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# CONTINUOUS EXTENSIONS OF SPECTRAL MEASURES

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One of the most important classes of operator-valued measures is the class of spectral measures. These are the natural extension to Banach spaces (and more general locally convex Hausdorff spaces, briefly lcHs) of the notion of the resolution of the identity of normal operators in a Hilbert space. In many applications, and in the general theory, the underlying lcHs $X$ in which the spectral measure acts may have very poor completeness properties. Consequently, the space of integrable functions may be too small for any reasonable analysis [11].

One natural response to such a situation might be to attempt to extend the spectral measure into acting in the completion, $\overline{X}$, of $X$. Indeed, certain problems have been treated quite effectively using this type of approach [14, 17, 19] by interpreting $X$ as a projective limit of seminormed spaces. For each continuous seminorm $q$ in $X$, the spectral measure $P$ acting in $X$ induces a spectral measure $P_q$ in the quotient normed space $X_q = X/q^{-1}(\{0\})$ which can then be extended to a spectral measure $\overline{P}_q$ in the (Banach space) completion $\overline{X}_q$. This reduces the problem to the more familiar Banach space setting by considering the family of spectral measures $\overline{P}_q$ acting in $\overline{X}_q$, as $q$ varies through the collection $\mathcal{P}(X)$ of all continuous seminorms in $X$.

For other types of problems it is more suitable to deal with $X$ directly and simply attempt to extend $P$ to $\overline{X}$. The main difficulties here are, firstly, that in practical examples of interest the completion $\overline{X}$ is difficult to identify and, secondly, from the point of view of analysis and integration theory, the space $\overline{X}$ may turn out to be unnecessarily large. It usually suffices to have $P$ extendable merely to the quasicompletion, $\widetilde{X}$, of $X$, or even the smaller sequential completion, $\widehat{X}$, of $X$ [2, 3, 11, 12, 16].

The aim of this note is to make a detailed study of the process of extending a given spectral measure $P$, acting in a lcHs $X$, to the various "completions" $\widehat{X}$, $\widetilde{X}$ and $\overline{X}$. Of particular interest is the determination of criteria which ensure that the extended $P$ is actually a spectral measure

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again (this is not always automatic) and to identify the integrable functions for the extended measure in terms of those for the original measure $P$. It turns out that the extension process to the sequential completion $\widehat{X}$ is the most satisfactory. This is somewhat unexpected since $\widetilde{X}$ and $\overline{X}$ often have desirable topological properties which $\widehat{X}$ may not share (e.g. the balanced, convex hull of a relatively compact subset of a quasicomplete space is again compact; this fails for sequentially complete spaces, [13; §2]).

**1. Preliminaries.** In this section we fix the notation, record some definitions and establish some basic facts needed later. Throughout, $X$ is a lcHs and $X'$ its continuous dual space. Every subspace of $X$ is equipped with the induced topology from $X$.

Let $\Sigma$ be a $\sigma$-algebra of subsets of a non-empty set $\Gamma$. Let $m : \Sigma \to X$ be a vector measure, meaning that the sequence $\{m(E_n)\}_{n=1}^{\infty}$ is unconditionally summable in $X$ with $m(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n)$, whenever $E_n \in \Sigma$, for $n \in \mathbb{N}$, are pairwise disjoint sets. Given $x' \in X'$, let $\langle x', m\rangle$ denote the complex measure defined by $\langle x', m\rangle(E) = \langle x', m(E)\rangle$ for each $E \in \Sigma$. Its variation measure is denoted by $|\langle x', m\rangle|$. The space of all $\mathbb{C}$-valued, $\Sigma$-simple functions on $\Gamma$ is denoted by $\mathrm{sim}(\Sigma)$.

A $\Sigma$-measurable function $f : \Gamma \to \mathbb{C}$ is called *$m$-integrable* if it is $\langle x', m\rangle$-integrable for every $x' \in X'$ and if there exists a unique set function $fm : \Sigma \to X$ such that $\langle x', (fm)(E)\rangle = \int_E f\, d\langle x', m\rangle$, for $x' \in X'$, $E \in \Sigma$. The set function $fm$, which is again a vector measure by the Orlicz–Pettis theorem [6; I, Theorem 1.3], is called the *indefinite integral* of $f$ with respect to $m$. Given an $m$-integrable function $f$, the element $(fm)(E)$ of $X$ is also denoted by $\int_E f\, dm$, for each $E \in \Sigma$. The linear space of all $m$-integrable functions is denoted by $\mathcal{L}^1(m)$. Clearly $\mathrm{sim}(\Sigma) \subseteq \mathcal{L}^1(m)$.

