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SCHATTEN CLASSES AND COMMUTATORS ON SIMPLE MARTINGALES

BY

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1. Introduction. Martingale transforms were first introduced and studied by Burkholder [2] and recently by Chao and Long [7]. Singular integral operators in the local field setting have been studied by Phillips and Taibleson [21], and Chao and Taibleson [9], which led to the study of matrix transforms on simple martingales by Janson [11], and Chao and Janson [6]. Fractional integral transforms and commutators with these singular integral operators for simple martingales were discussed in Chao and Ombe [8], and Chao, Daly and Ombe [5].

In this paper, we study the commutators with the above mentioned operators in the simple martingale setting and obtain their compactness and Schatten–von Neumann S_p-properties. These results extend those for the Euclidean case which has been studied by many authors, e.g. Janson and Wolff [14], Uchiyama [24], Janson and Peetre [12, 13], Peng [18, 19], Rochberg and Semmes [22]. These commutators are operators of Hankel type. For the study of Hankel operators, see Peller [16, 17]. The arguments used to obtain our results for simple martingales are quite different due to the nondegeneracy conditions for the singular integral operators involved.

In §2, we provide some preliminaries. Paraproducts and fractional integrals are discussed in §3. In §4, we study the compactness and S_p-properties ($1 \le p \le \infty$) of the commutators. Finally, S_p-properties for $0 < p < 1$ are studied in §5.

2. Preliminaries. Let $\Omega = [0,1)$ and $d \ge 2$ be a fixed integer. For each $n \ge 0$, let $\mathcal{F}_n$ be the σ-field generated by the d-adic intervals $Q_n^k = [kd^{-n}, (k+1)d^{-n}), 0 \le k < d^n$, of Ω. Let $\mathcal{F}$ be the σ-field generated by all such intervals, and dx be the Lebesgue measure on Ω. Then $(\Omega, \mathcal{F}, dx)$ is a d-adic probability space. A martingale on $(\Omega, \mathcal{F}, dx)$ is called a *simple*

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martingale (or *d-adic martingale*). For $f \in L^1(\Omega)$, we define $f_n = E(f \mid \mathcal{F}_n)$, $\Delta_n(f) = f_n - f_{n-1}$ for $n \geq 1$ and $\Delta_0(f) = f_0$; then $f = \sum_{n=0}^{\infty} \Delta_n(f)$.

2.1. *Operators on simple martingales.* Now we introduce four operators on simple martingales which have been studied by many authors.

1. *Paraproduct*:

$$\Pi_b(f) = \sum_{n=1}^{\infty} \Delta_n(b) f_{n-1}. \tag{1.1}$$

In [7] Chao and Long have shown that Π_b is bounded on L^p $(1 < p < \infty)$ if and only if $b \in \mathrm{BMO}$.

2. *Martingale transform*:

$$T_\nu(f) = \sum_{n=1}^{\infty} \nu_{n-1} \Delta_n(f), \tag{1.2}$$

where $\nu = \{\nu_n\}$ is an adapted process.

In fact, $T_\nu(f) = \Pi_f(\nu)$. In [2] Burkholder has shown that T_ν is bounded on L^p $(1 < p < \infty)$ if and only if $\|\nu\|_\infty = \sup_n \|\nu_n\|_\infty < \infty$.

3. *Fractional integral operator* I^α:

$$I^\alpha f = \sum_{k=0}^{\infty} d^{-k\alpha} \Delta_k(f). \tag{1.3}$$

In [8] Chao and Ombe have shown that I^α is bounded from H^p to H^q, where $1/q = 1/p - \alpha$.

4. *Singular integral operator* T_A. Here we consider only the case $d > 2$. When $d = 2$, a refinement of the arguments must be applied. See Chao [3]. For f an integrable function, we notice that on any $Q_n^k \in \mathcal{F}_n$, f_n is a constant and f_{n+1} has d values. Hence $f_{n+1} - f_n$ may be regarded as a vector in $\mathbb{C}^d$, which will be called the *local difference* of f on the atom Q_n^k. It is easy to see that every local difference actually belongs to the $(d-1)$-dimensional space $V = \{(x_i)_{i=1}^d : \sum x_i = 0\}$. Given a $d \times d$ matrix $A = (a_{ij})$, we can define a linear operator A on V which gives the singular integral operator T_A as follows:

$$T_A(f) = \sum_{k=0}^{\infty} A \Delta_k(f). \tag{1.4}$$

In [11] Janson has proved that T_A is bounded on H^p $(0 < p < \infty)$ (see also Chao [3]).

For a nice function b, let the operator of multiplication by b be denoted also by b. For any linear operator T, we may define the commutator $[b, T] = bT - Tb$. In this paper we study three kinds of commutator: $[b, T_\nu], [b, I^\alpha]$

and $[b, T_A]$. They have some similar properties to the paraproduct Π_b. Their boundedness has been obtained by Chao, Daly and Ombe [5]:

• $[b, T_\nu]$ is bounded on L^p $(1 < p < \infty)$ if and only if $b \in \mathrm{BMO}$, provided that ν satisfies the nondegeneracy condition (D_ν):

(D_ν) There is an $N > 0$ such that if $n \geq N$ and $Q_n^k \in \mathcal{F}_n$, then there is an m, $1 \leq m \leq n-1$, such that

$$\nu_0 + \frac{d-1}{d} \sum_{i=1}^{m-1} d^i \nu_i(x) - d^{m-1}\nu_m(x) \neq 0 \quad \text{for } x \in Q_n^k.$$

• $[b, I^\alpha]$ is bounded from H^p to H^q for $0 < p < q < \infty$, $q > 1$ and $\alpha = 1/p - 1/q$ if and only if $b \in \mathrm{BMO}$.

