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A LITTLE MORE ON THE PRODUCT OF TWO PSEUDOCOMPACT SPACES

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0. Introduction. The main aim of this paper is to answer the question of when $\beta X \times \beta Y$ is the Wallman compactification of $X \times Y$ with respect to the normal base consisting of the zero-sets of all those continuous real functions defined on $X \times Y$ which are continuously extendable over $\beta X \times \beta Y$. In passing, we shall obtain several new conditions which are necessary and sufficient for $X \times Y$ to be pseudocompact.

To provide a framework for our discussion, let us recall that a *normal base* $\mathcal{D}$ for a Tikhonov space X is a base for the closed sets of X which is stable under finite unions and finite intersections and has the following properties:

(i) $\emptyset, X \in \mathcal{D}$;

(ii) if $A \in \mathcal{D}$ and $x \in X \setminus A$, then there exists $B \in \mathcal{D}$ such that $x \in B \subseteq X \setminus A$;

(iii) if $A, B \in \mathcal{D}$ and $A \cap B = \emptyset$, then there exist $C, D \in \mathcal{D}$ such that $A \subseteq X \setminus C \subseteq D \subseteq X \setminus B$.

The *Wallman compactification* of X with respect to a normal base $\mathcal{D}$ is the space $w_{\mathcal{D}}X$ of all ultrafilters in $\mathcal{D}$ which has the collection

$$\{\{p \in w_{\mathcal{D}}X : D \in p\} : D \in \mathcal{D}\}$$

as a base for the closed sets (cf. [2; Section 8], [12; Section 4.4] or [6]). Let us mention that V. M. Ul'yanov gave in [14] a solution to the famous problem of O. Frink on Wallman compactifications (cf. [6]) by proving that a compactification of a Tikhonov space need not be of Wallman type.

All the spaces considered below are assumed to be completely regular and Hausdorff. As usual, the symbol $C(X)$ will stand for the algebra of continuous real functions defined on X, and $C^*(X)$ for the subalgebra of $C(X)$ consisting of bounded functions.

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One of the most natural normal bases associated with a compactification αX of a space X is the collection $Z_\alpha(X) = \{f^{-1}(0) : f \in C_\alpha(X)\}$ where $C_\alpha(X)$ is the family of all those functions $f \in C^*(X)$ which are continuously extendable over αX. For simplicity, we shall put $Z(X) = Z_\beta(X)$ with β standing for the Čech–Stone compactification. Denote by $w_\alpha X$ the Wallman compactification of X with respect to $Z_\alpha(X)$. It is well known that $\beta X = w_\beta X$ (cf. [12; 4.4(h)]). The inequality $\alpha X \le w_\alpha X$ always holds; however, in general, $\alpha X \neq w_\alpha X$ (cf. [16]). Corollary 3.4 of [16] asserts that $\alpha X = w_\alpha X$ for every compactification αX of X if and only if the space X is pseudocompact. This gives a full description of the structure of all compactifications of $X \times Y$ in the case when $X \times Y$ is pseudocompact.

For compactifications αX and γY of spaces X and Y, respectively, denote by $\alpha \times \gamma\langle X \times Y\rangle$ the compactification $\alpha X \times \gamma Y$ of $X \times Y$.

If we are given two pseudocompact spaces X and Y such that $X \times Y$ is not pseudocompact, we can deduce from the above-mentioned Corollary 3.4 of [16] that there exists a compactification $\alpha\langle X \times Y\rangle$ of $X \times Y$ such that $\alpha\langle X \times Y\rangle \neq w_\alpha\langle X \times Y\rangle$; however, we do not know which one of the compactifications $\alpha\langle X \times Y\rangle$ of $X \times Y$ fails to be equivalent to $w_\alpha\langle X \times Y\rangle$. In view of Glicksberg's theorem, for infinite spaces X and Y, the equality $\beta X \times \beta Y = \beta\langle X \times Y\rangle$ holds if and only if the product $X \times Y$ is pseudocompact (cf. [10]). Therefore, if X and Y are pseudocompact spaces such that the product $X \times Y$ is not pseudocompact, then $\beta X \times \beta Y \neq \beta\langle X \times Y\rangle$ and it seems natural to ask whether a compactification $\alpha\langle X \times Y\rangle \le \beta X \times \beta Y$ can be non-equivalent to $w_\alpha\langle X \times Y\rangle$. In the present paper, among other things, we shall prove that if X and Y are infinite Tikhonov spaces, then $\beta X \times \beta Y = w_{\beta\times\beta}\langle X \times Y\rangle$ if and only if both the spaces X and Y are pseudocompact, which holds if and only if $\alpha\langle X \times Y\rangle = w_\alpha\langle X \times Y\rangle$ for every compactification $\alpha\langle X \times Y\rangle \le \beta X \times \beta Y$. This result, together with Glicksberg's theorem, describes the structure of all compactifications of $X \times Y$ in the case when $X \times Y$ is pseudocompact, and the structure of all compactifications smaller than $\beta X \times \beta Y$ in the case when both X and Y are pseudocompact but their product $X \times Y$ is not necessarily pseudocompact. Our result seems a little striking if one recollects that F. Kost proved in [11] that the product of Wallman type compactifications is of Wallman type; furthermore, $\beta X \times \beta Y$ is always the Wallman compactification with respect a normal base consisting of some zero-sets.

1. $\beta X \times \beta Y$ as a Wallman type compactification. Before proceeding to the body of this section, let us establish some useful facts.