The characteristic function of $E \subseteq \Gamma$ is denoted by $\chi_E$. A set $E \in \Sigma$ is called *$m$-null* if $\chi_E m$ is the zero measure. A $\mathbb{C}$-valued, $\Sigma$-measurable function is said to be *$m$-essentially bounded* if it is bounded off an $m$-null set. The space of all $m$-essentially bounded functions is denoted by $\mathcal{L}^\infty(m)$. If $X$ is sequentially complete, then

$$\mathcal{L}^\infty(m) \subseteq \mathcal{L}^1(m); \tag{1.1}$$

see [9; p. 161]. The inclusion (1.1) is not always valid; see [11], for example.

LEMMA 1.1. *Let $m_\alpha : \Sigma \to X$, $\alpha \in A$, be a net of vector measures converging setwise to a vector measure $m : \Sigma \to X$ with* $\sup\{p(m_\alpha(E)) : \alpha \in A,\ E \in \Sigma\} < \infty$ *for each $p \in \mathcal{P}(X)$. Suppose that a function $f : \Gamma \to \mathbb{C}$ is bounded, $\Sigma$-measurable and integrable with respect to $m$ and each $m_\alpha$, $\alpha \in A$. Then* $\lim_\alpha \int_\Gamma f\, dm_\alpha = \int_\Gamma f\, dm$.

Proof. It is clear that $\int_\Gamma s\,dm_\alpha \to \int_\Gamma s\,dm$ whenever $s \in \operatorname{sim}(\Sigma)$. Suppose now that $f$ is as stated. It is known that

$$\sup_{E\in\Sigma} q\Big(\int_E f\,dn\Big) \le 4(\operatorname*{ess\,sup}_{\gamma\in\Gamma} |f(\gamma)|) \sup_{E\in\Sigma} q(n(E)),$$

for every $q \in \mathcal{P}(X)$ and any vector measure $n : \Sigma \to X$; see [6; I, Lemmas 1.2 & 2.2]. The desired conclusion follows from this inequality, the fact that $f$ can be approximated uniformly on $\Gamma$ by elements from $\operatorname{sim}(\Sigma)$, and the Nikodym boundedness theorem (apply [1; I, Theorem 3.1] in the seminormed space $(X, q)$, for each $q \in \mathcal{P}(X)$). ∎

We note that if $m : \Sigma \to X$ is a vector measure which takes its values in a subspace $Z \subseteq X$ and $f$ is an $m$-integrable function with $\int_E f\,dm \in Z$, for each $E \in \Sigma$, then $f$ is $m_Z$-integrable, where $m_Z$ denotes $m$ considered as taking its values in $Z$.

The space of all continuous linear operators of $X$ into itself is denoted by $L(X)$. When $L(X)$ is equipped with the strong operator topology $\tau_{\mathrm{s}}$ (i.e. the topology of pointwise convergence in $X$), we denote it by $L_{\mathrm{s}}(X)$. When a sequence is convergent in $L_{\mathrm{s}}(X)$ we will simply say that the sequence *strongly converges*.

Given a set function $P : \Sigma \to L(X)$ and $x \in X$, let $Px : \Sigma \to X$ denote the set function $Px : E \mapsto P(E)x$, for $E \in \Sigma$. A linear subspace $Y$ of $X$, not necessarily closed, is called *$P$-invariant* if $P(E)Y \subset Y$ for each $E \in \Sigma$. Let $J_Y : Y \to X$ be the natural injection. The *restriction* of $P$ to $Y$ is the set function $P_Y : \Sigma \to L(Y)$ such that