• $[b, T_A]$ is bounded on L^p $(1 < p < \infty)$ if and only if $b \in \mathrm{BMO}$, provided that A satisfies the nondegeneracy condition (D_A):

(D_A) For any i, there exist $j, k \neq i$ such that

$$a_{ij} \neq a_{ik} \quad \text{(row)} \quad \text{or} \quad a_{ji} \neq a_{ki} \quad \text{(column)}.$$

2.2. *Schatten–von Neumann ideal S_p.* Let H_1, H_2 be two Hilbert spaces and $\mathcal{L}(H_1, H_2)$ the set of all bounded linear operators from H_1 to H_2, and let $\mathcal{K}(H_1, H_2)$ be the set of all compact operators. For $T \in \mathcal{L}(H_1, H_2)$, we define the *singular number* $s_n = s_n(T)$ by

$$s_n = \inf\{\|T - F\| : \operatorname{rank}(F) \leq n\}, \tag{2.1}$$

and the *Schatten–von Neumann ideal* S_p by

$$\begin{aligned} S_p &= \Big\{T \in \mathcal{K}(H_1, H_2) : \Big(\sum_{n=0}^{\infty} s_n^p\Big)^{1/p} < \infty\Big\} \quad \text{for } 0 < p < \infty, \\ S_\infty &= \mathcal{L}(H_1, H_2). \end{aligned} \tag{2.2}$$

For the properties of S_p, see e.g. [13].

2.3. *Besov spaces and Triebel–Lizorkin spaces.* For $s \in \mathbb{R}$ and $0 < p, q \leq \infty$, the *Besov space B_p^{sq} of simple martingales* is defined by

$$B_p^{sq} = \Big\{f : \|f\|_{B_p^{sq}} = \Big\{\sum_{k=0}^{\infty} (d^{ks}\|\Delta_k(f)\|_p)^q\Big\}^{1/q} < \infty\Big\}. \tag{2.3}$$

Sometimes we adopt shorter notations $B_p^s = B_p^{sp}$ and $B_p = B_p^{1/p}$.

For $s \in \mathbb{R}$ and $0 < p, q \leq \infty$, the *Triebel–Lizorkin space F_p^{sq} of simple martingales* is defined by

$$F_p^{sq} = \Big\{f : \|f\|_{F_p^{sq}} = \Big\{E\Big[\sum_{k=0}^{\infty} (d^{ks}|\Delta_k(f)|)^q\Big]^{p/q}\Big\}^{1/p} < \infty\Big\}. \tag{2.4}$$

B_p^{sq} and F_p^{sq} on simple martingales have the same properties as those on $\mathbb{R}^n$ (for the latter, see Bergh and Löfström [1], Peetre [15] and Triebel [23]). In particular, we have

(1) B_p^{sq} and F_p^{sq} are Banach spaces for $1 \le p, q \le \infty$ and Fréchet spaces for $0 < p, q < 1$.

(2) $C_0 = \{\text{finite martingales}\}$ is dense in B_p^{sq} and F_p^{sq} for $0 < p, q < \infty$.

(3) $B_2^{s2} = F_2^{s2} = H_s^2 = I^s(H^2)$ and $F_p^{s2} = H_s^p = I^s(H^p)$ for $0 < p < \infty$.

(4) $[B_{p_0}^{s_0q_0}, B_{p_1}^{s_1q_1}]_\theta = B_{p^*}^{s^*q^*}$ and $[F_{p_0}^{s_0q_0}, F_{p_1}^{s_1q_1}]_\theta = F_{p^*}^{s^*q^*}$, where $0 < \theta < 1$, $s_0, s_1 \in \mathbb{R}$, $1 \le p_0, p_1, q_0, q_1 \le \infty$, $s^* = (1-\theta)s_0 + \theta s_1$ and

$$\frac{1}{p^*} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \qquad \frac{1}{q^*} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}.$$

(There are also results about real interpolation.)

(5) $(B_p^{sq})^* = B_{p'}^{-sq'}$ and $(F_p^{sq})^* = F_{p'}^{-sq'}$ for $s \in \mathbb{R}$, $1 \le p, q < \infty$, $1/p + 1/p' = 1$, $1/q + 1/q' = 1$.

Let $b_\infty^{s\infty} = b_\infty^s$ be the closure of C_0 in B_∞^s-norm and f_∞^{s2} be the closure of C_0 in F_∞^{s2}-norm. Then

$$(b_\infty^s)^* = B_1^{-s}, \quad (f_\infty^{s2})^* = F_1^{(-s)2}, \quad \text{BMO} = F_\infty^{02}, \quad \text{VMO} = f_\infty^{02}.$$

(6) I^α is an isometric isomorphism from B_p^{sq} to $B_p^{(s+\alpha)q}$ and from F_p^{sq} to $F_p^{(s+\alpha)q}$.

2.4. *Orthonormal, weakly orthonormal and nearly weakly orthonormal sequences* (see Rochberg and Semmes [22]). Let $\mathcal{P} = \{Q_n^k : n \ge 0,\ 0 \le k < d^n\}$ and $L^2(\Delta_n) = \{f : f \in L^2, \mathcal{F}_n\text{-measurable and } E_{n-1}(f) = 0\}$. Then

$$L^2(\Omega) = \bigoplus_{n=0}^{\infty} L^2(\Delta_n).$$

Let $\{e^1, \ldots, e^{d-1}\}$ be an orthonormal basis of V, i.e. $\{e^i = (c_1^i, \ldots, c_d^i)\}$ satisfies

$$c_1^i + c_2^i + \ldots + c_d^i = 0,$$
$$c_1^i \bar{c}_1^j + c_2^i \bar{c}_2^j + \ldots + c_d^i \bar{c}_d^j = \delta_{ij} \quad \text{for } i, j = 1, \ldots, d-1.$$

For $Q = Q_n^k \in \mathcal{P}$, let

$$\psi_Q^i(x) = d^{n/2}\{c_1^i \chi_{Q_n^{kd}}(x) + c_2^i \chi_{Q_n^{kd+1}}(x) + \ldots + c_d^i \chi_{Q_n^{kd+d-1}}(x)\}.$$

Then $\{\psi_Q^i\}_{Q \in \mathcal{P}, i \in \{1, \ldots, d-1\}}$ is an orthonormal basis of $L^2(\Omega)$, and

$$L^2(\Delta_n) = \operatorname{span}\{\psi_Q^i : |Q| = d^{-(n-1)},\ i = 1, \ldots, d-1\}.$$

Let Δ_n denote also the projection of $L^2(\Omega)$ onto $L^2(\Delta_n)$, and E_n the projection of $L^2(\Omega)$ onto $\bigoplus_{k=1}^n L^2(\Delta_k)$. Then $E_n(f) = f_n$ is just the conditional expectation of f, and $\Delta_n(f)$ is the martingale difference of f.