The following proposition is an immediate consequence of Lemmas 1.1 and 2.1 of [17]:

1.1. PROPOSITION. *For any compactifications αX of X and γY of Y, we have*

$$Z_{\alpha\times\gamma}(X\times Y)=\Big\{\bigcap_{i=1}^{\infty}\bigcup_{j=1}^{n_i}[f_{i,j}^{-1}(0)\times g_{i,j}^{-1}(0)] : f_{i,j}\in C_\alpha(X)\ \&\ g_{i,j}\in C_\gamma(Y)\ \text{for } i\in\mathbb{N},\ j=1,\ldots,n_i\ (n_i\in\mathbb{N})\Big\}.$$

Our next proposition can easily be deduced from Theorems 2.2 and 2.8 of [16].

1.2. PROPOSITION. *For every compactification αX of X, the following conditions are equivalent*:

(i) $\alpha X = w_\alpha X$;

(ii) *for any disjoint $Z_1, Z_2 \in Z_\alpha(X)$, we have*

$$\mathrm{cl}_{\alpha X} Z_1 \cap \mathrm{cl}_{\alpha X} Z_2 = \emptyset;$$

(iii) *for any $f, g \in C_\alpha(X)$ such that $f^{-1}(0)\cap g^{-1}(0)=\emptyset$, the function*

$$h=\frac{|f|}{|f|+|g|}$$

is continuously extendable over αX.

1.3. COROLLARY. *If $\alpha X = w_\alpha X$ and $X \subseteq T \subseteq \alpha X$, then αX is the Wallman compactification of T arising from the normal base $Z_\alpha(T) = \{f^{-1}(0)\cap T : f \in C(\alpha X)\}$.*

Proof. Take any $f, g \in C(\alpha X)$ such that $f^{-1}(0)\cap T\cap g^{-1}(0)=\emptyset$. Put $h(t) = |f(t)|/(|f(t)|+|g(t)|)$ for $t \in T$. Then, by 1.2, the function $h{\restriction}X$ has a continuous extension over αX, which, together with the density of X in T, implies that h is continuously extendable over αX. The proof is completed by applying 1.2 once again.

1.4. COROLLARY. *Let αX and γX be compactifications of X such that $\alpha X \le \gamma X$. If $\alpha X \ne w_\alpha X$, then there exists a set $Z \in Z(\gamma X)$ such that $\emptyset \ne Z \subseteq \gamma X \setminus X$.*

Proof. It follows from 1.2 that there exist functions $f_1, f_2 \in C_\alpha(X)$ such that $f_1^{-1}(0)\cap f_2^{-1}(0)=\emptyset$ but $\widetilde{f}_1^{-1}(0)\cap\widetilde{f}_2^{-1}(0)\ne\emptyset$, where $\widetilde{f}_i$ is the continuous extension of f_i over αX $(i=1,2)$. Put $Z=\pi^{-1}[\widetilde{f}_1^{-1}(0)\cap\widetilde{f}_2^{-1}(0)]$, where $\pi:\gamma X\to\alpha X$ is the quotient map showing that $\alpha X\le\gamma X$. Then $\emptyset\ne Z\in Z(\gamma X)$ and $Z\subseteq\gamma X\setminus X$.

We shall make use of the following theorem which can be deduced from Theorem 3.10 of [16] and Problem 3.12.16(a) of [5].

1.5. THEOREM. *A non-pseudocompact Tikhonov space X is Lindelöf if and only if $\alpha X \neq w_\alpha X$ for any compactification αX of X non-equivalent to βX.*

Let us say that a family $\mathcal{E}$ of subsets of X is *semicompact* if, for any sequence $\langle E_n\rangle$ of members of $\mathcal{E}$ with $\bigcap_{n=1}^{\infty} E_n = \emptyset$, there exists $m \in \mathbb{N}$ such that $\bigcap_{n=1}^{m} E_n = \emptyset$. Recall the well-known characterization of pseudocompactness which follows from [9; 5H(4)].

1.6. PROPOSITION. *A Tikhonov space X is pseudocompact if and only if the family $Z(X)$ is semicompact.*

Now, we are in a position to prove the main result of this section.

1.7. THEOREM. *For infinite Tikhonov spaces X and Y, the following conditions are equivalent*:

(i) *both X and Y are pseudocompact*;

(ii) *the collection $Z_{\beta\times\beta}(X \times Y)$ is semicompact*;

(iii) *X is pseudocompact and the projection $p_X : X \times Y \to X$ carries any member of $Z_{\beta\times\beta}(X \times Y)$ onto a closed subset of X*;

(iv) *X is pseudocompact and, for each $Z \in Z_{\beta\times\beta}(X \times Y)$,*

$$\mathrm{cl}_{X\times\beta Y}(Z) = \bigcup_{x\in X} \mathrm{cl}_{X\times\beta Y}[Z \cap (\{x\} \times Y)];$$

(v) *$\beta X \times \beta Y$ is the Wallman compactification of $X \times Y$ with respect to the normal base $Z_{\beta\times\beta}(X \times Y)$*;

(vi) *every compactification $\alpha\langle X \times Y\rangle$ of $X \times Y$ smaller than $\beta X \times \beta Y$ is the Wallman compactification of $X \times Y$ with respect to the normal base $Z_\alpha(X \times Y)$.*

Proof. We shall show that (i)⇔(ii)⇒(iii)⇒(iv)⇒(v) and that (ii)⇒(vi). The implication (vi)⇒(v) is obvious.