$$J_Y \circ P_Y(E) = P(E) \circ J_Y, \qquad E \in \Sigma. \tag{1.2}$$

Let $P : \Sigma \to L(X)$ be a *spectral measure*. In other words, $P$ is a multiplicative, operator-valued measure satisfying $P(\Gamma) = I$ (the identity operator in $X$). Of course, the countable additivity of $P$ is with respect to $\tau_{\mathrm{s}}$; this is often indicated explicitly by writing $P : \Sigma \to L_{\mathrm{s}}(X)$. By multiplicativity we mean that $P(E\cap F) = P(E)P(F)$ for all $E, F \in \Sigma$. For each $f \in \mathcal{L}^1(P)$, let $P(f) = (fP)(\Gamma) = \int_\Gamma f\,dP$. The multiplicativity of $P$ implies that $E \in \Sigma$ is $P$-null iff $P(E) = 0$. Integrability with respect to $P$ is determined in a simpler way than for general vector measures, due to the multiplicativity of $P$.

LEMMA 1.2. *Let $P : \Sigma \to L_s(X)$ be a spectral measure. The following statements for a $\mathbb{C}$-valued, $\Sigma$-measurable function $f$ on $\Gamma$ are equivalent*:

(o) *The function $f$ is $P$-integrable.*

(i) *The function $f$ is $\langle x', Px\rangle$-integrable, for $x' \in X'$ and $x \in X$, and there is $T_1 \in L(X)$ such that $\langle x', T_1x\rangle = \int_\Gamma f\,d\langle x', Px\rangle$, for $x' \in X'$, $x \in X$.*

(ii) *The function $f$ is $Px$-integrable for each $x \in X$, and there is an operator $T_2 \in L(X)$ such that $T_2 x = \int_\Gamma f\, dPx$, for $x \in X$.*

(iii) *There exist functions $s_n \in \mathrm{sim}(\Sigma)$, for $n \in \mathbb{N}$, converging pointwise to $f$, such that the sequence $\{P(s_n)\}_{n=1}^\infty$ strongly converges to some operator $T_3 \in L(X)$.*

(iv) *There exist functions $f_n \in \mathcal{L}^1(P)$, for $n \in \mathbb{N}$, converging pointwise to $f$, such that the sequence $\{P(f_n)\}_{n=1}^\infty$ strongly converges to some operator $T_4 \in L(X)$.*

*In this case $T_j = P(f)$ for each $j = 1, 2, 3, 4$, and*

$$P(f\chi_E) = P(f)P(E) = P(E)P(f), \qquad E \in \Sigma. \tag{1.3}$$

Proof. For the equivalence of (o) and (i) see [3; Proposition 1.2]. The definition of integrability ensures the equivalence of (i) and (ii). The equivalence of (o), (iii) and (iv) follows from [9; Lemma 2.3 & Theorem 2.4] and the equivalence of (o) and (i). ∎

The following result, of interest in its own right, will be needed in Section 2.

LEMMA 1.3. *Let $X$ be a lcHs and $Q : \Sigma \to L(X)$ be a finitely additive set function defined on a $\sigma$-algebra $\Sigma$ and such that its range $Q(\Sigma)$ is a bounded subset of $L_\mathrm{s}(X)$. Then $\sup\{q(Q(E)x) : x \in B,\ E \in \Sigma\} < \infty$, for $q \in \mathcal{P}(X)$ and each bounded set $B \subset X$.*

Proof. Fix $q \in \mathcal{P}(X)$ and a bounded set $B$ in $X$. Let $\overline{X}_q$ denote the (Banach space) completion of the normed space $X/q^{-1}(\{0\})$ and $\pi_q : X \to \overline{X}_q$ be the canonical map. For each $x \in X$, let $\mu_x = \pi_q \circ Qx : \Sigma \to \overline{X}_q$, which is a finitely additive set function with bounded range. For each $E \in \Sigma$, the set $Q(E)B$ is bounded in $X$ (as $P(E) \in L(X)$) and hence $(\pi_q \circ Q(E))B$ is bounded in $\overline{X}_q$, that is, $\sup\{\overline{q}(\mu_x(E)) : x \in B\} < \infty$. Here $\overline{q}$ denotes the norm in $\overline{X}_q$ induced from the quotient norm in $X/q^{-1}(\{0\})$. By the Nikodym boundedness theorem [1; I, Theorem 3.1],

$$\sup_{\substack{x \in B \\ E \in \Sigma}} q(Q(E)x) = \sup_{E \in \Sigma} \sup_{x \in B} \overline{q}(\mu_x(E)) < \infty. \ ∎$$