It should be pointed out that $\{\psi_Q^i\}_{Q\in\mathcal{P},i\in\{1,\dots,d-1\}}$ is the universal unconditional basis for all B_p^{sq} and F_p^{sq}.

Now we introduce a frame of $L^2(\Omega)$. (For the notion of frames, see e.g. Peng [20].) For $Q_n^k \in \mathcal{P}$, let $\overline{Q}_n^k$ denote its mother interval, i.e. the smallest interval properly containing Q_n^k. Let

$$\phi_{Q_n^k}(x) = d^{n/2}\biggl(\chi_{Q_n^k}(x) - \frac{1}{d}\chi_{\bar{Q}_n^k}(x)\biggr).$$

Then $\{\phi_{Q_n^k}\}$ becomes a tight frame. So for every $f \in L^2(\Omega)$, we have

$$\Delta_n(f) = \sum_{|Q_n^k|=d^{-n}} \langle f, \phi_{Q_n^k}\rangle d^{n/2}\chi_{Q_n^k}(x),$$

$$f(x) = \sum_{Q_n^k}\langle f, \phi_{Q_n^k}\rangle d^{n/2}\chi_{Q_n^k}(x) \quad \text{and} \quad \|f\|_{L^2}^2 = \sum_{Q_n^k} |\langle f, \phi_{Q_n^k}\rangle|^2.$$

Again $\{\phi_Q\}$ is also a universal unconditional basis for B_p^{sq} and F_p^{sq} (see also [20]). Moreover, both $\{\psi_Q^i\}$ and $\{\phi_Q\}$ are bases of BMO and VMO. We have:

- $f \in \mathrm{BMO}(\Omega)$ if and only if $\{\lambda_Q\} \in \mathrm{BMO}(\mathcal{P})$, i.e.
$$\sup_{P\in\mathcal{P}} \frac{1}{|P|}\sum_{Q\subset P} |\lambda_Q|^2|Q| < \infty,$$
- $f \in \mathrm{VMO}(\Omega)$ if and only if $\{\lambda_Q\} \in \mathrm{VMO}(\mathcal{P})$, i.e.
$$\{\lambda_Q\} \in \mathrm{BMO}(\mathcal{P}) \quad \text{and} \quad |\lambda_Q| \to 0 \text{ as } |Q| \to 0,$$

for $\lambda_Q = \langle f, \psi_Q\rangle$ or $\langle f, \phi_Q\rangle$.

Let H be a Hilbert space. A sequence $\{e_i\} \subset H$ is called *weakly orthonormal* (WO) if $\|\sum \lambda_i e_i\| \le C(\sum |\lambda_i|^2)^{1/2}$. In fact, $\{e_i\}$ is a WO sequence if and only if it is the image of an orthonormal sequence under a bounded linear map. (See Rochberg and Semmes [22].)

A *nearly weakly orthonormal* (NWO) *sequence* $\{e_Q\}_{Q\in\mathcal{P}}$ is a sequence in $L^2(\Omega)$ indexed by $\mathcal{P}$ such that the following maximal operator estimate holds. Set

$$f^*(x) = \sum_{x\in Q}\{|Q|^{1/2}|\langle f, e_Q\rangle|\}.$$

Then

$$\|f^*\|_2 \le C\|f\|_2.$$

For example, if $\mathrm{supp}(e_Q) \subset \overline{Q}$ and $\|e_Q\|_\infty \le |Q|^{-1/2}$ or $\|e_Q\|_p \le |Q|^{1/p-1/2}$ for some $p > 2$, then $\{e_Q\}$ is a NWO sequence. (See again Rochberg and Semmes [22].)

LEMMA 2.1. *Suppose that there exist two NWO sequences $\{e_Q\}$ and $\{f_Q\}$ such that $T = \sum_{Q\in\mathcal{P}} \lambda_Q\langle\cdot, e_Q\rangle f_Q$. Then*

(1) $\{\lambda_Q\} \in \mathrm{BMO}(\mathcal{P})$ *implies that* $T \in S_\infty$ *and* $\|T\| \le C\|\{\lambda_Q\}\|_{\mathrm{BMO}}$.

(2) $\{\lambda_Q\} \in \mathrm{VMO}(\mathcal{P})$ *implies that* T *is compact.*

(3) $\{\lambda_Q\} \in l^p(\mathcal{P})$ *implies that* $T \in S_p$ *and* $\|T\|_{S_p} \le C_p(\sum_{Q\in\mathcal{P}} |\lambda_Q|^p)^{1/p}$, $0 < p < \infty$.

LEMMA 2.2. *If* $\{e_Q\}, \{f_Q\}$ *are two NWO sequences, then*

$$\Big(\sum_{Q\in\mathcal{P}} |\langle Te_Q, f_Q\rangle|^p\Big)^{1/p} \le C_p\|T\|_{S_p} \quad \text{for } 1 < p < \infty.$$

LEMMA 2.3. *If* T *is a compact operator on* $L^2(\Omega)$ *and* $e_i \to 0$ *weakly as* $i \to \infty$, *then* $\|Te_i\|_2 \to 0$.

