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GENERALIZED PROJECTIONS OF BOREL AND ANALYTIC SETS

BY

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For a σ-ideal $\mathcal{I}$ of sets in a Polish space X and for $A \subseteq X^2$, we consider the generalized projection $\Phi(A)$ of A given by $\Phi(A) = \{x \in X : A_x \notin \mathcal{I}\}$, where $A_x = \{y \in X : \langle x, y\rangle \in A\}$. We study the behaviour of Φ with respect to Borel and analytic sets in the case when $\mathcal{I}$ is a Σ^0_2-supported σ-ideal. In particular, we give an alternative proof of the recent result of Kechris showing that $\Phi[\Sigma^1_1(X^2)] = \Sigma^1_1(X)$ for a wide class of Σ^0_2-supported σ-ideals.

1. Introduction. Throughout the paper, X is a fixed uncountable Polish space. We denote by $\mathcal{P}(X)$ the power set of X and by $\mathcal{B}(X)$ the family of all Borel sets in X. Let $\Sigma^0_\alpha(X)$ and $\Pi^0_\alpha(X)$ $(0 < \alpha < \omega_1)$ stand for subclasses of $\mathcal{B}(X)$ defined as in [Mo, 1B, 1F]. The families of all analytic sets and of all coanalytic sets in X will be written as $\Sigma^1_1(X)$ and $\Pi^1_1(X)$. Denote by 2^ω the Cantor space and by ω^ω the Baire space.

We consider proper σ-ideals of subsets of X, containing all singletons. A σ-ideal $\mathcal{I}$ is called *Σ^0_2-supported* if each set $A \in \mathcal{I}$ is contained in a set from $\mathcal{I} \cap \Sigma^0_2(X)$. A closed set $F \subseteq X$ is called *$\mathcal{I}$-perfect* if, for each open set $U \subseteq X$, the condition $U \cap F \neq \emptyset$ implies $\mathrm{cl}(U \cap F) \notin \mathcal{I}$ (where $\mathrm{cl}(E)$ denotes the closure of E). The family of all $\mathcal{I}$-perfect sets will be written as $\mathcal{M}_{\mathcal{I}}$. We say that $\mathcal{I}$ satisfies the *countable chain condition* (in short ccc) if each disjoint subfamily of $\mathcal{B}(X) \setminus \mathcal{I}$ is countable. Following [KS], for a family $\mathcal{F} \subseteq \mathcal{P}(X)$, we define

$$\mathrm{MGR}(\mathcal{F}) = \{E \subseteq X : (\forall A \in \mathcal{F})(E \cap A \text{ is meager in } A)\}.$$

Let us quote two latest results on Σ^0_2-supported σ-ideals.

THEOREM 1.1 [KS, Th. 2]. *Let $\mathcal{I} \subseteq \mathcal{P}(X)$ be a Σ^0_2-supported σ-ideal. Then precisely one of the following possibilities holds*:

(i) $\mathcal{I} = \mathrm{MGR}(\mathcal{F})$ *for a countable family $\mathcal{F}$ of closed subsets of X*,

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(ii) *there is a homeomorphic embedding* $h : 2^\omega \times \omega^\omega \to X$ *such that* $h[\{x\} \times \omega^\omega] \notin \mathcal{I}$ *for each* $x \in 2^\omega$.

Observe that (i) implies that $\mathcal{I}$ satisfies ccc, and (ii) implies that it does not. Thus (i) yields a characterization of Σ^0_2-supported σ-ideals satisfying ccc, and (ii) yields the characterization of Σ^0_2-supported σ-ideals without ccc.

THEOREM 1.2 [So]. *If* $\mathcal{I} \subseteq \mathcal{P}(X)$ *is a* Σ^0_2*-supported* σ*-ideal then*, *for each* $A \in \Sigma^1_1(X)$, *either* $A \in \mathcal{I}$ *or there is an* $\mathcal{I}$*-perfect set* $F \subseteq X$ *such that* $A \cap F$ *is comeager in* F.

Theorem 1.2 is an equivalent version of the original formulation (cf. [So, Th. 1; Remark (2), p. 1024]) and it generalizes the result of Petruska [P] dealing with the σ-ideal of sets that can be covered by F_σ Lebesgue null sets in $[0, 1]$.