LEMMA 1.4. *Let $P : \Sigma \to L_\mathrm{s}(X)$ be a spectral measure. Then the restriction $P_Y$ of $P$ to a $P$-invariant subspace $Y$ of $X$ is an $L_\mathrm{s}(Y)$-valued spectral measure on $\Sigma$ satisfying the following statements*:

(i) *If $f \in \mathcal{L}^1(P) \cap \mathcal{L}^1(P_Y)$ and $J_Y : Y \to X$ denotes the natural injection, then*

$$P(f\chi_E) \circ J_Y = J_Y \circ P_Y(f\chi_E), \qquad E \in \Sigma. \tag{1.4}$$

*In particular, $P(f)Y \subset Y$.*

(ii) $\mathcal{L}^1(P) \cap \mathcal{L}^1(P_Y) = \{f \in \mathcal{L}^1(P) : P(f)Y \subset Y\}$.

Proof. Clearly $P_Y$ is a spectral measure. It follows from (1.2) that

$$(J_Y \circ P_Y(f))y = \int_\Gamma f\, d(J_Y \circ P_Y y) = (P(f) \circ J)y, \qquad y \in Y.$$

So (1.4) holds by Lemma 1.2. This establishes (i).

For (ii), let $f \in \mathcal{L}^1(P)$ and suppose that $P(f)Y \subseteq Y$. Let $T \in L(Y)$ be the restriction of $P(f)$ to $Y$, i.e. $P(f) \circ J_Y = J_Y \circ T$. Fix $y \in Y$ and $y' \in Y'$. By the Hahn–Banach theorem $y'$ has an extension $x' \in X'$ satisfying $y' = x' \circ J_Y$. Then, for each $E \in \Sigma$, we have $\langle y', P_Y y\rangle(E) = \langle x', J_Y(P_Y y)\rangle(E) = \langle x', P(E)(J_Y y)\rangle$. So, $f \in \mathcal{L}^1(\langle y', P_Y y\rangle)$ and

$$\int_\Gamma f\, d\langle y', P_Y y\rangle = \langle x', P(f)(J_Y y)\rangle = \langle y', Ty\rangle.$$

Lemma 1.2 implies that $f \in \mathcal{L}^1(P_Y)$. ■

Remark 1.5. (i) If $Y$ is a $P$-invariant subspace of $X$ with the property that it contains the limits of all of its convergent sequences, then $\mathcal{L}^1(P) \subseteq \mathcal{L}^1(P_Y)$. For, if $f \in \mathcal{L}^1(P)$, we can choose a sequence $\{s_n\}_{n=1}^\infty \subseteq \mathrm{sim}(\Sigma)$ such that $\{P(s_n)\}_{n=1}^\infty$ strongly converges to $P(f)$; see Lemma 1.2. Clearly $P(s_n)Y \subseteq Y$, for $n \in \mathbb{N}$. Since $P(s_n) \to P(f)$ in $L_{\mathrm{s}}(X)$ it follows that $P(f)Y \subseteq Y$ and hence $f \in \mathcal{L}^1(P_Y)$ by Lemma 1.4(ii).

(ii) The inclusion $\mathcal{L}^1(P) \subseteq \mathcal{L}^1(P_Y)$ of (i) does not hold in general. Let $X$ be the Banach space $\mathcal{L}^1([0,1])$ and, for each $E \in \Sigma$ (the Borel subsets of $\Gamma = [0,1]$), let $P(E) \in L(X)$ be the operator in $\mathcal{L}^1([0,1])$ of multiplication by $\chi_E$. Then $\mathcal{L}^1(P) = \mathcal{L}^\infty([0,1])$ and, for each $f \in \mathcal{L}^\infty([0,1])$, the element $P(f) \in L(X)$ is the operator in $\mathcal{L}^1([0,1])$ of multiplication by $f$. Let $Y_1 = \mathrm{sim}(\Sigma)$ considered as a subspace of $X$. Then $Y_1$ is $P$-invariant and $\mathcal{L}^1(P_Y) = \mathrm{sim}(\Sigma)$. So, if $g \in \mathcal{L}^\infty([0,1])$ is not a $\Sigma$-simple function, then $g \in \mathcal{L}^1(P)$ but $g \notin \mathcal{L}^1(P_{Y_1})$. We note that $Y_1$ is even dense in $X$.