Assume that (i) holds and suppose that $\langle Z_n\rangle$ is a sequence of members of $Z_{\beta\times\beta}(X \times Y)$ such that $\bigcap_{n=1}^{m} Z_n \neq \emptyset$ for each $m \in \mathbb{N}$. By 1.1, there exist functions $f_{i,j,n} \in C(X)$ and $g_{i,j,n} \in C(Y)$ such that

$$Z_n = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{m(n,i)} [f_{i,j,n}^{-1}(0) \times g_{i,j,n}^{-1}(0)].$$

Put

$$A_k = \bigcap_{n=1}^{k} \bigcap_{i=1}^{k} \bigcup_{j=1}^{m(n,i)} [f_{i,j,n}^{-1}(0) \times g_{i,j,n}^{-1}(0)].$$

A straightforward calculation shows that $\bigcap_{n=1}^{\infty} Z_n = \bigcap_{k=1}^{\infty} A_k$ and $A_{k+1} \subseteq A_k$ for $k \in \mathbb{N}$. Let $B_k = p_X(A_k)$ for $k \in \mathbb{N}$. As $\emptyset \neq \bigcap_{n=1}^{k} Z_n \subseteq A_k$, we have $B_k \neq \emptyset$ for any $k \in \mathbb{N}$. Clearly, $B_{k+1} \subseteq B_k$ for $k \in \mathbb{N}$. Since A_k can be represented in the form $\bigcup_{p=1}^{r} [f_p^{-1}(0) \times g_p^{-1}(0)]$ for some $f_p \in C(X)$ and $g_p \in C(Y)$ $(p = 1, \ldots, r)$, the sets B_k are zero-sets in X. It follows from 1.6 that there exists $x_0 \in \bigcap_{k=1}^{\infty} B_k$. Then $(\{x_0\} \times Y) \cap A_k \neq \emptyset$ for each $k \in \mathbb{N}$. Since Y is pseudocompact, we have $(\{x_0\} \times Y) \cap \bigcap_{k=1}^{\infty} A_k \neq \emptyset$ by 1.6, which implies that $\bigcap_{n=1}^{\infty} Z_n \neq \emptyset$. This proves that (i)⇒(ii).

Assume (ii). Suppose that $Z_n \in Z(X)$ and $\bigcap_{n=1}^{\infty} Z_n = \emptyset$. Then $Z_n \times Y \in Z_{\beta\times\beta}(X \times Y)$ and $\bigcap_{n=1}^{\infty} (Z_n \times Y) = \emptyset$. Hence there is $m \in \mathbb{N}$ such that $\bigcap_{n=1}^{m} (Z_n \times Y) = \emptyset$, which shows that $Z(X)$ is semicompact. By 1.6, X is pseudocompact. Similarly, Y is pseudocompact, too.

Let $Z \in Z_{\beta\times\beta}(X \times Y)$ be represented in the form

$$Z = \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{n_i} [f_{i,j}^{-1}(0) \times g_{i,j}^{-1}(0)],$$

where $f_{i,j} \in C(X)$ and $g_{i,j} \in C(Y)$ for $i \in \mathbb{N}$ and $j = 1, \ldots, n_i$ $(n_i \in \mathbb{N})$ (cf. 1.1). Suppose that $x_0 \notin p_X(Z)$. Put $C_k = \bigcap_{i=1}^{k} \bigcup_{j=1}^{n_i} [f_{i,j}^{-1}(0) \times g_{i,j}^{-1}(0)]$ for $k \in \mathbb{N}$. Since $(\{x_0\} \times Y) \cap Z = \emptyset$ and $Z = \bigcap_{k=1}^{\infty} C_k$, it follows from the pseudocompactness of Y that there is $k_0 \in \mathbb{N}$ such that $(\{x_0\} \times Y) \cap C_{k_0} = \emptyset$ (cf. 1.6). Obviously, $p_X(C_{k_0})$ is a zero-set in X, $p_X(Z) \subseteq p_X(C_{k_0})$ and $x_0 \notin p_X(C_{k_0})$. Therefore $\mathrm{cl}_X\, p_X(Z) \subseteq p_X(C_{k_0})$ and, in consequence, $x_0 \notin \mathrm{cl}_X\, p_X(Z)$. Hence (ii)⇒(iii).

The proof that (iii)⇒(iv) is a slight modification of the proof of the implication (1)⇒(2) of Theorem 1.1 in [3]. We include it below for completeness.

Suppose that, for some $Z \in Z_{\beta\times\beta}(X \times Y)$, there exists

$$\langle x_0, y_0 \rangle \in \mathrm{cl}_{X\times\beta Y}\, Z \setminus \bigcup_{x\in X} \mathrm{cl}_{X\times\beta Y}[Z \cap (\{x\} \times Y)].$$

In particular, $\langle x_0, y_0 \rangle \notin \mathrm{cl}_{X\times\beta Y}[Z \cap (\{x_0\} \times Y)]$. There exists $H \in Z_{\beta\times\beta}(X \times \beta Y)$ such that $H \cap Z \cap (\{x_0\} \times Y) = \emptyset$ and $\langle x_0, y_0 \rangle \in \mathrm{int}_{X\times\beta Y} H$. Then $\emptyset \neq H \cap Z \in Z_{\beta\times\beta}(X \times Y)$, $x_0 \notin p_X(H \cap Z)$ and $x_0 \in \mathrm{cl}_X\, p_X(H \cap Z)$, which contradicts (iii). Hence (iii)⇒(iv).