For a σ-ideal $\mathcal{I} \subseteq \mathcal{P}(X)$ we consider the generalized projection $\Phi_{\mathcal{I}} : \mathcal{P}(X^2) \to \mathcal{P}(X)$ (denoted further by Φ) given by

$$\Phi(E) = \{x \in X : E_x \notin \mathcal{I}\}, \quad E \in \mathcal{P}(X^2),$$

where $E_x = \{y \in X : \langle x, y\rangle \in E\}$ for $x \in X$. Note that if $\mathcal{I} = \{\emptyset\}$ then $\Phi(E)$ is exactly the projection of E onto the first factor. If $\mathcal{I}$ is one of the following σ-ideals:

- of all meager sets in X,
- of all Lebesgue null sets in $\mathbb{R}$,
- of all countable sets in X,

then

$$(*) \qquad \Phi[\Sigma^1_1(X^2)] = \Sigma^1_1(X).$$

These are classical results; compare [Ke, 29.E]. Note that the inclusion "$\supseteq$" in $(*)$ is obvious since for each $A \in \Sigma^1_1(X)$ we have $A \times X \in \Sigma^1_1(X^2)$ and $A = \Phi(A \times X)$. Following [Sh], if $(*)$ holds, $\mathcal{I}$ is called Σ^1_1-*definable*. For the first two σ-ideals listed above, we additionally have

$$(**) \qquad \Phi[\Sigma^0_\alpha(X^2)] = \Sigma^0_\alpha(X), \quad 0 < \alpha < \omega_1$$

(cf. e.g. [G, Th. 2.2]). For Mycielski σ-ideals [My] in $X = 2^\omega$, the behaviour of Φ with respect to Borel and projective subclasses was studied in [BR]; then $(*)$ does not hold since $\Phi[\Sigma^1_1(X^2)] = \Pi^1_2(X)$. Further results for generalized Mycielski σ-ideals are contained in [R]. For special product σ-ideals, condition $(*)$ was proved in [Sh]. We are going to verify conditions $(*)$ and $(**)$ for Σ^0_2-supported σ-ideals.

2. An alternative proof of a theorem of Kechris. We denote by $\mathrm{CL}(X)$ the space of all closed subsets of X. It is known [Ke, Th. 12.6] that there exists a Polish topology τ on $\mathrm{CL}(X)$ such that the σ-algebra of Borel sets with respect to τ is identical with the σ-algebra generated by the sets

$$\boldsymbol{W}(G) = \{F \in \mathrm{CL}(X) : F \cap G \neq \emptyset\},$$

where G varies over open subsets of X. That is the *Effros Borel structure* of $\mathrm{CL}(X)$. We also consider the sets

$$\boldsymbol{V}(G) = \{F \in \mathrm{CL}(X) : F \subseteq G\}$$

for open sets $G \subseteq X$. Recall that, if X is compact, the topology generated by the subbase consisting of the sets $\boldsymbol{V}(G), \boldsymbol{W}(G)$ (where G varies over open subsets of X) is the *Vietoris topology* on the hyperspace $\mathcal{K}(X)$ of compact subsets of X. In that case $\mathcal{K}(X)$ is compact (and metrizable by the Hausdorff distance), and the Effros Borel structure of $\mathrm{CL}(X)$ is identical with $\mathcal{B}(\mathcal{K}(X))$ (cf. [Ke, 12.11]). Consequently, for a compact X, we may assume that the above-mentioned topology τ is equal to the Vietoris topology (then we will treat the topological spaces $\mathrm{CL}(X)$ and $\mathcal{K}(X)$ as identical). Note that, for a general Polish space X, sets $\boldsymbol{V}(G)$ are coanalytic in τ and they need not be Borel (cf. [Ke, 27.7]).

From a recent result of Kechris [Ke, Th. 35.38] one immediately obtains the following theorem.