3. Paraproducts and fractional integrals

3.1. *Paraproducts*

THEOREM 3.1. (1) *For* $1 < p < \infty$, Π_b *is bounded on* $L^p(\Omega)$ *if and only if* $b \in \mathrm{BMO}$ *and* $\|\Pi_b\| \equiv \|b\|_{\mathrm{BMO}}$.

(2) *For* $1 < p < \infty$, Π_b *is compact on* $L^p(\Omega)$ *if and only if* $b \in \mathrm{VMO}$.

(3) *For* $0 < p < \infty$, $\Pi_b \in S_p(L^2, L^2)$ *if and only if* $b \in B_p$ *and* $\|\Pi_b\|_{S_p} \equiv \|b\|_{B_p}$.

Proof. (1) is known (see Chao and Long [7]). It can also be obtained from Lemma 2.1. Here we give the proofs of (2) and the main result of (3). We postpone the proof for the converse result of (3) ($\Pi_b \in S_p$ implies $b \in B_p$ for $0 < p < 1$) to §5.

Instead of the operator Π_b, we consider the equivalent associated bilinear form $\Pi_b(f, g) = E(\Pi_b f, g)$. Then we have

$$\begin{aligned}\Pi_b(f,g) &= E\Big(\sum_{n=1}^{\infty} \Delta_n(b) f_{n-1}\overline{\Delta_n(g)}\Big) \\ &= \sum_{Q_n^k} \langle b, \phi_{Q_n^k}\rangle\langle f, \chi_{Q_n^k}\rangle|\overline{Q}_n^k|^{-1}\overline{\langle g, \phi_{Q_n^k}\rangle}.\end{aligned} \tag{3.1}$$

If $b = \sum_{k=1}^n \Delta_k(b)$ is a finite martingale, it is easy to see that Π_b is of finite rank, and therefore compact on $L^p(\Omega)$. The set of all finite martingales is dense in VMO, so if $b \in \mathrm{VMO}$, then Π_b is compact.

Conversely, if Π_b is compact on $L^p(\Omega)$, let us show that $b \in \mathrm{VMO}$. By Lemma 2.3, it suffices to show that $|\langle b, \phi_Q\rangle| \to 0$ as $|Q| \to 0$, where $\{\phi_Q\}$ is the frame of §2.4. If that is not true, then there exists a subsequence Q_j such that $|\langle b, \phi_{Q_j}\rangle| \ge C \ge 0$; we may assume that $|\langle b, \phi_Q\rangle| \ge C \ge 0$. Note that

$$C \le |\langle b, \phi_Q\rangle| = |\langle \Pi_b(\chi_Q), \phi_Q\rangle| \le d^{n(1/2-1/p')} \sup_{\|g\|_{p'}\le 1} E(\Pi_b(\chi_Q)\overline{g})$$
$$= d^{n(1/2-1/p')}\|\Pi_b(\chi_Q)\|_p = C\|\Pi_b(|Q|^{1/p'-1/2}\chi_Q)\|_p.$$

But $|Q|^{1/p'-1/2}\chi_Q \to 0$ weakly in $L^p(\Omega)$ as $|Q| \to 0$ by Lemma 2.3, and the compactness of Π_b implies that $\|\Pi_b(|Q|^{1/p'-1/2}\chi_Q)\|^p \to 0$. This contradiction shows that $b \in \mathrm{VMO}$.

If $b \in B_p$, then

$$\|b\|_{B_p} \equiv \Big\{\sum_{Q\in\mathcal{P}} |Q|^{-p/2}|\langle b, \phi_Q\rangle|^p\Big\}^{1/p} < \infty.$$

By Lemma 2.1 and (3.1), we have

$$\|\Pi_b\|_{S_p}^p \le C\sum_Q (|Q|^{-1/2}|\langle b, \phi_Q\rangle|)^p = C\|b\|_{B_p}^p.$$

Conversely, if $1 < p < \infty$ and $\Pi_b \in S_p$, then by Lemma 2.2 we have

$$\|b\|_{B_p}^p \le C\sum_{Q\in\mathcal{P}} |Q|^{-p/2}|\langle b, \phi_Q\rangle|^p$$
$$= C\sum_Q |Q|^{-1/p}|\langle \Pi_b(\chi_Q), \phi_Q\rangle|^p \le C\|\Pi_b\|_{S_p}^p.$$

3.2. *Fractional integrals*. Let $\alpha > 0$ and $f \in L^2(\Omega)$. The fractional integral I^α can be written as

$$I^\alpha f = \sum_{n=1}^{\infty} d^{-n\alpha}\Delta_n(f) = \sum_{Q,i} d^{-(n+1)\alpha}\langle f, \psi_Q^i\rangle\psi_Q^i,$$

where $\{\psi_Q^i\}$ is the orthonormal basis of §2.3. This means that I^α has a Schmidt decomposition, so

$$\|I^\alpha\|_{S_p}^p = \sum_{Q,i} d^{-(n+1)\alpha p} = (d-1)\sum_n d^{n-(n+1)\alpha p}.$$

Thus we get

THEOREM 3.2. *If $\alpha > 0$, then $I^\alpha \in S_p(L^2, L^2)$ if and only if $p > 1/\alpha$, and*

$$\|I^\alpha\|_{S_p} = \Big\{(d-1)\sum_n d^{n-(n+1)\alpha p}\Big\}^{1/p}.$$

Remark. Theorem 3.2 says that I^α has a cut off at $p = 1/\alpha$.

4. Commutators. Now we return to the commutators $[b, T_\nu]$, $[b, I^\alpha]$ and $[b, T_A]$. The main results for them are the following three theorems.

THEOREM 4.1. (1) *For* $1 < p < \infty$, $[b, T_\nu]$ *is bounded on* L^p *if and only if* $b \in \mathrm{BMO}$.

(2) *For* $1 < p < \infty$, $[b, T_\nu]$ *is compact on* L^p *if and only if* $b \in \mathrm{VMO}$.