Assume (iv). Take any functions $f, g \in C_{\beta\times\beta}(X \times Y)$ such that $f^{-1}(0) \cap g^{-1}(0) = \emptyset$. Put $h = |f|/(|f| + |g|)$ and, for $a, b \in [0; 1]$ with $a < b$, consider the sets $Z_a = \{\langle x, y \rangle \in X \times Y : h(\langle x, y \rangle) \leq a\}$ and $Z_b = \{\langle x, y \rangle \in X \times Y : h(\langle x, y \rangle) \geq b\}$. Then $Z_a, Z_b \in Z_{\beta\times\beta}(X \times Y)$. Hence

$$\begin{aligned} &\mathrm{cl}_{X\times\beta Y}(Z_a) \cap \mathrm{cl}_{X\times\beta Y}(Z_b) \\ &\quad = \bigcup_{x\in X} (\mathrm{cl}_{X\times\beta Y}[Z_a \cap (\{x\} \times Y)] \cap \mathrm{cl}_{X\times\beta Y}[Z_b \cap (\{x\} \times Y)]) = \emptyset \end{aligned}$$

because the zero-sets $Z_a \cap (\{x\} \times Y)$ and $Z_b \cap (\{x\} \times Y)$ in $\{x\} \times Y$ have disjoint closures in the Čech–Stone compactification of $\{x\} \times Y$. In view of

[1; Corollary 3], the function h has a continuous extension $\widetilde{h}$ over $X\times\beta Y$. Since X is pseudocompact, so is $X\times\beta Y$ (cf. [5; 3.10.27]). By Glicksberg's theorem (cf. [10]), $\beta\langle X\times\beta Y\rangle=\beta X\times\beta Y$; hence $\widetilde{h}$ has a continuous extension over $\beta X\times\beta Y$. This, together with 1.2, gives that (iv)⇒(v).

Assume (v) and suppose, if possible, that X is not pseudocompact. Take an unbounded continuous function $f:X\to[0;\infty)$. There exists an increasing sequence $\langle m_n\rangle$ of positive integers such that $f^{-1}((m_n;m_{n+1}))\neq\emptyset$ for each $n\in\mathbb{N}$. Choose $d_n\in f^{-1}((m_n;m_{n+1}))$ and put $D=\{d_n:n\in\mathbb{N}\}$. Let $E=D\times\beta Y$ and $\gamma E=\mathrm{cl}_{\beta X}D\times\beta Y$. We shall show that γE is the Wallman compactification of E with respect to the normal base $Z_\gamma(E)$.

Take any $Z_1,Z_2\in Z_\gamma(E)$ such that $Z_1\cap Z_2=\emptyset$. By 1.1, there are functions $f_{i,j,k}\in C^*(D)$ and $g_{i,j,k}\in C(\beta Y)$ such that

$$Z_k=\bigcap_{i=1}^{\infty}\bigcup_{j=1}^{n(i,k)}[f_{i,j,k}^{-1}(0)\times g_{i,j,k}^{-1}(0)]\quad \text{for } k=1,2.$$

For each $n\in\mathbb{N}$, choose $\varepsilon_n>0$ such that $[f(d_n)-\varepsilon_n;f(d_n)+\varepsilon_n]\subset(m_n;m_{n+1})$. Let $D_n=f^{-1}([f(d_n)-\varepsilon_n;f(d_n)+\varepsilon_n])$ for $n\in\mathbb{N}$. Observe that $H_{i,j,k}=\bigcup\{D_n:f_{i,j,k}(d_n)=0\}$ is a zero-set in X. Let

$$H_k=\bigcap_{i=1}^{\infty}\bigcup_{j=1}^{n(i,k)}[H_{i,j,k}\times g_{i,j,k}^{-1}(0)]\quad \text{for } k=1,2.$$

Then $H_1\cap H_2=\emptyset$ and $H_1,H_2\in Z_{\beta\times\beta}(X\times\beta Y)$. According to 1.3, $\beta X\times\beta Y$ is the Wallman compactification of $X\times\beta Y$ arising from the normal base $\{h^{-1}(0):h\in C_{\beta\times\beta}(X\times\beta Y)\}$. Hence, by 1.2, $\mathrm{cl}_{X\times\beta Y}H_1\cap\mathrm{cl}_{X\times\beta Y}H_2=\emptyset$. Therefore $\mathrm{cl}_{\gamma E}Z_1\cap\mathrm{cl}_{\gamma E}Z_2=\emptyset$ because $Z_k\subseteq H_k$ for $k=1,2$. Thus, by 1.2, γE is the Wallman compactification of E with respect to $Z_\gamma(E)$. Since E is Lindelöf and non-pseudocompact, it follows from 1.5 that $\gamma E=\beta E$. On the other hand, $\gamma E=\beta D\times\beta Y$, so $\beta\langle D\times\beta Y\rangle=\beta D\times\beta Y$. By Glicksberg's theorem, $D\times\beta Y$ is pseudocompact, which is absurd. Hence (v)⇒(i).

Assume now that (vi) does not hold. By 1.4, there is a function $\psi\in C(\beta X\times\beta Y)$ such that $\emptyset\neq\psi^{-1}(0)\subseteq(\beta X\times\beta Y)\setminus(X\times Y)$. Put $Z_n=\psi^{-1}([-1/n;1/n])\cap(X\times Y)$. Then $Z_n\in Z_{\beta\times\beta}(X\times Y)$ and $\bigcap_{n=1}^{m}Z_n\neq\emptyset$ for each $m\in\mathbb{N}$. Obviously, $\bigcap_{n=1}^{\infty}Z_n=\emptyset$, which contradicts (ii). Hence (ii)⇒(vi) and the proof of 1.7 is complete.

