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AN EXTENSION OF AN INEQUALITY DUE TO STEIN AND LEPINGLE

BY

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Hardy spaces consisting of adapted function sequences and generated by the q-variation and by the conditional q-variation are considered. Their dual spaces are characterized and an inequality due to Stein and Lepingle is extended.

1. Introduction. It is known that the dual of the martingale Hardy space $H_1^{S_2}$ generated by the quadratic variation is $\mathcal{BMO}_2^-$ and that of the Hardy space $H_1^{s_2}$ generated by the conditional quadratic variation is $\mathcal{BMO}_2$ (see Garsia [4], Herz [5]). The first result is extended by Dellacherie and Meyer [3] to the space $h_1^{S_2}$ containing adapted function sequences. The inequality

$$\Big\|\Big(\sum_{n=0}^{\infty}|E_{\mathcal{F}_{n-1}}X_n|^q\Big)^{1/q}\Big\|_p \le C_p\Big\|\Big(\sum_{n=0}^{\infty}|X_n|^q\Big)^{1/q}\Big\|_p \quad (1<p<\infty) \tag{1}$$

was proved by Stein [9] for $q=2$ and by Asmar and Montgomery-Smith [1] for $1\le q\le\infty$, where X_n ($n\in\mathbb{N}$) are arbitrary measurable functions. Using the latter duality result Lepingle [8] verified (1) for $p=1$, $q=2$ and for adapted functions. The two-parameter analogue of Lepingle's result can be found in Weisz [10].

Lepingle [7] proved that the dual of the martingale Hardy space $H_1^{s_q}$ is $\mathcal{BMO}_{q'}$ and more recently the author [12] verified that the dual of $H_1^{S_q}$ is $\mathcal{BMO}_{q'}^-$, where $1\le q<\infty$, $1/q+1/q'=1$ and s_q (resp. S_q) denotes the conditional q-variation (resp. the q-variation).

In this paper the Hardy spaces of adapted function sequences are embedded isometrically in martingale Hardy spaces and so the dual of $h_p^{S_q}$ generated by the q-variation and, moreover, the dual of $h_p^{s_q}$ generated by the conditional q-variation are characterized ($1\le p,q<\infty$). Applying the

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duality result with respect to $h_1^{S_q}$ we extend inequality (1) to $1 = p \le q < \infty$. Moreover, if $(\mathcal{F}_n)$ is regular then (1) holds also for $0 < p < 1 \le q < \infty$.

2. Preliminaries and notations. Let $(\Omega, \mathcal{A}, P)$ be a probability space and let $(\mathcal{F}_n, n \in \mathbb{N})$ be a non-decreasing sequence of σ-algebras. For simplicity, we suppose that

$$\sigma\Big(\bigcup_{n=0}^{\infty} \mathcal{F}_n\Big) := \bigvee_{n=0}^{\infty} \mathcal{F}_n = \mathcal{A}.$$

The expectation operator and the conditional expectation operator relative to a σ-algebra $\mathcal{C}$ are denoted by E and $E_{\mathcal{C}}$, respectively. We briefly write L_p for the $L_p(\Omega, \mathcal{A}, P)$ space with the norm (or quasinorm) $\|f\|_p := (E|f|^p)^{1/p}$ $(0 < p \le \infty)$.

In this paper we consider sequences $X = (X_n, n \in \mathbb{N})$ of integrable and *adapted* (i.e. X_n is $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$) functions. We always suppose that $X_0 = 0$. The *q-variation* $S_q(X)$ and the *conditional q-variation* $s_q(X)$ $(0 < q < \infty)$ of X are defined by

$$S_q(X) := \Big(\sum_{n=0}^{\infty} |X_n|^q\Big)^{1/q} \quad \text{and} \quad s_q(X) := \Big(\sum_{n=0}^{\infty} E_{\mathcal{F}_{n-1}} |X_n|^q\Big)^{1/q},$$

respectively, while for $q = \infty$ we let

$$S_\infty(X) := s_\infty(X) := \sup_{n \in \mathbb{N}} |X_n|.$$

Let us introduce the Hardy spaces $h_p^{S_q}$ and $h_p^{s_q}$ $(0 < p, q \le \infty)$ consisting of the sequences $X = (X_n)$ of adapted functions for which