(3) *For* $0 < p < \infty$, $[b, T_\nu] \in S_p(L^2, L^2)$ *if and only if* $b \in B_p$.

THEOREM 4.2. *Let* $\alpha \neq 0$.

(1) *For* $1 < p < \infty$, $[b, I^\alpha]$ *is bounded on* L^p *if and only if* $I^\alpha b \in \mathrm{BMO}$.

(2) *For* $1 < p < \infty$, $[b, I^\alpha]$ *is compact on* L^p *if and only if* $I^\alpha b \in \mathrm{VMO}$.

(3) *For* $0 < p < \infty$, $[b, I^\alpha] \in S_p(L^2, L^2)$ *if and only if* $b \in B_p^{1/p-\alpha}$.

THEOREM 4.3. (1) *For* $1 < p < \infty$, $[b, T_A]$ *is bounded on* L^p *if and only if* $b \in \mathrm{BMO}$.

(2) *For* $1 < p < \infty$, $[b, T_A]$ *is compact on* L^p *if and only if* $b \in \mathrm{VMO}$.

(3) *For* $0 < p < \infty$, $[b, T_A] \in S_p(L^2, L^2)$ *if and only if* $b \in B_p$.

We postpone the proof of the partial converse results in part (3) of these theorems for $0 < p < 1$ to §5. Now we give the proofs of the rest of the theorems.

The boundedness results in part (1) of the above three theorems are known. Theorems 4.1(1) and 4.3(1) are due to Chao, Daly and Ombe [5]. Theorem 4.2(1) can be proved in the same way. It can also be obtained from the proof of the S_p-estimates given below.

4.1. *Compactness*. We start with a general linear operator T on $L^2(\Omega)$. Let T denote again its associate bilinear form, T' denote its adjoint in the sense $E(g\overline{T'(h)}) = E(T(g)\overline{h})$ and T_b denote the commutator $[b, T]$. Formally we have (see [5], p. 63)

$$E([b, T](f)\overline{g}) = E(b(T(f)\overline{g} - f\overline{T'(g)})). \tag{4.1}$$

Now we prove the compactness results in part (2) of Theorems 4.1, 4.2 and 4.3. Let b be a finite martingale. Then $[b, T_\nu], [b, I^\alpha]$ and $[b, T_A]$ are of finite rank; this implies that if $b \in \mathrm{VMO}$ (for Theorems 4.1 and 4.3) or $I^\alpha b \in \mathrm{VMO}$ (for Theorem 4.2), then $[b, T_\nu], [b, I^\alpha]$ and $[b, T_A]$ are compact.

To get the converse results we need the following fact, which is easily shown from the proof of Theorems 4, 6, 8 in [4].

Let $\{\psi_Q^i\}$ be an orthonormal basis in L^2 of §2.3. Then there exists $\{h_Q^i\}$ with $\mathrm{supp}(h_Q^i) \subset \overline{Q}$ and $\|h_Q^i\|_\infty \leq 1$ such that

$$\psi_Q^i = T_\nu(h_Q^i)\overline{g}_Q^i - h_Q^i\overline{T'_\nu(g_Q^i)},$$

where $g_Q^i = C\psi_Q^i$. Similarly we have

$$I^\alpha\psi_Q^i = I^\alpha(h_Q^i)\overline{g}_Q^i - h_Q^i\overline{I^\alpha(g_Q^i)}, \qquad \psi_Q^i = T_A(h_Q^i)\overline{g}_Q^i - h_Q^i\overline{T'_A(g_Q^i)}.$$

From this fact and (4.1) we have

$$
\begin{aligned}
|\langle b,\psi_Q^i\rangle| &= |E(b(T_\nu(h_Q^i)\overline{g}_Q^i - h_Q^i\overline{T_\nu'(g_Q^i)}))| = |E([b,T_\nu](h_Q^i)\overline{g}_Q^i)|\\
&\le \|[b,T_\nu](|Q|^{1/p'-1/2}h_Q^i)\|_p,\\
|\langle I^\alpha b,\psi_Q^i\rangle| &= |\langle b, I^\alpha\psi_Q^i\rangle| = |E(b(I^\alpha(h_Q^i)\overline{g}_Q^i - h_Q^i\overline{I^\alpha(g_Q^i)}))|\\
&= |E([b,I^\alpha](h_Q^i)\overline{g}_Q^i)| \le \|[b,I^\alpha](|Q|^{1/p'-1/2}h_Q^i)\|_p,\\
|\langle b,\psi_Q^i\rangle| &= |E(b(T_A(h_Q^i)\overline{g}_Q^i - h_Q^i\overline{T_A'(g_Q^i)}))|\\
&= |E([b,T_A](h_Q^i)\overline{g}_Q^i)| \le \|[b,T_A](|Q|^{1/p'-1/2}h_Q^i)\|_p.
\end{aligned}
$$

If $[b,T_\nu]$, $[b,I^\alpha]$ or $[b,T_A]$ is compact, then $|\langle b,\psi_Q^i\rangle|\to 0$ or $|\langle I^\alpha b,\psi_Q^i\rangle|\to 0$ as $|Q|\to 0$, and therefore $b\in \mathrm{VMO}$ (for Theorem 4.1 and 4.3) or $I^\alpha b\in \mathrm{VMO}$ (for Theorem 4.2).

