle > 0$. But condition $(*)$ above insures that the limit exists, so for each $\omega \in \Omega'$, there exists $T \in \mathcal{L}(c_0, E)_1$ such that $\lim_{n\to\infty} \langle g_n(\omega), Te_n \rangle > 0$. We now choose the operator T measurably using the same argument as in the above lemma: i.e., there exists $T : \Omega \mapsto \mathcal{L}(c_0, E)_1$ strongly measurable such that $T(\omega) = 0$ for $\omega \notin \Omega'$ and $\lim_{n\to\infty} \langle g_n(\omega), T(\omega)e_n \rangle > 0$ for all $\omega \in \Omega'$. Let $\delta(\omega) = \lim_{n\to\infty} \langle g_n(\omega), T(\omega)e_n \rangle$ for $\omega \in \Omega'$ and 0 otherwise.

The map $\omega \mapsto \delta(\omega)$ is measurable and we obtain

$$\lim_{n\to\infty} \int \langle g_n(\omega), T(\omega)e_n \rangle \, d\lambda(\omega) = \int \delta(\omega) \, d\lambda(\omega) = \delta > 0.$$

The proof of Proposition 5 is complete. ∎

To complete the proof of Theorem 1, fix $(g_n) \ll (f_n)$, $T : \Omega \to \mathcal{L}(c_0, E)_1$ strongly measurable and $\delta > 0$ as in Proposition 5.

For each $n \in \mathbb{N}$, let $G_n : \Sigma \to E^*$ be the measure in $M(\Omega, E^*)$ defined by

$$G_n(A) = \text{weak*-}\int_A g_n(\omega)\, d\lambda(\omega).$$

Since $f_n(\omega) = \varrho(m'_n)(\omega)\chi_{\Omega'}(\omega)$ for every $\omega \in \Omega$ and $m'_n \in \operatorname{conv}\{m_n, m_{n+1}, \ldots\}$, it is clear that $G_n \in \operatorname{conv}\{m_n\chi_{\Omega'}, m_{n+1}\chi_{\Omega'}, \ldots\}$ and we will show that $\{G_n : n \geq 1\}$ is not a (V)-subset of $M(\Omega, E^*)$ to get a contradiction by virtue of Proposition 3. Since $\omega \mapsto T(\omega)e_n$ is norm-measurable for each $n \in \mathbb{N}$, one can choose (using Lusin's Theorem) a compact subset $\Omega'' \subset \Omega$ with $\lambda(\Omega \setminus \Omega'') < \delta/3$ and such that the map $\omega \to T(\omega)e_n$ $(\Omega'' \to E)$ is continuous for each $n \in \mathbb{N}$.

Let $\Lambda : C(\Omega'', E) \to C(\Omega, E)$ be an extension operator (the existence of such an operator is given by Theorem 21.1.4 of [21]) and consider $t_n = \Lambda(T(\cdot)e_n|_{\Omega''})$. The series $\sum_{n=1}^\infty t_n$ is a WUC series in $C(\Omega, E)$. In fact, the operator $S : c_0 \to C(\Omega'', E)$ given by $Se = T(\cdot)e|_{\Omega''}$ is easily checked to be linear and bounded and $t_n = \Lambda \circ S(e_n)$, so $\sum_{n=1}^\infty t_n$ is a WUC series.

The following estimate concludes the proof:

$$\begin{aligned}\langle t_n, G_n\rangle &= \int \langle g_n(\omega), t_n(\omega)\rangle\, d\lambda(\omega)\\ &= \int_{\Omega''} \langle g_n(\omega), T(\omega)e_n\rangle\, d\lambda(\omega) + \int_{\Omega\setminus\Omega''} \langle g_n(\omega), t_n(\omega)\rangle\, d\lambda(\omega),\end{aligned}$$

so

$$\begin{aligned}&\langle t_n, G_n\rangle - \int \langle g_n(\omega), T(\omega)e_n\rangle\, d\lambda(\omega)\\ &\qquad = \int_{\Omega\setminus\Omega''} \langle g_n(\omega), t_n(\omega)\rangle\, d\lambda(\omega) - \int_{\Omega\setminus\Omega''} \langle g_n(\omega), T(\omega)e_n\rangle\, d\lambda(\omega)\end{aligned}$$

and

$$\Big|\langle t_n, G_n\rangle - \int \langle g_n(\omega), T(\omega)e_n\rangle\, d\lambda(\omega)\Big| \leq 2\frac{\delta}{3},$$

which implies that

$$\Big|\int \langle g_n(\omega), T(\omega)e_n\rangle\, d\lambda(\omega)\Big| \leq 2\frac{\delta}{3} + |\langle t_n, G_n\rangle|.$$

Hence

$$\liminf_{n\to\infty} |\langle t_n, G_n\rangle| \geq \frac{\delta}{3}.$$

This of course shows that $\{G_n : n \geq 1\}$ is not a (V)-set.

Theorem 1 is proved. ■

Theorem 1 above has the following consequences relative to Banach spaces of compact operators. In what follows, if X and Y are Banach spaces, then $K_{\mathrm{w}^*}(X^*, Y)$ denotes the Banach space of weak* to weakly continuous compact operators from X^* to Y equipped with the operator norm and $K(X,Y)$ the space of compact operators from X to Y with the operator norm. We have the following corollaries.

COROLLARY 2. *Let X and Y be Banach spaces. If X is injective and Y is separable and has property* (V) *then $K_{w^*}(X^*,Y)$ has property* (V).