$$\|X\|_{h_p^{S_q}} := \|S_q(X)\|_p < \infty \quad \text{and} \quad \|X\|_{h_p^{s_q}} := \|s_q(X)\|_p < \infty,$$

respectively. Note that $h_p^{S_q}$ is a subspace of the well-known space $L_p(l_q)$ that contains sequences $\xi = (\xi_n, n \in \mathbb{N})$ of $\mathcal{A}$-measurable functions and is equipped with the norm

$$\|\xi\|_{L_p(l_q)} := \Big[E\Big(\sum_{n=0}^{\infty} |\xi_n|^q\Big)^{p/q}\Big]^{1/p}.$$

Now we introduce the corresponding *bmo* spaces. For $1 \le q < \infty$, bmo_q and bmo_q^- consist of all sequences $X = (X_n)$ of adapted functions for which

$$\|X\|_{bmo_q} = \Big\| \sup_{n \in \mathbb{N}} \Big(E_{\mathcal{F}_n} \sum_{k=n+1}^{\infty} |X_k|^q\Big)^{1/q} \Big\|_\infty < \infty$$

and

$$\|X\|_{bmo_q^-} = \Big\| \sup_{n \in \mathbb{N}} \Big(E_{\mathcal{F}_n} \sum_{k=n}^{\infty} |X_k|^q\Big)^{1/q} \Big\|_\infty < \infty,$$

respectively. Furthermore, let $bmo_\infty = bmo_\infty^- = h_\infty^{S_\infty}$.

Dellacherie and Meyer [3] showed that the dual of $h_1^{S_2}$ is bmo_2^-. It will be proved that the dual of $h_1^{S_q}$ is $bmo_{q'}^-$ and the one of $h_1^{s_q}$ is $bmo_{q'}$ ($1 \le q < \infty$, $1/q + 1/q' = 1$).

3. Duality results. We shall embed the spaces $h_p^{S_q}$ and $h_p^{s_q}$ in Hardy spaces of martingales, the duals of which are known. Let

$$\mathcal{D}_k := \sigma(r_0, \ldots, r_{k-1}) = \sigma\{[l2^{-k}, (l+1)2^{-k}) : 0 \le l < 2^k\}$$

be the dyadic σ-algebras (see Weisz [11]), where r_k is the Rademacher function on $[0,1)$, i.e.

$$r_k(x) := \begin{cases} 1 & \text{if } x \in [2l/2^{k+1}, (2l+1)/2^{k+1}) \quad (0 \le l < 2^k), \\ -1 & \text{if } x \in [(2l+1)/2^{k+1}, (2l+2)/r2^{k+1}) \quad (0 \le l < 2^k). \end{cases}$$

Set

$$\mathcal{A}_n := \sigma(\mathcal{F}_n \times \mathcal{D}_n) \qquad (n \in \mathbb{N}).$$

Consider the probability space $(\Omega \times [0,1), \sigma(\mathcal{A} \times \mathcal{B}), P \times \lambda)$ and the stochastic basis $(\mathcal{A}_n, n \in \mathbb{N})$, where $\mathcal{B}$ denotes the Borel measurable sets and λ is Lebesgue measure.

We investigate the martingales relative to $(\mathcal{A}_n)$, i.e. the sequences $f = (d_n f, n \in \mathbb{N})$ of adapted functions relative to $(\mathcal{A}_n)$ for which $E_{\mathcal{A}_{n-1}} d_n f = 0$ ($n \in \mathbb{N}$). The *q-variation* and *conditional q-variation* ($0 < q < \infty$) of a martingale f is given by

$$S_q(f) := \Big(\sum_{n=0}^{\infty} |d_n f|^q\Big)^{1/q} \quad \text{and} \quad s_q(f) := \Big(\sum_{n=0}^{\infty} E_{\mathcal{A}_{n-1}} |d_n f|^q\Big)^{1/q},$$