4.2. *S_p-direct estimates.* For $T_b=[b,T]$, let

$$T_b^{nm}(f,g) = E(T_b(\Delta_n(f))\overline{\Delta_m(g)}).$$

Then

$$T_b = T_b^{(1)} + T_b^{(2)} + T_b^{(3)}, \tag{4.2}$$

where

$$T_b^{(1)} = \sum_{n=1}^{\infty}\sum_{m=1}^{n-1} T_b^{nm},\quad T_b^{(2)} = \sum_{m=1}^{\infty}\sum_{n=1}^{m-1} T_b^{nm}\quad\text{and}\quad T_b^{(3)} = \sum_{n=1}^{\infty} T_b^{nn}.$$

Note that

$$E_nT_\nu = T_\nu E_n,\quad E_nI^\alpha = I^\alpha E_n\quad\text{and}\quad E_nT_A = T_AE_n.$$

Then we have

$$
\begin{aligned}
E([b,T_\nu]f\overline{g}) &= E(T_\nu(f)\Pi_b(\overline{g})) - E(f\Pi_b(\overline{T_\nu' g}))\\
&\quad + E(\Pi_b(T_\nu f)\overline{g}) - E(\Pi_b(f)\overline{T_\nu' g}),
\end{aligned}\tag{4.3}
$$

$$
\begin{aligned}
E([b,I^\alpha]f\overline{g}) &= E(I^\alpha(f)\Pi_b(\overline{g})) - E(f\Pi_b(\overline{I^\alpha g}))\\
&\quad + E(\Pi_b(I^\alpha f)\overline{g}) - E(\Pi_b(f)\overline{I^\alpha g}),
\end{aligned}\tag{4.4}
$$

and

$$
\begin{aligned}
E([b,A]f\overline{g}) &= E(T_A(f)\Pi_b(\overline{g})) - E(f\Pi_b(\overline{T_A' g}))\\
&\quad + E(\Pi_b(T_A f)\overline{g}) - E(\Pi_b(f)\overline{T_A' g}) + T_b^{(3)}(f,g),
\end{aligned}\tag{4.5}
$$

where

$$T_b^{(3)}(f,g) = \sum_{n=1}^{\infty} E(b(T_A\Delta_n(f)\overline{\Delta_n(g)} - \Delta_n(f)\overline{T_A\Delta_n(g)}))$$

$$= \sum_{Q\in\mathcal{P}}\sum_{i=1}^{d-1}\sum_{i'=1}^{d-1} E\Big((b-b_{\bar{Q}})\sum_{s=1}^{d}(a_{st}c_t^i\overline{c}_s^{i'} - \overline{a}_{ts}c_s^i\overline{c}_t^{i'})\chi_Q\Big)$$
$$\times\, d^n\langle f,\psi_Q^i\rangle\overline{\langle g,\psi_Q^i\rangle}$$
$$= \sum_{Q\in\mathcal{P}}\sum_{i=1}^{d-1}\sum_{i'=1}^{d-1}\sum_{s=1}^{d} A_{ii's}\Delta_n(b)\chi_Q\langle f,\psi_Q^i\rangle\overline{\langle g,\psi_Q^i\rangle}.$$

Then Theorem 3.1 gives us

$$\begin{aligned}
&\|[b,T_\nu]\|_{L^p\to L^p} \le C\|b\|_{\mathrm{BMO}} && \text{for } 1<p<\infty,\\
&\|[b,T_\nu]\|_{S_p} \le C\|b\|_p && \text{for } 0<p<\infty,\\
&\|[b,I^\alpha]\|_{L^p\to L^p} \le C\|I^\alpha b\|_{\mathrm{BMO}} && \text{for } 1<p<\infty,\\
&\|[b,I^\alpha]\|_{S_p} \le C\|I^\alpha b\|_p = C\|b\|_{B_p^{1/p-\alpha}} && \text{for } 0<p<\infty,
\end{aligned}$$

and

$$\begin{aligned}
&\|T_b^{(3)}\|_{L^p\to L^p} \le C\|b\|_{\mathrm{BMO}} && \text{for } 1<p<\infty,\\
&\|T_b^{(3)}\|_{S_p} \le C\|b\|_p && \text{for } 0<p<\infty.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\|[b,T_A]\|_{L^p\to L^p} \le C\|b\|_{\mathrm{BMO}} && \text{for } 1<p<\infty,\\
&\|[b,T_A]\|_{S_p} \le C\|b\|_p && \text{for } 0<p<\infty.
\end{aligned}$$

4.3. *S_p-converse estimates for $1\le p\le\infty$.* If an operator S has the Schmidt decomposition $S=\sum\lambda_i\langle\cdot,e_i\rangle f_i$, then $\mathrm{tr}(ST^*)=\sum\lambda_i\langle T(f_i),e_i\rangle$. By this fact and Theorem 4.1, for $g\in B_{p'}^{-1/p}$, we have

$$|E(b\overline{g})| = \Big|E\Big(b\sum_{Q,i}|Q|^{1/2}(T_\nu(h_Q^i)g_Q^i - h_Q^iT'_\nu(g_Q^i))\overline{\langle g,\psi_Q^i\rangle}\Big)\Big|$$
$$= \Big|\sum_{Q,i}|Q|^{1/2}\overline{\langle g,\psi_Q^i\rangle}E([b,T_\nu](h_Q^i)g_Q^i)\Big| = |\mathrm{tr}(S[b,T_\nu]^*)|,$$

where

$$S=\sum_{Q,i}|Q|^{1/2}\langle g,\psi_Q^i\rangle\langle\cdot,g_Q^i\rangle h_Q^i,$$

and $\{g_Q^i\}$ and $\{h_Q^i\}$ are NWO. Therefore

$$|E(bg)| \le \|[b,T_\nu]\|_{S_p}\|S\|_{S_{p'}} \le \|[b,T_\nu]\|_{S_p}\Big\{\sum_{Q,i}|Q|^{1/2}|\langle g,\psi_Q^i\rangle|^{p'}\Big\}^{1/p'}$$
$$\le C\|[b,T_\nu]\|_{S_p}\|g\|_{B_{p'}^{-1/p}}.$$

So we get $\|b\|_{B_p}\le C\|[b,T_\nu]\|_{S_p}$ for $1<p<\infty$.