Let $Y_2 = \mathcal{L}^\infty([0,1])$, considered as a (dense) subspace of $X$, in which case $\mathcal{L}^1(P_{Y_2}) = \mathcal{L}^\infty([0,1])$. So, it can happen that $\mathcal{L}^1(P_Y) = \mathcal{L}^1(P)$ for a *proper* subspace $Y$ of $X$.

(iii) If $Y$ is a dense subspace of $X$, then the $P$-null sets coincide with the $P_Y$-null sets. This is not so in general. In the notation of (ii), let $Y$ denote the closed subspace $P([0,1/2])X$, in which case $Y$ is $P$-invariant. Then every $E \in \Sigma$ which is a subset of $[1/2,1]$ is $P_Y$-null. If $E$ has positive Lebesgue measure, then $E$ is not $P$-null. ■

Let $\Lambda$ be a topological Hausdorff space and $Z \subseteq \Lambda$. Then $[Z]$ denotes the set of all elements in $\Lambda$ which are the limit of some sequence of points from $Z$. A set $Z \subseteq \Lambda$ is called *sequentially closed* if $Z = [Z]$. The *sequential closure* of a set $Z \subseteq \Lambda$ is the smallest sequentially closed subset of $\Lambda$ which contains $Z$. Alternatively, let $Z_0 = Z$. Let $\Omega_1$ be the smallest uncountable ordinal. Suppose that $0 < \alpha < \Omega_1$ and that $Z_\beta$ has been defined for all ordinals $\beta$ satisfying $0 \le \beta < \alpha$. Define $Z_\alpha = [\bigcup_{0\le\beta<\alpha} Z_\beta]$. Then $\bigcup_{0\le\alpha<\Omega_1} Z_\alpha$ is the sequential closure of $Z$ in $\Lambda$; it is equipped with the relative topology.

Let $X$ be a lcHs. The *sequential completion*, $\widehat{X}$, of $X$ is defined to be the sequential closure of $X$ in its completion $\overline{X}$. Of course, $\widehat{X}$ is the intersection in $\overline{X}$ of all sequentially complete subspaces of $\overline{X}$ which contain $X$. It is classical that typically the inclusions $X_\alpha \subseteq X_\beta \subseteq \widehat{X}$ are proper whenever $0 \le \alpha < \beta < \Omega_1$. Indeed, let $X$ be the space of continuous, $\mathbb{C}$-valued functions on $[0,1]$ equipped with the topology of pointwise convergence. Then $\overline{X} = \mathbb{C}^{[0,1]}$ is the lcHs of all $\mathbb{C}$-valued functions on $[0,1]$ equipped with the "same topology". For $1 \le \alpha < \Omega_1$, the space $X_\alpha$ consists of the Baire functions of class $\alpha$ and $\widehat{X}$ is the space of all Borel measurable functions on $[0,1]$. This example shows that $X$ need *not* be sequentially dense in $\widehat{X}$, although it is always dense in $\widehat{X}$ (being dense in $\overline{X}$). If $X$ is metrizable, then $\widehat{X} = [X] = X_1$ and $X$ is sequentially dense in $\widehat{X}$. This can also occur in non-metrizable spaces. Let $X$ denote the Banach space $c_0$ equipped with its weak topology $\sigma(c_0, \ell^1)$. Then $\widehat{X}$ is $\ell^\infty$ equipped with its weak-star topology $\sigma(\ell^\infty, \ell^1)$ and $X$ is sequentially dense in $\widehat{X}$, even though $X$ is not metrizable.

Recall that a lcHs $X$ is called *quasicomplete* if all closed and bounded subsets are complete for the relative topology. The *quasicompletion* $\widetilde{X}$ of a lcHs $X$ is defined to be the quasiclosure of $X$ taken in its completion $\overline{X}$; see [7; §23.1]. Since convergent sequences are bounded it follows easily that $\widehat{X} \subseteq \widetilde{X} \subseteq \overline{X}$ and $X$ is dense in each of $\widehat{X}$, $\widetilde{X}$ and $\overline{X}$.