THEOREM 2.1. *If $\mathcal{I} \subseteq \mathcal{P}(X)$ is a Σ^0_2-supported σ-ideal such that $\mathcal{I} \cap \mathrm{CL}(X) \in \Pi^1_1(\mathrm{CL}(X))$ then $\Phi[\Sigma^1_1(X^2)] = \Sigma^1_1(X)$.*

In this section we give an alternative proof of Theorem 2.1. Our argument uses Theorem 1.2 and some descriptive set-theoretic facts involving $\mathrm{CL}(X)$ and meager sets which can be of independent interest. Our previous version of Theorem 2.1 working with $\mathcal{K}(X)$ had a similar proof. At the time we were not aware of the existence of its general version in [Ke]. We would like to thank J. Pawlikowski who has informed us about it.

From now on, fix countable bases $\langle U_n\rangle_{n\in\omega}$ and $\langle V_n\rangle_{n\in\omega}$ of nonempty open sets in X and ω^ω, respectively. Fix also a bijection $r : \omega \times \omega \to \omega$.

PROPOSITION 2.1. *If $\mathcal{I} \subseteq \mathcal{P}(X)$ is a σ-ideal such that $\mathcal{I} \cap \mathrm{CL}(X) \in \Pi^1_1(\mathrm{CL}(X))$ then the set $\mathcal{M}_{\mathcal{I}}$ of all $\mathcal{I}$-perfect sets in X belongs to $\Sigma^1_1(\mathrm{CL}(X))$.*

Proof. For a fixed open set $U \subseteq X$, consider the mapping $g_U : \mathrm{CL}(X) \to \mathrm{CL}(X)$ given by $g_U(F) = \mathrm{cl}(U \cap F)$ for $F \in \mathrm{CL}(X)$. Thus, for an open set $G \subseteq X$, we have

$$g_U^{-1}[\boldsymbol{W}(G)] = \boldsymbol{W}[U \cap G].$$

Hence g_U is Borel measurable. If $F \in \mathrm{CL}(X)$ then

$$F \in \mathcal{M}_{\mathcal{I}} \Leftrightarrow (\forall n \in \omega)(U_n \cap F = \emptyset \vee g_{U_n}(F) \notin \mathcal{I} \cap \mathrm{CL}(X)).$$

Now, the assertion follows from the assumption and the Borelness of g_{U_n}. ∎

Remark. In some cases the conclusion of Proposition 2.1 is not sharp. For instance, if X is metric and compact, and $\mathcal{I}$ consists of all countable sets in X then $\mathcal{I}\cap\mathcal{K}(X)$ is in $\Pi^1_1(\mathcal{K}(X))\setminus\Sigma^1_1(\mathcal{K}(X))$ [Ku, §42,III]. But $\mathcal{M}_{\mathcal{I}}$ consists of all perfect sets in X and it forms a G_δ set in $\mathcal{K}(X)$ [Ku, §42,II, Th. 3].

The next two propositions are modified versions of classical results.

For a Polish space Z and $A\subseteq Z\times X$, we define

$$A^*=\{\langle z,F\rangle\in Z\times \mathrm{CL}(X): A_z\cap F \text{ is nonmeager in } F\},$$
$$A^{**}=\{\langle z,F\rangle\in Z\times \mathrm{CL}(X): A_z\cap F \text{ is comeager in } F\}.$$

Proposition 2.2. *If* $A\in\mathcal{B}(Z\times X)$ *then* $A^*,A^{**}\in\mathcal{B}(Z\times\mathrm{CL}(X))$.

Proof. First let $A\in\Sigma^0_1(Z\times X)$. Then for $\langle z,F\rangle\in Z\times\mathrm{CL}(X)$ we have

$$\langle z,F\rangle\in A^*\Leftrightarrow(\exists m,n\in\omega)(F\cap U_m\neq\emptyset \ \&\ z\in U_n\ \&\ U_n\times U_m\subseteq A).$$