Let us observe that, in view of [3; Thms. 4.3 & 1.1], conditions (iii) and (iv) of Theorem 1.7 will be equivalent to the pseudocompactness of $X\times Y$ if one replaces $Z_{\beta\times\beta}(X\times Y)$ by $Z_\beta(X\times Y)$.

It follows from the results of F. Kost obtained in [11] that, for any Tikhonov spaces X and Y, $\beta X\times\beta Y$ is the Wallman compactification of $X\times Y$ with respect to the normal base $\mathcal{B}$ consisting of all finite unions of sets of the form $f^{-1}(0) \times g^{-1}(0)$, where $f \in C^*(X)$ and $g \in C^*(Y)$. Denote by $\mathcal{B}_\delta$ the smallest family which contains $\mathcal{B}$ and is closed under countable intersections. In the light of 1.1, $\mathcal{B}_\delta = Z_{\beta\times\beta}(X \times Y)$; thus Theorem 1.7 shows that the Wallman compactification with respect to $\mathcal{B}$ can be equivalent to the Wallman compactification with respect to $\mathcal{B}_\delta$ only under very restrictive conditions.

The referee has posed the following problem:

PROBLEM. *If X and Y are pseudocompact spaces such that the product $X \times Y$ is not pseudocompact, must every compactification of $X \times Y$ be of Wallman type?*

A satisfactory answer to the referee's question is unknown to the author; however, under MA and the negation of CH, we shall show that there exist pseudocompact spaces X and Y such that the space $X \times Y$ has a compactification which is not of Wallman type. To this end, we shall need the following

1.8. THEOREM. *Under the negation of CH, every normal non-pseudocompact space has a compactification which is not of Wallman type.*

Proof. Let X be a normal non-pseudocompact space. The space X being non-pseudocompact, it contains a closed copy of the space $\mathbb{N}$ of positive integers. Without loss of precision, we may assume that $\mathbb{N}$ is a closed subspace of X. If we assume the negation of CH, then $2^\omega \geq \omega_2$ and, according to Corollary 2 of [14], there exists a compactification $\gamma\mathbb{N}$ of $\mathbb{N}$ which is not of Wallman type. Obviously, $\beta\mathbb{N} = \mathrm{cl}_{\beta X}\mathbb{N}$. Let $\pi : \mathrm{cl}_{\beta X}\mathbb{N} \to \gamma\mathbb{N}$ be the natural quotient map which witnesses that $\gamma\mathbb{N} \leq \beta\mathbb{N}$. Then the decomposition

$$\mathcal{A} = \{\pi^{-1}(z) : z \in \gamma\mathbb{N} \setminus \mathbb{N}\} \cup \{\{y\} : y \in \beta X \setminus (\beta\mathbb{N} \setminus \mathbb{N})\}$$

of βX is upper semicontinuous. Therefore, by the Alexandrov theorem (cf. [5; 3.2.11]), the quotient space $\alpha X = \beta X/\mathcal{A}$ obtained from βX by identifying each element of $\mathcal{A}$ with a point is a compactification of X. Suppose, if possible, that there exists a normal base $\mathcal{D}$ for X such that $\alpha X = w_{\mathcal{D}}X$. Let $\mathcal{F} = \{D \cap \mathbb{N} : D \in \mathcal{D}\}$. To show that $\mathcal{F}$ is a normal base for $\mathbb{N}$ and that $w_{\mathcal{F}}\mathbb{N} = \mathrm{cl}_{\alpha X}\mathbb{N}$, it suffices to check that

$$\mathrm{cl}_{\alpha X}(D \cap \mathbb{N}) = \mathrm{cl}_{\alpha X} D \cap \mathrm{cl}_{\alpha X} \mathbb{N}$$

for each $D \in \mathcal{D}$. Let us consider any $D \in \mathcal{D}$ and suppose that $y \in \mathrm{cl}_{\alpha X} D \cap \mathrm{cl}_{\alpha X} \mathbb{N}$ but $y \notin \mathrm{cl}_{\alpha X}(D \cap \mathbb{N})$. There exists $C \in \mathcal{D}$ such that $y \in \mathrm{cl}_{\alpha X} C$ and $\mathrm{cl}_{\alpha X} C \cap \mathrm{cl}_{\alpha X}(D \cap \mathbb{N}) = \emptyset$. Then $C \cap D \cap \mathbb{N} = \emptyset$ and it follows from the normality of X that $\mathrm{cl}_{\beta X}(C \cap D) \cap \mathrm{cl}_{\beta X} \mathbb{N} = \emptyset$. This implies that $\emptyset = \mathrm{cl}_{\alpha X}(C \cap D) \cap \mathrm{cl}_{\alpha X} \mathbb{N} = \mathrm{cl}_{\alpha X} C \cap \mathrm{cl}_{\alpha X} D \cap \mathrm{cl}_{\alpha X} \mathbb{N}$ (cf. [12; 4.4(f)]), which is absurd. The contradiction obtained proves that $\mathcal{F}$ is a normal base for $\mathbb{N}$ and that $w_{\mathcal{F}}\mathbb{N} = \mathrm{cl}_{\alpha X}\mathbb{N}$. But this is impossible because $\mathrm{cl}_{\alpha X}\mathbb{N} = \gamma\mathbb{N}$. Accordingly, αX cannot be of Wallman type.