LEMMA 1.6. *Let $X$ be a lcHs and $T \in L(X)$. Then $T$ has unique extensions $\widehat{T} \in L(\widehat{X})$, $\widetilde{T} \in L(\widetilde{X})$ and $\overline{T} \in L(\overline{X})$.*

Proof. For the existence and uniqueness of $\widetilde{T}$ and $\overline{T}$ we refer to [7; (4) in §23.1]. To establish the uniqueness and existence of $\widehat{T}$ it suffices to show that $\widehat{X}$ is an invariant subspace of $\overline{T}$ as then $\widehat{T}$ is the restriction of $\overline{T}$ to $\widehat{X}$. But, if $\overline{x} \in X_1$ then there is a sequence $\{x_n\}_{n=1}^\infty$ from $X$ which converges (in $\overline{X}$) to $\overline{x}$. The continuity of $\overline{T}$ implies that $Tx_n = \overline{T}x_n \to \overline{T}\overline{x}$ in $\overline{X}$. Since $Tx_n \in X$, for $n \in \mathbb{N}$, it follows that $\overline{T}\overline{x} \in X_1$, i.e. $\overline{T}X_1 \subseteq X_1$. This argument can be repeated via transfinite induction to establish that $\overline{T}\widehat{X}_\alpha \subseteq \widehat{X}_\alpha$, for each $0 \le \alpha < \Omega_1$, and hence $\overline{T}\widehat{X} \subseteq \widehat{X}$. ■

Remark 1.7. It follows from the uniqueness part of Lemma 1.6 that if $T, S \in L(X)$, then the continuous extension of $TS$ to $\widehat{X}$, $\widetilde{X}$ and $\overline{X}$ is the operator $\widehat{T}\widehat{S}$, $\widetilde{T}\widetilde{S}$ and $\overline{T}\,\overline{S}$, respectively. ■

LEMMA 1.8. *Let $Z$ be a lcHs and $Y$ be a dense subspace of $Z$. Let $H \subseteq L(Y)$ be equicontinuous. Suppose that each $T \in L(Y)$ has a (unique)* *extension* $T_Z \in L(Z)$. *Then* $H_Z = \{T_Z : T \in H\}$ *is an equicontinuous subset of* $L(Z)$.

Proof. Let $V$ be a closed neighbourhood of 0 in $Z$. Since $H$ is equicontinuous at 0 in $Y$, there is an open neighbourhood $U$ of 0 in $Z$ such that $H(U \cap Y) \subseteq V \cap Y$. Now

$$\begin{aligned} H_Z(U) &= H_Z(U \cap \mathrm{Cl}(Y)) \subseteq H_Z(\mathrm{Cl}(U \cap Y)) \\ &\subseteq \mathrm{Cl}(H_Z(U \cap Y)) \subseteq \mathrm{Cl}(V) = V, \end{aligned}$$

where Cl denotes closure. Hence, $H_Z$ is equicontinuous. ■

**2. $\sigma$-additive extensions.** Throughout this section let $X$ be a lcHs and $P : \Sigma \to L_{\mathrm{s}}(X)$ be a spectral measure defined on a $\sigma$-algebra $\Sigma$ of subsets of a set $\Gamma$. For each $E \in \Sigma$, let $\widehat{P}(E)$, $\widetilde{P}(E)$ and $\overline{P}(E)$ denote the continuous extension of $P(E)$ from $X$ to $\widehat{X}$, $\widetilde{X}$ and $\overline{X}$, respectively; see Lemma 1.6. It follows from Remark 1.7 that the set functions $\widehat{P} : \Sigma \to L(\widehat{X})$, $\widetilde{P} : \Sigma \to L(\widetilde{X})$ and $\overline{P} : \Sigma \to L(\overline{X})$ so defined are finitely additive and multiplicative and assign the identity operator (in $\widehat{X}$, $\widetilde{X}$ and $\overline{X}$, respectively) to $\Gamma$. This section is concerned with the following question: When are the extended set functions $\widehat{P}$, $\widetilde{P}$ and $\overline{P}$ again spectral measures, i.e. when are they $\sigma$-additive?