Similarly we have

$$
\begin{aligned}
|E(I^\alpha(b)\overline{g})| &= \Big|E\Big(b\sum_{Q,i}|Q|^{1/2}(I^\alpha(h_Q^i)g_Q^i - h_Q^i I^\alpha(g_Q^i))\overline{\langle g,\psi_Q^i\rangle}\Big)\Big| \\
&= \Big|\sum_{Q,i}|Q|^{1/2}\overline{\langle g,\psi_Q^i\rangle}E([b,I^\alpha](h_Q^i)g_Q^i)\Big| = |\mathrm{tr}(S[b,I^\alpha]^*)|,
\end{aligned}
$$

where S, $\{g_Q^i\}$ and $\{h_Q^i\}$ are as before. Therefore

$$
\begin{aligned}
|E(I^\alpha(b)g)| &\le \|[b,I^\alpha]\|_{S_p}\|S\|_{S_{p'}} \le \|[b,I^\alpha]\|_{S_p}\Big\{\sum_{Q,i}|Q|^{1/2}|\langle g,\psi_Q^i\rangle|^{p'}\Big\}^{1/p'} \\
&\le C\|[b,I^\alpha]\|_{S_p}\|g\|_{B_{p'}^{-1/p}}.
\end{aligned}
$$

Thus $\|I^\alpha b\|_{B_p} \le C\|[b,I^\alpha]\|_{S_p}$ for $1<p<\infty$.

Finally, we have

$$
\begin{aligned}
|E(b\overline{g})| &= \Big|E\Big(b\sum_{Q,i}|Q|^{1/2}(T_A(h_Q^i)g_Q^i - h_Q^i T_A'(g_Q^i))\overline{\langle g,\psi_Q^i\rangle}\Big)\Big| \\
&= \Big|\sum_{Q,i}|Q|^{1/2}\overline{\langle g,\psi_Q^i\rangle}E([b,T_A](h_Q^i)g_Q^i)\Big| = |\mathrm{tr}(S[b,T_A]^*)|,
\end{aligned}
$$

where S, $\{g_Q^i\}$ and $\{h_Q^i\}$ are as before. Hence

$$
\begin{aligned}
|E(bg)| &\le \|[b,T_A]\|_{S_p}\|S\|_{S_{p'}} \le \|[b,T_A]\|_{S_p}\Big\{\sum_{Q,i}|Q|^{1/2}|\langle g,\psi_Q^i\rangle|^{p'}\Big\}^{1/p'} \\
&\le C\|[b,T_A]\|_{S_p}\|g\|_{B_{p'}^{-1/p}}.
\end{aligned}
$$

Therefore $\|b\|_{B_p} \le C\|[b,T_A]\|_{S_p}$ for $1<p<\infty$.

We can also get the BMO-estimates for these three commutators by using $g\in H^1$ as in [5].

5. S_p-converse estimates for $0<p<1$. Here we just follow the argument in Peng [19].

LEMMA 5.1. *Suppose that $b\mapsto T_b$ is a linear map from* BMO *to $S_\infty(L^2,L^2)$, define $T_b^{n,m}=\Delta_m T_b\Delta_n$, and suppose that*

(1) $E_nT_b = T_{b_n}$,

(2) $\|T_b^{n+1,n}\|_{S_p} \ge Cd^{(n+1)/p}\|\Delta_n(b)\|_p$
(*or* $\|T_b^{n,n+1}\|_{S_p} \ge Cd^{(n+1)/p}\|\Delta_n(b)\|_p$),

(3) $\|\sum_{n=1}^\infty \Delta_nT_bE_{n-N}\|_{S_p} \le Cd^{-N/2}\|b\|_{B_p}$ *and* $\|\sum_{n=1}^\infty E_{n-N}T_b\Delta_n\|_{S_p} \le Cd^{-N/2}\|b\|_{B_p}$.

Then $T_b\in S_p$ implies $b\in B_p$ and

$$\|b\|_{B_p} \le C\|T_b\|_{S_p}. \tag{5.1}$$

Proof. It suffices to show (5.1) for $b \in B_p$. Since any finite martingale $b \in B_p$, by (1) we have

$$\|b_N\|_{B_p} \le C\|T_{b_N}\|_{S_p} \le C\|E_N T_b\|_{S_p} \le C\|T_b\|_{S_p}.$$

Letting $N \to \infty$, we get (5.1) for general b.

Now assume $b \in B_p$, and let us show (5.1). For N large enough, let

$$T_{b,k} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} T_b^{Nn+k+1,Nm+k} \quad \text{for } k = 0, 1, \ldots, N-1.$$

Then $\|T_{b,k}\|_{S_p} \le \|T_b\|_{S_p}$. Define

$$T_{b,k}^{(0)} = \sum_{n=0}^{\infty} T_b^{Nn+k+1,Nn+k},$$

$$T_{b,k}^{(1)} = \sum_{n=0}^{\infty}\sum_{m=0}^{n-1} T_b^{Nn+k+1,Nm+k},$$

$$T_{b,k}^{(2)} = \sum_{m=0}^{\infty}\sum_{n=0}^{m-1} T_b^{Nn+k+1,Nn+k}.$$

Then

$$N\|T_b\|_{S_p}^p \ge \sum_{k=0}^{N-1} \|T_{b,k}\|_{S_p}^p \ge \sum_{k=0}^{N-1} \{\|T_{b,k}^{(0)}\|_{S_p}^p - \|T_{b,k}^{(1)}\|_{S_p}^p - \|T_{b,k}^{(2)}\|_{S_p}^p\},$$

$$\|T_{b,k}^{(0)}\|_{S_p}^p \ge \sum_{k=0}^{\infty} \|T_b^{Nn+k+1,Nn+k}\|_{S_p}^p \ge Cd^{Nn+k+1}\|\varDelta_{Nn+k+1}(b)\|_p^p \quad \text{(by (2))},$$

$$\|T_{b,k}^{(1)}\|_{S_p}^p \le Cd^{-Np/2}\|b\|_{B_p}^p, \qquad \|T_{b,k}^{(2)}\|_{S_p}^p \le Cd^{-Np/2}\|b\|_{B_p}^p.$$

Thus we get

$$N\|T_b\|_{S_p}^p \ge C_1\|b\|_{B_p}^p - C_2 N d^{-Np/2}\|b\|_{B_p}^p.$$

Choosing N so large that $C_1 - C_2 N d^{-Np/2} = C > 0$, we obtain (5.1).