By Corollary 1 of [14], every compactification of every separable Tikhonov space is of Wallman type if and only if the continuum hypothesis holds; hence the assumption of the negation of CH cannot be omitted in Theorem 1.8. However, the author does not know whether the assumption of normality is essential in 1.8. Clearly, every compactification of every pseudocompact space is of Wallman type.

E. K. van Douwen proved in [4] that, under MA, there exist normal pseudocompact spaces X and Y such that $X \times Y$ is normal but not pseudocompact (cf. also [15; 3.2, p. 577]). If, in addition, we assume the negation of CH, then van Douwen's construction and Theorem 1.8 will give us a negative answer to the above-mentioned problem of the referee.

2. The pseudocompactness of $X \times Y$. Let $\mathcal{E}$ be a family of subsets of a set X and let T be a topological space. A mapping $K : T \to \mathcal{E}$ will be called *$\mathcal{E}$-upper semicontinuous* (abbr. *$\mathcal{E}$-u.sc.*) if, for any $t_0 \in T$ and $E \in \mathcal{E}$ such that $K(t_0) \cap E = \emptyset$, there exists an open neighbourhood U of t_0 in T such that $K(t) \cap E = \emptyset$ for any $t \in U$. We shall say that $\mathcal{E}$ *semiseparates* a set $A \subseteq X$ if, for any $E \in \mathcal{E}$ with $A \cap E = \emptyset$, there exists $F \in \mathcal{E}$ such that $A \subseteq F$ and $F \cap E = \emptyset$. When $\mathcal{A}$ is a collection of subsets of X, we shall say that $\mathcal{E}$ *semiseparates* $\mathcal{A}$ if $\mathcal{E}$ semiseparates any set $A \in \mathcal{A}$.

In what follows, the algebra $C^*(T)$ will always be considered with the topology of uniform convergence.

2.1. LEMMA. *Suppose that both X and Y are pseudocompact*, *and a set $A \subseteq X \times Y$ has the property that*, *for any $f \in C(X)$ and $g \in C(Y)$ with $A \cap [f^{-1}(0) \times g^{-1}(0)] = \emptyset$*, *there is $Z \in Z_{\beta\times\beta}(X \times Y)$ such that $A \subseteq Z$ and $Z \cap [f^{-1}(0) \times g^{-1}(0)] = \emptyset$. Then $Z_{\beta\times\beta}(X \times Y)$ semiseparates A.*

Proof. Take any $C \in Z_{\beta\times\beta}(X \times Y)$ such that $C \cap A = \emptyset$. Then, by 1.1, C has a Suslin representation in the form

$$C = \bigcup_{\sigma\in\mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{\infty} [f_{\sigma\restriction n}^{-1}(0) \times g_{\sigma\restriction n}^{-1}(0)]$$

for some $f_{\sigma\restriction n} \in C(X)$ and $g_{\sigma\restriction n} \in C(Y)$. For any $\sigma \in \mathbb{N}^{\mathbb{N}}$, there is $Z_\sigma \in Z_{\beta\times\beta}(X\times Y)$ such that $A \subseteq Z_\sigma$ and $Z_\sigma \cap \bigcap_{n=1}^{\infty}[f_{\sigma\restriction n}^{-1}(0) \times g_{\sigma\restriction n}^{-1}(0)] = \emptyset$. Since, by 1.7, the collection $Z_{\beta\times\beta}(X \times Y)$ is semicompact, there is $m(\sigma) \in \mathbb{N}$ such that $Z_\sigma \cap \bigcap_{n=1}^{m(\sigma)}[f_{\sigma\restriction n}^{-1}(0) \times g_{\sigma\restriction n}^{-1}(0)] = \emptyset$. Put

$$D = \bigcup_{\sigma\in\mathbb{N}^{\mathbb{N}}} \bigcap_{n=1}^{m(\sigma)} [f_{\sigma\restriction n}^{-1}(0) \times g_{\sigma\restriction n}^{-1}(0)].$$

Then $D \cap A = \emptyset$ and D can be represented as $D = \bigcup_{n=1}^{\infty}[f_n^{-1}(0) \times g_n^{-1}(0)]$ for some $f_n \in C(X)$ and $g_n \in C(Y)$. Choose $Z_n \in Z_{\beta\times\beta}(X \times Y)$ such that $A \subseteq Z_n$ and $Z_n \cap [f_n^{-1}(0) \times g_n^{-1}(0)] = \emptyset$. Then, for $Z = \bigcap_{n=1}^{\infty} Z_n$, we have $A \subseteq Z$ and $Z \cap C = \emptyset$.

2.2. Theorem. *For any non-void Tikhonov spaces X and Y, the following conditions are equivalent*:

(i) *$X \times Y$ is pseudocompact*;

(ii) *both X and Y are pseudocompact and $Z_{\beta\times\beta}(X \times Y) = Z(X \times Y)$*;

(iii) *the mapping $K : C^*(X) \times C^*(Y) \to Z(X \times Y)$ defined by*

$$K(f, g) = f^{-1}(0) \times g^{-1}(0)$$

is $Z(X \times Y)$-u.sc.;

(iv) *for any $Z \in Z(X\times Y)$ and $\langle f, g\rangle \in C^*(X)\times C^*(Y)$ with $Z\cap[f^{-1}(0)\times g^{-1}(0)] = \emptyset$, there exists $\varepsilon > 0$ such that $Z\cap[f^{-1}([-\varepsilon;\varepsilon])\times g^{-1}([-\varepsilon;\varepsilon])] = \emptyset$*;

(v) *both X and Y are pseudocompact and $Z_{\beta\times\beta}(X \times Y)$ semiseparates $Z(X \times Y)$.*

Proof. In view of Glicksberg's theorem (cf. [10]), the implication (i)⇒(ii) is obvious.