PROPOSITION 2.1. *Let* $X$ *be a lcHs and* $P : \Sigma \to L_{\mathrm{s}}(X)$ *be a spectral measure. Then* $\widehat{P} : \Sigma \to L_{\mathrm{s}}(\widehat{X})$ *is also a spectral measure.*

Proof. For each $x \in X_0 = X$, the set function $\widehat{P}x = Px : \Sigma \to \widehat{X}$ is $\sigma$-additive. Suppose that $\alpha \in (0, \Omega_1)$ is an ordinal number such that $\widehat{P}x$ is $\sigma$-additive in $\widehat{X}$ for every $x \in \bigcup_{0 \le \beta < \alpha} X_\beta$. Let $x \in X_\alpha$. Choose a sequence $\{x_n\}_{n=1}^\infty$ from $\bigcup_{0 \le \beta < \alpha} X_\beta$ which converges to $x$ in $\widehat{X}$. For each $E \in \Sigma$ we have $\widehat{P}(E)x_n \to \widehat{P}(E)x$ (by continuity of $\widehat{P}(E) \in L(\widehat{X})$), i.e. $\widehat{P}x_n \to \widehat{P}$ setwise in $\widehat{X}$. Then the Vitali–Hahn–Saks theorem [4; IV, Theorem 10.6] implies the $\sigma$-additivity of $\widehat{P}x$. Hence, $\widehat{P}x$ is $\sigma$-additive in $\widehat{X}$ for each $x \in X_\alpha$. Transfinite induction implies the $\sigma$-additivity of $\widehat{P}$ for each $x \in \widehat{X}$. ■

It turns out that the analogue of Proposition 2.1 fails for $\widetilde{P}$ and $\overline{P}$ in general.

EXAMPLE 2.2. Let $\Sigma = 2^{\mathbb{N}}$ and $X$ be the space $c_{00}$ of all functions $x : \mathbb{N} \to \mathbb{C}$ which are finitely supported. Equip $X$ with the weak topology $\sigma(c_{00}, \ell^\infty)$ induced by the natural duality (of pointwise summation) between $c_{00}$ and $\ell^\infty$. Then $\widetilde{X}$ is the dual space $(\ell^\infty)'$ of the (Banach) space $\ell^\infty$, equipped with the weak-star topology $\sigma((\ell^\infty)', \ell^\infty)$. Let $P : \Sigma \to L_{\mathrm{s}}(X)$ be the spectral measure defined by

$$P(E)x = x\chi_E, \quad x \in X,\ E \in \Sigma. \tag{2.1}$$

For each $E \in \Sigma$, let $Q(E) \in L(\ell^\infty)$ be the projection given by $Q(E) : \varphi \mapsto \chi_E\varphi$, for $\varphi \in \ell^\infty$. Then $\widetilde{P}(E) \in L(\widetilde{X})$ is precisely the dual operator $Q(E)' : (\ell^\infty)' \to (\ell^\infty)'$. There exists $x_0 \in \widetilde{X}$ such that the complex measure $E \mapsto \langle x_0, \chi_E\rangle$, for $E \in \Sigma$, is not $\sigma$-additive [7; §31.1]. Since the function $\mathbf{1}$ (constantly equal to 1 on $\mathbb{N}$) belongs to $\ell^\infty = (\widetilde{X})'$ and $\langle \mathbf{1}, \widetilde{P}(E)x_0\rangle = \langle \mathbf{1}, Q(E)'x_0\rangle = \langle x_0, \mathbf{1}\chi_E\rangle = \langle x_0, \chi_E\rangle$, for $E \in \Sigma$, it follows from the Orlicz–Pettis theorem that the set function $\widetilde{P}x_0$ is not $\sigma$-additive and, hence, neither is $\widetilde{P} : \Sigma \to L_s(\widetilde{X})$. Since $\widetilde{X} \subseteq \overline{X}$ it follows that $\overline{P} : \Sigma \to L_s(\overline{X})$ is also not $\sigma$-additive. ■

An operator-valued measure $Q : \Sigma \to L_s(X)$ is called *equicontinuous* if its range $Q(\Sigma) = \{Q(E) : E \in \Sigma\}$ is an equicontinuous subset of $L(X)$. The following result provides a sufficient condition for the $\sigma$-additivity of $\widetilde{P}$ and $\overline{P}$.

PROPOSITION 2.3. *Let $P : \Sigma \to L_s(X)$ be an equicontinuous spectral measure. Then both $\widetilde{P} : \Sigma \to L_s(\widetilde{X})$ and $\overline{P} : \Sigma \to L_s(\overline{X})$ are also spectral measures* (*i.e. $\sigma$-additive*).