Proof. $K_{w^*}(X^*,Y)$ is isometrically isomorphic to $K_{w^*}(Y^*,X)$, which is a complemented subspace of $K_{w^*}(Y^*,C(B_{X^*}))\approx C(B_{X^*},Y)$, and has property (V) by Theorem 1. ■

COROLLARY 3. *Let X be an $\mathcal{L}_\infty$-space and Y a separable Banach space with property* (V). *Then $K(X^*,Y)$ has property* (V).

Proof. The space $K(X^*,Y)$ is isomorphic to $K_{w^*}(X^{***},Y)$ (see [19]) and it is well known that X^{**} is injective and so $K_{w^*}(X^{***},Y)$ has property (V) by Corollary 2. ■

We now turn our attention to Bochner spaces. In [1], Bombal observed that if E is a closed subspace of an order-continuous Banach lattice, then $L^p(\mu,E)$ has property (V) if $1<p<\infty$ and E has property (V). Our next result shows that for the separable case, property (V) can be lifted to the Bochner space $L^p(\mu,E)$.

THEOREM 2. *Let E be a separable Banach space and (Ω,Σ,μ) be a finite measure space. If $1<p<\infty$, then the space $L^p(\mu,E)$ has property* (V) *if and only if E does.*

Proof. Without loss of generality, we will assume that Ω is a compact Hausdorff space, μ is a Borel measure and Σ is the completion of the field of Borel-measurable subsets of Ω. For $1<p<\infty$, let q such that $1/p+1/q=1$.

It is a well-known fact that the dual of $L^p(\mu,E)$ is isometrically isomorphic to the space $M^q(\mu,E^*)$ of all vector measures $F:\Sigma\to E^*$ with

$$\|F\|_q=\sup_\pi\Big\{\sum_{A\in\pi}\frac{\|F(A)\|^q}{\mu(A)^q}\mu(A)\Big\}^{1/q}<\infty$$

(see for instance [9], p. 115).

Let H be a (V)-subset of $M^q(\mu,E^*)$. We need to show that H is relatively weakly compact in $M^q(\mu,E^*)$. Since $C(\Omega,E)\subset L^p(\mu,E)$ and $C(\Omega,E)$ has property (V) by Theorem 1, H is relatively weakly compact in $M(\Omega,E^*)$. Let $(m_n)_n\subset H$ and let $g_n:\Omega\to E^*$ be the weak*-density of m_n with respect to μ. There exist $G_n\in\operatorname{conv}\{m_n,m_{n+1},\ldots\}$ and $G\in M(\Omega,E^*)$ such that G_n converges to G in $M(\Omega,E^*)$. If we denote by g the weak*-density of G with respect to μ, then we get

$$\lim_{n\to\infty}\int\|g_n(\omega)-g(\omega)\|\,d\mu(\omega)=0,$$

and by passing to a subsequence, we can assume that $\|g_n(\omega) - g(\omega)\|$ converges to zero μ-a.e. Since $\sup_n \int \|g_n(\omega)\|^q \, d\mu(\omega) < \infty$, it is clear that $\int \|g(\omega)\|^q \, d\mu(\omega) < \infty$ and therefore $G \in M^q(\mu, E^*)$.

Now for each $\varphi \in L^p(\mu)$, the sequence $(\varphi(\cdot)\|g_n(\cdot) - g(\cdot)\|)$ is uniformly integrable and therefore

$$\lim_{n\to\infty} \int \varphi(\omega)\|g_n(\omega) - g(\omega)\| \, d\mu(\omega) = 0.$$

This shows that the sequence $(\|g_n(\cdot) - g(\cdot)\|)$ converges weakly to zero in $L^q(\mu)$. There exists $\psi_n \in \operatorname{conv}\{\|g_n(\cdot) - g(\cdot)\|, \|g_{n+1}(\cdot) - g(\cdot)\|, \ldots\}$ so that ψ_n converges to zero in norm in $L^q(\mu)$. Let $\psi_n = \sum_{i=p_n}^{q_n} \lambda_i \|g_i(\cdot) - g(\cdot)\|$. We define the following sequence of measures:

$$F_n(A) = \text{weak*-}\int \sum_{i=p_n}^{q_n} \lambda_i g_i(\omega) \, d\mu(\omega).$$

Clearly $F_n \in \operatorname{conv}\{m_n, m_{n+1}, \ldots\}$ and

$$\begin{aligned}\|F_n - G\|_q^q &= \int \Big(\Big\| \sum_{i=p_n}^{q_n} \lambda_i g_i(\omega) - g(\omega)\Big\|\Big)^q \, d\mu(\omega)\\ &\le \int \Big(\sum_{p_n}^{q_n} \lambda_i \|g_i(\omega) - g(\omega)\|\Big)^q \, d\mu(\omega) = \int (\psi_n(\omega))^q \; d\mu(\omega),\end{aligned}$$

so $\lim_{n\to\infty} \|F_n - G\|_q = 0$, which proves that H is relatively weakly compact in $M^q(\mu, E^*)$ (see for instance [23]). Theorem 2 is proved. ∎

Remark. As was observed in [20], the property (V) cannot be lifted from a Banach space E to the Bochner space $L^\infty(\mu, E)$. In fact, the space $E = (\Sigma \oplus \ell_1^n)_{c_0}$ has property (V) but $L^\infty(\mu, E)$ contains a complemented copy of ℓ^1, hence failing property (V).

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