Assume (ii). Take any $f_0 \in C^*(X), g_0 \in C^*(Y)$ and $Z \in Z(X \times Y)$ such that $Z \cap [f_0^{-1}(0) \times g_0^{-1}(0)] = \emptyset$. By 1.7(ii), there exists $n \in \mathbb{N}$ such that $Z \cap [f_0^{-1}([-1/n; 1/n]) \times g_0^{-1}([-1/n; 1/n])] = \emptyset$. If $\langle f, g\rangle \in C^*(X) \times C^*(Y)$, $|f - f_0| < 1/n$ and $|g - g_0| < 1/n$, then $Z \cap [f^{-1}(0) \times g^{-1}(0)] = \emptyset$ because $f^{-1}(0) \times g^{-1}(0) \subseteq f_0^{-1}([-1/n; 1/n]) \times g_0^{-1}[-1/n; 1/n])$. Hence (ii)⇒(iii).

Assume (iii). Now, let $\langle f_1, g_1\rangle \in C^*(X) \times C^*(Y)$, $Z \in Z(X \times Y)$ and $Z\cap[f_1^{-1}(0)\times g_1^{-1}(0)] = \emptyset$. Since K is $Z(X\times Y)$-u.sc., there is $\varepsilon > 0$ such that if $\langle f, g\rangle \in C^*(X)\times C^*(Y)$ has the property that $|f-f_1| \le \varepsilon$ and $|g-g_1| \le \varepsilon$, then $K(f, g) \cap Z = \emptyset$. Let $\langle x_1, y_1\rangle \in f_1^{-1}([-\varepsilon; \varepsilon]) \times g_1^{-1}([-\varepsilon; \varepsilon])$. Then, for $f = f_1 - f_1(x_1)$ and $g = g_1 - g_1(y_1)$, we have $|f - f_1| \le \varepsilon$ and $|g - g_1| \le \varepsilon$; hence $Z \cap [f^{-1}(0) \times g^{-1}(0)] = \emptyset$. Since $\langle x_1, y_1\rangle \in f^{-1}(0) \times g^{-1}(0)$, we have $\langle x_1, y_1\rangle \notin Z$. Altogether this yields $Z \cap [f_1^{-1}([-\varepsilon; \varepsilon]) \times g_1^{-1}([-\varepsilon; \varepsilon])] = \emptyset$ and we conclude that (iii)⇒(iv).

Assume (iv) and suppose, if possible, that X is not pseudocompact. There exists a sequence $\langle f_n\rangle$ of continuous functions $f_n : X \to [0; 1]$ such that $f_{n+1}^{-1}(0) \subseteq f_n^{-1}(0) \neq \emptyset$ for any $n \in \mathbb{N}$ but $\bigcap_{n=1}^{\infty} f_n^{-1}(0) = \emptyset$. Put $f = \sum_{n=1}^{\infty}(1/2^n)f_n$ and $g(y) = 0$ for any $y \in Y$. Then, for $Z_0 = X \times Y$, we have $Z_0 \cap [f^{-1}(0) \times g^{-1}(0)] = \emptyset$; thus there exists $\varepsilon > 0$ such that $Z_0 \cap [f^{-1}([-\varepsilon; \varepsilon]) \times g^{-1}([-\varepsilon; \varepsilon])] = \emptyset$. Hence $f^{-1}([-\varepsilon; \varepsilon]) = \emptyset$. Take $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0+1}^{\infty} 1/2^n < \varepsilon$. Then $f_{n_0}^{-1}(0) \subseteq f^{-1}([-\varepsilon; \varepsilon])$ because $f_{n_0}^{-1}(0) \subseteq f_n^{-1}(0)$ for each $n \le n_0$. Hence $f^{-1}([-\varepsilon; \varepsilon]) \neq \emptyset$ and we obtain a contradiction which shows that X is pseudocompact. Similarly, Y is pseudocompact, too. Now, an application of 2.1 shows that (iv)⇒(v).

Assume (v) and suppose that $X \times Y$ is not pseudocompact. There is an unbounded continuous function $h : X \times X \to [0;\infty)$. We can define by induction an increasing sequence $\langle m_n \rangle$ of positive integers, numbers $\varepsilon_n > 0$ and functions $f_n \in C^*(X)$ and $g_n \in C^*(Y)$ such that $f_n^{-1}([-\varepsilon_n;\varepsilon_n]) \times g_n^{-1}([-\varepsilon_n;\varepsilon_n]) \subseteq h^{-1}((m_n;m_{n+1}))$ and $f_n^{-1}(0) \times g_n^{-1}(0) \neq \emptyset$ for each $n \in \mathbb{N}$. By the equivalence (i)⇔(v) of 1.7 and by 1.2, there exist functions $h_n \in C_{\beta\times\beta}(X \times Y)$ such that

$$h_n[(X \times Y) \setminus (f_n^{-1}[(-\varepsilon_n;\varepsilon_n)] \times g_n^{-1}[(-\varepsilon_n;\varepsilon_n)])] = \{0\}$$

and

$$h_n[f_n^{-1}([-\varepsilon_n/2;\varepsilon_n/2]) \times g_n^{-1}([-\varepsilon_n/2;\varepsilon_n/2])] = \{1\}.$$

Put $\psi_n = \sum_{m=n}^{\infty} h_m$ for $n \in \mathbb{N}$. Clearly, $\psi_n \in C^*(X \times Y)$ and $\psi_n^{-1}(1) = \bigcup_{m=n}^{\infty} h_m^{-1}(1)$ for $n \in \mathbb{N}$. We now show that $p_X(\psi_n^{-1}(1))$ is closed in X for any $n \in \mathbb{N}$.