Hence $A^*\in\mathcal{B}(Z\times\mathrm{CL}(X))$. Assume that $1<\alpha<\omega_1$ and that the assertion holds for sets from $\bigcup_{\beta<\alpha}\Sigma^0_\beta(Z\times X)$. For instance, let α be a successor. If $A\in\Sigma^0_\alpha(Z\times X)$, $A=\bigcup_{n\in\omega}A_n$ and $A_n\in\Pi^0_{\alpha-1}(Z\times X)$ for $n\in\omega$, then

$$\begin{aligned}A^*&=\bigcup_{n\in\omega}A^*_n\\&=\bigcup_{n,k\in\omega}\{\langle z,F\rangle\in Z\times\mathrm{CL}(X):U_k\cap F\neq\emptyset\\&\qquad\qquad\&\ U_k\cap F\setminus(A_n)_z\text{ is meager in }F\}\\&=\bigcup_{n,k\in\omega}((Z\times\boldsymbol{W}(U_k))\setminus((Z\times U_k)\setminus A_n)^*).\end{aligned}$$

Hence $A^*\in\mathcal{B}(Z\times\mathrm{CL}(X))$, by the induction hypothesis. If α is a limit number, the proof is similar.

The assertion for A^{**} follows from $A^{**}=(Z\times\mathrm{CL}(X))\setminus((Z\times X)\setminus A)^*$. ■

Proposition 2.3. *If* $A\in\Sigma^1_1(X^2)$ *then* $A^*,A^{**}\in\Sigma^1_1(X\times\mathrm{CL}(X))$.

Proof (cf. [Mo, 4F.19]). First we show the assertion for A^{**}. Assume that A is the projection of a closed set $B\subseteq X^2\times\omega^\omega$ along ω^ω. Define H as the set of all $\langle\varepsilon,x,F\rangle\in\omega^\omega\times X\times\mathrm{CL}(X)$ satisfying the formula

$$(\forall k,n\in\omega)(((\varepsilon\circ r)(k,n)=1\ \&\ F\cap U_k\neq\emptyset)\Rightarrow B_x\cap((F\cap U_k)\times V_n)\neq\emptyset).$$

Let D consist of all $\langle\varepsilon,y\rangle\in\omega^\omega\times X$ satisfying the formula

$$\begin{aligned}&(\exists k,n\in\omega)((\varepsilon\circ r)(k,n)=1\ \&\ y\in U_k)\ \&\\&(\forall k,n,p\in\omega)(((\varepsilon\circ r)(k,n)=1\ \&\ y\in U_k)\\&\Rightarrow(\exists k',n'\in\omega)((\varepsilon\circ r)(k',n')=1\ \&\ y\in U_{k'}\\&\&\ U_{k'}\subseteq U_k\ \&\ V_{n'}\subseteq V_n\ \&\ \operatorname{diam}(U_{k'})<2^{-p}\ \&\ \operatorname{diam}(V_{n'})<2^{-p})).\end{aligned}$$

(Here $\mathrm{diam}(E)$ denotes the diameter of a set E.)

Now, we will prove that for $\langle x,F\rangle \in X\times \mathrm{CL}(X)$ we have

$$(\triangle)\qquad \langle x,F\rangle \in A^{**} \Leftrightarrow (\exists \varepsilon \in \omega^\omega)(\langle \varepsilon,x,F\rangle \in H \ \&\ \langle \varepsilon,F\rangle \in D^{**}).$$

To show "$\Rightarrow$" in $(\triangle)$, consider $\langle x,F\rangle \in A^{**}$. Hence $A_x\cap F$ is comeager in F. By the Jankov–von Neumann selection theorem [Mo, 4E.9] we can find a function $f: F\to \omega^\omega$ with the Baire property which uniformizes $B_x\cap(F\times\omega^\omega)$. Choose a G_δ set $C\subseteq F$ comeager in F and such that $f|_C$ is continuous. Pick any $\varepsilon\in\omega^\omega$ such that

$$(\forall k,n\in\omega)((\varepsilon\circ r)(k,n)=1 \Leftrightarrow (U_k\cap C\neq\emptyset \ \&\ f[U_k\cap C]\subseteq V_n)).$$