respectively, and for $q = \infty$ we let

$$S_\infty(f) := s_\infty(f) := \sup_{k \in \mathbb{N}} |d_k f|.$$

The martingale Hardy spaces $H_p^{S_q}$ and $H_p^{s_q}$ ($0 < p, q \le \infty$) containing martingales relative to $(\mathcal{A}_n)$ are defined with the help of the norms

$$\|f\|_{H_p^{S_q}} := \Big(\int_\Omega \int_0^1 S_q(f)^p \, dP \, d\lambda\Big)^{1/p} \quad \text{and} \quad \|f\|_{H_p^{s_q}} := \Big(\int_\Omega \int_0^1 s_q(f)^p \, dP \, d\lambda\Big)^{1/p},$$

respectively. The corresponding dual spaces are equipped with the norms

$$\|f\|_{\mathcal{BMO}_q} = \Big\| \sup_{n \in \mathbb{N}} \Big(E_{\mathcal{A}_n} \sum_{k=n+1}^{\infty} |d_k f|^q\Big)^{1/q} \Big\|_\infty$$

and

$$\|f\|_{\mathcal{BMO}_q^-} = \Big\| \sup_{n \in \mathbb{N}} \Big(E_{\mathcal{A}_n} \sum_{k=n}^{\infty} |d_k f|^q\Big)^{1/q} \Big\|_\infty .$$

Set

$$\mathcal{BMO}_\infty = \mathcal{BMO}_\infty^- = H_\infty^{S_\infty}.$$

It is easy to see that the operator

$$X \mapsto f^X := (X_n r_{n-1},\ n \in \mathbb{N}) \tag{2}$$

maps $h_p^{S_q}$ in $H_p^{S_q}$ isometrically ($0 < p \le \infty$, $1 \le q \le \infty$). Indeed, the function $d_n f^X := X_n r_{n-1}$ is $\mathcal{A}_n$-measurable and integrable, of course. On the other hand,

$$E_{\mathcal{A}_{n-1}}(d_n f^X) = E_{\mathcal{F}_{n-1}}(X_n) E_{\mathcal{D}_{n-1}}(r_{n-1}) = 0$$

because the Rademacher functions are independent. Since $S_q(f^X) = S_q(X)$, our statement is proved. As $E_{\mathcal{A}_{n-1}}|d_n f^X|^q = E_{\mathcal{F}_{n-1}}|X_n|^q$, we have $s_q(f^X) = s_q(X)$, and so (2) is isometric from $h_p^{s_q}$ to $H_p^{s_q}$ ($0 < p \le \infty$, $1 \le q \le \infty$). Similarly, we can show that (2) is an isometry from bmo_q to $\mathcal{BMO}_q$ and from bmo_q^- to $\mathcal{BMO}_q^-$ ($1 \le q \le \infty$).

We can prove in the same way as Theorem 14 in Weisz [12] that the dual of $h_p^{S_q}$ is $h_{p'}^{S_{q'}}$, where $1 < p, q < \infty$ or $1 = q \le p < \infty$ and $1/p + 1/p' = 1/q + 1/q' = 1$. The following result, due to Dellacherie and Meyer [3] for $q = 2$, extends this result to $p = 1$.

THEOREM 1. *The dual of* $h_1^{S_q}$ *is* $bmo_{q'}^-$ *whenever* $1 \le q < \infty$ *and* $1/q + 1/q' = 1$.

Proof. Since the proof is similar to that of Theorem 1 in Weisz [10], we sketch it only. For $Y \in bmo_{q'}^-$ consider the functional

$$l_Y(X) := E\Big(\sum_{n=0}^{\infty} X_n Y_n\Big) \qquad (X \in h_q^{S_q}).$$

Notice that $h_q^{S_q}$ is dense in $h_1^{S_q}$. We verified in [12] that the dual of $H_1^{S_q}$ is $\mathcal{BMO}_{q'}^-$ with the same assumption on q and q' as in the theorem. Using this we conclude that

$$\begin{aligned} |l_Y(X)| &= \Big| \int_\Omega \int_0^1 \sum_{n=0}^{\infty} d_n f^X d_n f^Y \, dP \, d\lambda \Big| \\ &\le C \|f^X\|_{H_1^{S_q}} \|f^Y\|_{\mathcal{BMO}_{q'}^-} = C \|X\|_{h_1^{S_q}} \|Y\|_{bmo_{q'}^-}, \end{aligned}$$

which yields that l_Y is bounded on $h_1^{S_q}$.