Now we check that $\varPi_b$, $[b, T_\nu]$, $[I^{-\alpha}, I^\alpha]$ and $[b, T_A]$ satisfy the conditions of Lemma 5.1. In fact our main task is to verify (2) and (3), the others are trivial.

For $\varPi_b$,

$$\|T_b^{n+1,n}\| = \Big\| \sum_{k=0}^{d^{n-1}+1}\sum_{i=1}^{d-1}\sum_{k'=kd}^{(k+1)d-1}\sum_{i'=1}^{d-1} \langle b, \psi_{Q_n^{k'}}^{i'}\rangle\langle \cdot, \psi_{Q_{n-1}^k}^i\rangle \psi_{Q_{n-1}^k}^i \psi_{Q_n^{k'}}^{i'} \Big\|_{S_p}$$

$$\ge Cd^{n/2}\Big\{\sum_{|Q|=d^{-n}} |\langle b, \psi_Q^i\rangle|^p\Big\}^{1/p} = Cd^{n/p}\|\varDelta_n(b)\|_p,$$

$$T_b^{n,n+1} = 0,$$

$$\sum_{n=1}^{\infty} \Delta_n \Pi_b E_{n-N} = \sum_{Q,i} \langle b, \psi^i_{Q^k_{n-1}} \rangle \Big\langle \cdot, \frac{1}{|Q^{k'}_{n-N}|^{1/2}} \chi_{Q^{k'}_{n-N}} \Big\rangle d^{(n-N)/2} \psi^i_{Q^k_{n-1}}.$$

By Lemma 3.2, we have

$$\Big\| \sum_{n=1}^{\infty} \Delta_n \Pi_b E_{n-N} \Big\|_{S_p} \le C d^{-N/(2p)} \|b\|_{B_p}, \qquad \sum_{m=0}^{\infty} E_{m_n} \Pi_b \Delta_m = 0.$$

The verifications of (3) for $[b, T_\nu]$, $[I^{-\alpha} b, I^\alpha]$ and $[b, T_A]$ are similar: just use Lemma 3.2 and (4.3)–(4.5).

Now we verify (2) for the three commutators. First we consider $[b, T_\nu]$. Suppose that ν satisfies the nondegeneracy condition (D_ν). Then for any Q^k_n, there exists $Q^{k'}_n \ne Q^k_n$ such that $T_\nu(\chi_{Q^{k'}_n}) = C \ne 0$ for $x \in Q^k_n$ (see [5]). Thus

$$\begin{aligned}
T^{n+1,n}_b(f,g) &= E(b(T_\nu \Delta_{n+1}(f)\overline{\Delta_n(g)} - \Delta_{n+1}(f) T'_\nu \overline{\Delta_n(g)})) \\
&= \sum_{k=0}^{d^n-1} \sum_{i=1}^{d-1} E(b(T_\nu \psi^i_{Q^k_n} \overline{\Delta_n(g)} - \psi^i_{Q^k_n} T'_\nu \overline{\Delta_n(g)})) \langle f, \psi^i_{Q^k_n} \rangle \\
&= C \sum_{k=0}^{d^n-1} \sum_{i=1}^{d-1} \langle b, \psi^i_{Q^k_n} \rangle \langle f, \psi^i_{Q^k_n} \rangle \overline{\Delta_n(g)}|_{Q^k_n}, \\
\|T^{n+1,n}_b\|^p_{S_p} &= C \sum_{k=0}^{d^n-1} \sum_{i=1}^{d-1} |\langle b, \psi^i_{Q^k_n} \rangle|^p d^{np/2} = C d^{n+1} \|\Delta_{n+1}(b)\|^p_p.
\end{aligned}$$

The verification for $[I^{-\alpha} b, I^\alpha]$ is similar.

Finally, we verify $[b, T_A]$. Suppose that A satisfies the nondegeneracy condition (D_A). Then for any k, there exist i and j such that $a_{ki} \ne a_{kj}$. Thus

$$\begin{aligned}
T^{n+1,n}_b(f,g) &= E(b(T_A \Delta_{n+1}(f)\overline{\Delta_n(g)} - \Delta_{n+1}(f) T'_A \overline{\Delta_n(g)})) \\
&= \sum_{k=0}^{d^n-1} \sum_{i=1}^{d-1} E(b(T_A \psi^i_{Q^k_n} \overline{\Delta_n(g)} - \psi^i_{Q^k_n} T'_A \overline{\Delta_n(g)})) \langle f, \psi^i_{Q^k_n} \rangle.
\end{aligned}$$

Notice that

$$\begin{aligned}
&\sum_{k=0}^{d^n-1} \sum_{i=1}^{d-1} \langle [b, T_A] \Delta_n(f)|_{Q^k_n}, P_{\bar{Q}^k_n} g \rangle \\
&\qquad = \sum_{k=0}^{d^n-1} \sum_{i=1}^{d-1} (a_{k\,(\mathrm{mod}\, d) i} - a_{k\,(\mathrm{mod}\, d) j}) \langle b, \psi^i_{Q^k_n} \rangle \overline{(g_{Q^i_n} - g_{Q^j_n})} \langle f, \psi^i_{Q^k_n} \rangle.
\end{aligned}$$

Therefore

$$\|T_b^{n+1,n}\|_{S_p}^p \geq C \sum_{k=0}^{d^n-1} \sum_{i=1}^{d-1} |\langle b, \psi^i_{Q_n^k}\rangle|^p d^{np/2} = C\|b\|_{B_p}^p.$$

This completes the proof.

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