Using the fact that $A_x\cap F$ is comeager in F, we see that $\langle \varepsilon,x,F\rangle\in H$. Additionally, $C\subseteq D_\varepsilon\cap F$ by the continuity of $f|_C$. Since C is comeager in F, therefore $D_\varepsilon\cap F$ is comeager in F. Hence $\langle\varepsilon,F\rangle\in D^{**}$. To show "$\Leftarrow$" in $(\triangle)$, assume that $\langle \varepsilon,x,F\rangle\in H$ and $\langle\varepsilon,F\rangle\in D^{**}$ for some $\varepsilon\in\omega^\omega$. Let $y\in D_\varepsilon\cap F$. Thus we can define inductively subsequences $\langle U_{k_i}\rangle_{i\in\omega}$ and $\langle V_{n_i}\rangle_{i\in\omega}$ such that

$$y\in F\cap U_{k_i},\quad U_{k_{i+1}}\subseteq U_{k_i},\quad V_{n_{i+1}}\subseteq V_{n_i}$$

and

$$\mathrm{diam}(U_{k_i})<2^{-i},\quad \mathrm{diam}(V_{n_i})<2^{-i},\quad B_x\cap((F\cap U_{k_i})\times V_{n_i})\neq\emptyset$$

for each $i\in\omega$. Hence there is a Cauchy sequence $\langle y_i,z_i\rangle\in B_x\cap(F\times\omega^\omega)$ and it tends to $\langle y,z\rangle$ for some $z\in\omega^\omega$. Since B_x and F are closed, we have $\langle y,z\rangle\in B_x\cap(F\times\omega^\omega)$ and thus $y\in A_x\cap F$. We have shown that $D_\varepsilon\cap F\subseteq A_x\cap F$. Now, from $\langle\varepsilon,F\rangle\in D^{**}$ it follows that $\langle x,F\rangle\in A^{**}$.

Finally, observe that $H\in\Sigma^1_1(\omega^\omega\times X\times\mathrm{CL}(X))$ and $D\in\mathcal{B}(\omega^\omega\times X)$. Thus $D^{**}\in\mathcal{B}(\omega^\omega\times\mathrm{CL}(X))$ by Proposition 2.2, and $(\triangle)$ yields the conclusion.

To show the assertion for A^*, notice that for $\langle x,F\rangle\in X\times\mathrm{CL}(X)$ we have

$$\langle x,F\rangle\in A^*\Leftrightarrow(\exists n\in\omega)(U_n\cap F\neq\emptyset \ \&\ \langle x,g_{U_n}(F)\rangle\in A^{**}),$$

where $g_{U_n}(F)=\mathrm{cl}(U_n\cap F)$. Since g_{U_n} is Borel measurable (compare the proof of Proposition 2.1), the proof is finished. ∎

Now, we are ready to prove Theorem 2.1. By Theorem 1.2, for any $A\in\Sigma^1_1(X^2)$ and $x\in X$, we have

$$A_x\notin\mathcal{I}\Leftrightarrow(\exists F\in\mathrm{CL}(X))(F\in\mathcal{M}_{\mathcal{I}} \ \&\ \langle x,F\rangle\in A^{**}).$$

By Propositions 2.1 and 2.3, the formula $F\in\mathcal{M}_{\mathcal{I}}$ & $\langle x,F\rangle\in A^{**}$ defines a set in $\Sigma^1_1(X\times\mathrm{CL}(X))$. Thus $\Phi(A)\in\Sigma^1_1(X)$.

Remarks. (a) In the case when X is metric and compact, one can assume in Theorem 2.1 that $\mathcal{I}\cap\mathcal{K}(X)\in\Sigma^1_1(\mathcal{K}(X))\cup\Pi^1_1(\mathcal{K}(X))$ since, by [KLW, Th. 11], if $\mathcal{I}\cap\mathcal{K}(X)\in\Sigma^1_1(\mathcal{K}(X))$ then $\mathcal{I}\cap\mathcal{K}(X)\in\Pi^0_2(\mathcal{K}(X))$.

Note that the collection of Π^1_1 σ-ideals of compact sets is quite wide (cf. [Ke, 33.C]).

(b) Observe that there are Σ^1_1-definable σ-ideals which need not be Σ^0_2-supported. For instance, the σ-ideal of Lebesgue null sets in $\mathbb{R}$ is not Σ^0_2-supported but it satisfies the statement of Theorem 2.1. Nevertheless, the assumption that $\mathcal{I}$ is Σ^0_2-supported cannot be omitted, which follows from [BR, Th.3.1(b)], where $\Phi[\Sigma^1_1(X^2)] = \Pi^1_2(X)$ and $\mathcal{I} \cap \mathcal{K}(X) \in \Pi^0_2(\mathcal{K}(X))$ [BR, Corollary 2.2].