Conversely, if l is in the dual of $h_1^{S_q}$ then it is also in the dual of $h_q^{S_q}$. Consequently, there exists $Y \in h_{q'}^{S_{q'}}$ such that

$$l(X) = E\Big(\sum_{n=0}^{\infty} X_n Y_n\Big) \qquad (X \in h_q^{S_q}). \tag{3}$$

On the other hand, l can be extended preserving its norm onto $H_1^{S_q}$. Therefore there exists $g \in \mathcal{BMO}_{q'}^-$ such that

$$(4) \qquad l(X) = l(f^X) = \int_\Omega \int_0^1 \sum_{n=0}^\infty X_n r_{n-1} d_n g \, dP \, d\lambda \qquad (X \in h_q^{S_q})$$

and

$$\|g\|_{\mathcal{BMO}_{q'}^-} \le C\|l\|.$$

It follows from (3) and (4) that

$$Y_n(\omega) = \int_0^1 r_{n-1}(x) d_n g(\omega, x) \, d\lambda(x).$$

Applying this we obtain

$$\begin{aligned} \|Y\|_{bmo_{q'}^-} &= \Big\| \sup_{n\in\mathbb{N}} \Big(E_{\mathcal{F}_n} \sum_{k=n}^\infty |Y_k|^{q'} \Big)^{1/q'} \Big\|_\infty \\ &\le \Big\| \sup_{n\in\mathbb{N}} \Big(E_{\mathcal{A}_n} \sum_{k=n}^\infty |d_k g|^{q'} \Big)^{1/q'} \Big\|_\infty = \|g\|_{\mathcal{BMO}_{q'}^-} \end{aligned}$$

and the theorem is proved. ∎

The following theorem can be proved similarly.

THEOREM 2. *The dual of* $h_p^{s_q}$ *is* $h_{p'}^{s_{q'}}$, *where* $1 < p \le q < \infty$ *or* $p \ge q \ge 2$ *and* $1/p+1/p' = 1/q+1/q' = 1$. *Moreover, the dual of* $h_1^{s_q}$ *is* $bmo_{q'}$ *provided that* $1 \le q < \infty$ *and* $1/q + 1/q' = 1$.

It is interesting to note that the duals of $bmo_{q'}$ and $bmo_{q'}^-$ are not $h_1^{s_q}$ and $h_1^{S_q}$, respectively. However, a kind of special subspaces of $bmo_{q'}$ and $bmo_{q'}^-$ can be defined, having duals $h_1^{s_q}$ and $h_1^{S_q}$, respectively.

Let vmo_q (resp. vmo_q^-) contain all elements $X \in bmo_q$ (resp. $X \in bmo_q^-$) for which

$$\lim_{n\to\infty} \Big\| \Big(E_{\mathcal{F}_n} \sum_{k=n+1}^\infty |X_k|^q \Big)^{1/q} \Big\|_\infty = 0$$

$$\Big(\text{resp.} \quad \lim_{n\to\infty} \Big\| \Big(E_{\mathcal{F}_n} \sum_{k=n}^\infty |X_k|^q \Big)^{1/q} \Big\|_\infty = 0\Big).$$

With the method used in Weisz [12] one can show that if every σ-algebra $\mathcal{F}_n$ is generated by finitely many atoms then the dual of $vmo_{q'}$ is $h_1^{s_q}$ and the dual of $vmo_{q'}^-$ is $h_1^{S_q}$ whenever $1 < q' < \infty$ and $1/q + 1/q' = 1$.