Take any $x \in X$ such that $x \notin p_X(\psi_n^{-1}(1))$. Then $(\{x\} \times Y) \cap \psi_n^{-1}(1) = \emptyset$. Since Y is pseudocompact, it follows from 1.6 that, for each $m \geq n$, there exists $\delta_m > 0$ such that $(\{x\} \times Y) \cap h_m^{-1}((1-\delta_m;1+\delta_m)) = \emptyset$. Put $A_m = (X \times Y) \setminus h_m^{-1}((1-\delta_m;1+\delta_m))$ and $A = \bigcap_{m=n}^{\infty} A_m$. Then $A \in Z_{\beta\times\beta}(X \times Y)$, $\{x\} \times Y \subseteq A$ and $A \cap \psi_n^{-1}(1) = \emptyset$. Since $Z_{\beta\times\beta}(X \times Y)$ semiseparates $Z(X \times Y)$, there is $D \in Z_{\beta\times\beta}(X \times Y)$ such that $\psi_n^{-1}(1) \subseteq D$ and $D \cap A = \emptyset$. Then $x \notin p_X(D)$. By (i)⇔(iii) of 1.7, $x \notin \mathrm{cl}_X\, p_X(D)$, which implies that $x \notin \mathrm{cl}_X\, p_X(\psi_n^{-1}(1))$. Hence $p_X(\psi_n^{-1}(1))$ is closed in X.

Put $U_n = \mathrm{int}_X\, p_X(\psi_n^{-1}(1))$ for $n \in \mathbb{N}$. Then $U_n \neq \emptyset$ and $U_{n+1} \subseteq U_n$ for any $n \in \mathbb{N}$. Since X is pseudocompact, it follows from Theorem 3.10.23 of [5] that there exists $x_0 \in \bigcap_{n=1}^{\infty} \mathrm{cl}_X\, U_n$. Then $x_0 \in \bigcap_{n=1}^{\infty} p_X(\psi_n^{-1}(1))$. This implies that $(\{x_0\} \times Y) \cap \psi_n^{-1}(1) \neq \emptyset$ for each $n \in \mathbb{N}$. Since Y is pseudocompact and $\psi_{n+1}^{-1}(1) \subseteq \psi_n^{-1}(1)$ for any $n \in \mathbb{N}$, it follows from 1.6 that $(\{x_0\} \times Y) \cap \bigcap_{n=1}^{\infty} \psi_n^{-1}(1) \neq \emptyset$, which is absurd. Hence (v)⇒(i).

A variety of other conditions equivalent to the pseudocompactness of $X \times Y$ have been found by many authors (cf., for instance, [3], [7], [10] & [13]).

If X and Y are pseudocompact and $X \times Y$ is not pseudocompact, then the semiseparation of $Z(X \times Y)$ by $Z_{\beta\times\beta}(X \times Y)$ is spoilt by a set $Z \in Z(X \times Y)$ which is a countable union of members of $Z_{\beta\times\beta}(X \times Y)$. Therefore one may suspect that there exist pseudocompact spaces X and Y such that $X \times Y$ is not pseudocompact but the smallest σ-algebra containing $Z_{\beta\times\beta}(X \times Y)$ is equal to the smallest σ-algebra containing $Z(X \times Y)$. Such an example is not known to the author.

Let us observe that the implication (ii)⇒(i) of 2.2 is an immediate consequence of 1.6 and the implication (i)⇒(ii) of 1.7.

The proof of 2.2 shows that the following proposition holds:

2.3. Proposition. *For non-void Tikhonov spaces X and Y, the following conditions are equivalent*:

(i) *both X and Y are pseudocompact*;

(ii) *the mapping $K : C^*(X) \times C^*(Y) \to Z_{\beta\times\beta}(X \times Y)$ defined by*

$$K(f, g) = f^{-1}(0) \times g^{-1}(0)$$

is $Z_{\beta\times\beta}(X \times Y)$-u.sc.;

(iii) *for any $Z \in Z_{\beta\times\beta}(X \times Y)$ and $\langle f, g\rangle \in C^*(X) \times C^*(Y)$ with $Z \cap [f^{-1}(0) \times g^{-1}(0)] = \emptyset$, there exists $\varepsilon > 0$ such that $Z \cap [f^{-1}([-\varepsilon; \varepsilon]) \times g^{-1}([-\varepsilon; \varepsilon])] = \emptyset$.*

Finally, let us notice that the following pseudocompact version of Lemma 8.6 of [8] can easily be drawn from 2.2:

2.4. Proposition. *A Tikhonov space X is pseudocompact if and only if the mapping $K : C^*(X) \to Z(X)$ defined by $K(f) = f^{-1}(0)$ is $Z(X)$-u.sc.*

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