3. Further results

THEOREM 3.1. *Let $\mathcal{I} \subseteq \mathcal{P}(X)$ be a Σ^0_2-supported σ-ideal.*

(a) *If $\mathcal{I}$ satisfies ccc then*

$$\Phi[\Sigma^0_\alpha(X^2)] = \Sigma^0_\alpha(X) \quad \text{for } \alpha < \omega_1 \quad \text{and} \quad \Phi[\Sigma^1_1(X^2)] = \Sigma^1_1(X).$$

(b) *If $\mathcal{I}$ does not satisfy ccc then $\Sigma^1_1(X) \subseteq \Phi[\Pi^0_3(X^2)]$.*

(c) *If $\mathcal{I} \cap \mathrm{CL}(X) \in \Pi^1_1(\mathrm{CL}(X))$ and $\mathcal{I}$ does not satisfy ccc then*

$$\Phi[\Pi^0_3(X^2)] = \Phi[\Sigma^1_1(X^2)] = \Sigma^1_1(X).$$

Proof. (a) Since $\mathcal{I}$ satisfies ccc, condition (i) of Theorem 1.1 holds. Let the $\mathcal{F}$ appearing there consist of closed sets F_n, $n \in \omega$. For $A \subseteq F_n \times X$ put

$$\Phi_n(A) = \{x \in F_n : A_x \notin \mathrm{MGR}(F_n)\}.$$

Since

$$\Phi(E) = \bigcup_{n\in\omega} \Phi_n(E \cap (F_n \times X)) \quad \text{for } E \subseteq X^2,$$

the assertion follows from the analogous properties of the operators Φ_n.

(b) Since $\mathcal{I}$ does not satisfy ccc, condition (ii) of Theorem 1.1 holds. If $h : 2^\omega \times \omega^\omega \to X$ is the embedding appearing in that condition, the set $B = h[2^\omega \times \omega^\omega]$ is of type G_δ in X [Ku, §35,III]. We can extend the continuous function $\mathrm{pr}_1 \circ h^{-1} : B \to 2^\omega$ to a Baire 1 function $f : X \to 2^\omega$ [Ku, §35,VI]. (Here $\mathrm{pr}_1 : 2^\omega \times \omega^\omega \to 2^\omega$ stands for the projection on the first factor.) Then

$$f^{-1}[\{t\}] \supseteq h[\mathrm{pr}_1^{-1}[\{t\}]] = h[\{t\} \times \omega^\omega] \notin \mathcal{I}$$

for each $t \in 2^\omega$. Let $A \in \Sigma^1_1(X)$. Pick $D \in \Pi^0_2(X \times 2^\omega)$ so that A is the projection of D along 2^ω. Put

$$E = \{\langle x, y\rangle \in X^2 : \langle x, f(y)\rangle \in D\}.$$

Then $E \in \Pi^0_3(X^2)$ and $A = \Phi(E)$. (The final part of that argument is derived from [B, Proposition 2.4].)

Assertion (c) is a consequence of (b) and Theorem 2.1. ∎

Let us show one simple application.

COROLLARY 3.1. *If $\mathcal{I}$ is the σ-ideal of all sets in $X = \mathbb{R}$ that can be covered by F_σ Lebesgue null sets then*

$$\Phi[\Pi^0_3(X^2)] = \Phi[\Sigma^1_1(X^2)] = \Sigma^1_1(X).$$

Proof. The σ-ideal $\mathcal{I}$ is Σ^0_2-supported, not-ccc (cf. [B]), and $\mathcal{I} \cap \mathrm{CL}(X) \in \Pi^1_1(\mathrm{CL}(X))$ (cf. [Ke, p. 292]).

Remark. We do not know whether Π^0_3 can be replaced by Π^0_2 in the above corollary. Obviously that is possible when $\mathcal{I} = \{\emptyset\}$ and also when $\mathcal{I}$ consists of all countable sets in X (cf. [Ke, Example 29.21]).

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