4. Inequalities. It follows from the convexity and concavity lemma (see Garsia [4], pp. 113–114) that

$$\Big\|\Big(\sum_{n=0}^{\infty} E_{\mathcal{F}_{n-1}}|X_n|^q\Big)^{1/q}\Big\|_p \le C_p\Big\|\Big(\sum_{n=0}^{\infty} |X_n|^q\Big)^{1/q}\Big\|_p \quad (q \le p < \infty), \tag{5}$$

and

$$\Big\|\Big(\sum_{n=0}^{\infty} |X_n|^q\Big)^{1/q}\Big\|_p \le C_p\Big\|\Big(\sum_{n=0}^{\infty} E_{\mathcal{F}_{n-1}}|X_n|^q\Big)^{1/q}\Big\|_p \quad (0 < p \le q),$$

where (X_n) is a sequence of $\mathcal{A}$-measurable functions. Note that by Hölder's inequality (1) follows from (5) for $q \le p < \infty$.

In case there exists a constant $R > 0$ such that for all $f \in L_1$ one has $E_{\mathcal{F}_n}|f| \le RE_{\mathcal{F}_{n-1}}|f|$ $(n \in \mathbb{N})$, the stochastic basis $(\mathcal{F}_n)$ is said to be *regular*. Since the sequence of dyadic σ-algebras is regular, it can easily be seen that whenever $(\mathcal{F}_n)$ is regular, so is $(\mathcal{A}_n)$. It is proved in [12] that in this case the spaces $H_p^{s_q}$ and $H_p^{S_q}$ are equivalent $(0 < p < \infty, 1 \le q < \infty)$. Hence $h_p^{s_q}$ and $h_p^{S_q}$ are also equivalent. This means, amongst other things, that if $(\mathcal{F}_n)$ is regular then (5) also holds for $0 < p < \infty$ and $1 \le q < \infty$ when (X_n) is an adapted function sequence. Consequently, under these conditions we obtain (1) for the parameters $0 < p < \infty$ and $1 \le q < \infty$.

If $(\mathcal{F}_n)$ is not regular then (1) is not true for $p = 1$ (see Lepingle [8]). However, if we take again adapted sequences then it holds for $p = 1$, too. The case $q = 2$ can also be found in Lepingle [8].

THEOREM 3. *If $(X_n, n \in \mathbb{N})$ is a sequence of adapted functions and $1 \le q < \infty$ then*

$$\Big\|\Big(\sum_{n=0}^{\infty} |E_{\mathcal{F}_{n-1}}X_n|^q\Big)^{1/q}\Big\|_1 \le C\Big\|\Big(\sum_{n=0}^{\infty} |X_n|^q\Big)^{1/q}\Big\|_1.$$

Proof. Since the dual of $L_1(l_q)$ is $L_\infty(l_{q'})$ $(1 \le q < \infty,\ 1/q + 1/q' = 1)$ we have

$$E\Big(\sum_{n=0}^{\infty} |E_{\mathcal{F}_{n-1}}X_n|^q\Big)^{1/q} = \sup_{\substack{Y \in L_\infty(l_{q'}) \\ \|Y\|_{L_\infty(l_{q'})} \le 1}} \Big|E\Big[\sum_{n=0}^{\infty} (E_{\mathcal{F}_{n-1}}X_n)Y_n\Big]\Big|.$$

By Theorem 1,

$$\Big|E\Big[\sum_{n=0}^{\infty} (E_{\mathcal{F}_{n-1}}X_n)Y_n\Big]\Big| \le C\|X\|_{h_1^{S_q}}\|(E_{\mathcal{F}_{n-1}}Y_n, n \in \mathbb{N})\|_{bmo_{q'}^-}.$$

The inequality $\|Y\|_{L_\infty(l_{q'})} \le 1$ implies

$$E_{\mathcal{F}_n} \sum_{k=n}^{\infty} |E_{\mathcal{F}_{k-1}} Y_k|^{q'} \le |E_{\mathcal{F}_{n-1}} Y_n|^{q'} + E_{\mathcal{F}_n} \sum_{k=n+1}^{\infty} |Y_k|^{q'} \le 2,$$

which shows that

$$\|(E_{\mathcal{F}_{n-1}} Y_n, n \in \mathbb{N})\|_{bmo^-_{q'}} \le 2^{1/q'}.$$

The proof of the theorem is complete. ■

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