Proof. On the equicontinuous subset $\widetilde{P}(\Sigma)$ of $L(\widetilde{X})$ (cf. Lemma 1.8), the pointwise convergence topologies over $X$ and $\widetilde{X}$ coincide [8; (1) in §39.4]. Since $X$ is dense in $\widetilde{X}$, the set function $\widetilde{P}$ is $\sigma$-additive in $L_s(\widetilde{X})$ because the $\widetilde{X}$-valued measure $\widetilde{P}x = Px$ is $\sigma$-additive for each $x \in X$. A similar proof applies to $\overline{P}$ in $\overline{X}$. ■

Remark 2.4. The equicontinuity in Proposition 2.3 is not necessary. Let $X = (c_0, \sigma(c_0, \ell^1))$, in which case $\widetilde{X} = (\ell^\infty, \sigma(\ell^\infty, \ell^1))$. For $E \in 2^{\mathbb{N}}$, define a projection $P(E) \in L(X)$ by $P(E)x = x\chi_E$, for $x \in X$. Then $P : 2^{\mathbb{N}} \to L_s(X)$ so defined is a spectral measure. For each $E \in 2^{\mathbb{N}}$, the projection $\widetilde{P}(E) \in L(\widetilde{X})$ is given by $\widetilde{P}(E)\varphi = \chi_E\varphi$, for $\varphi \in \widetilde{X}$, and it is routine to verify that $\widetilde{P} : 2^{\mathbb{N}} \to L_s(\widetilde{X})$ is $\sigma$-additive. Since $\widetilde{P}$ is not equicontinuous [10; Proposition 4(i)] neither is $P$ (by Lemma 1.8). ■

Since a subset of a lcHs $X$ is bounded iff it is weakly bounded the Orlicz–Pettis theorem, together with the fact that every complex measure has bounded range, implies that the range of any measure $Q : \Sigma \to L_s(X)$ is a bounded subset of the lcHs $L_s(X)$. If $X$ is barrelled, then every bounded subset of $L_s(X)$ is equicontinuous [8; (2) in §39.3], and hence every measure with values in $L_s(X)$ is necessarily equicontinuous. So, the hypothesis of Proposition 2.3 holds in all barrelled spaces. Unfortunately, the class of barrelled spaces which are not complete or quasicomplete is rather restrictive. A more extensive class of spaces is the *quasibarrelled* spaces [7; Ch.6]; it includes all bornological lcH-spaces, and hence all metrizable lcH-spaces [7; §28.1]. For further examples of the non-metrizable quasibarrelled spaces we refer to [7], for example. Accordingly, the following result shows that Proposition 2.3 has some generality.

PROPOSITION 2.5. *Let $X$ be a quasibarrelled lcHs and $P : \Sigma \to L_{\mathrm{s}}(X)$ be a spectral measure. Then $P$ is necessarily equicontinuous.*

Proof. This follows from Lemma 1.3 and [8; (3) p. 137]. ∎

Proposition 2.3 provides a sufficient condition on a spectral measure $P : \Sigma \to L_{\mathrm{s}}(X)$ which ensures that $\widetilde{P} : \Sigma \to L_{\mathrm{s}}(\widetilde{X})$ is also $\sigma$-additive. We end this section with another sufficient condition of a different kind which is simple but quite effective in some cases. A subspace $Y$ of a lcHs $Z$ is called *sequentially dense* if $[Y] = Z$.

PROPOSITION 2.6. *Let $X$ be a lcHs such that $X$ is sequentially dense in $\widetilde{X}$ (resp. $\overline{X}$). Then, for every spectral measure $P$ in $X$, the set function $\widetilde{P}$ (resp. $\overline{P}$) is a spectral measure in $\widetilde{X}$ (resp. $\overline{X}$).*

Proof. The sequential denseness of $X$ in $\widetilde{X}$ means that $[X] = \widetilde{X}$, and hence $\widetilde{X} \subseteq \widehat{X}$ as $[X] \subseteq \widehat{X}$. Since always $\widehat{X} \subseteq \widetilde{X}$ we have $\widehat{X} = \widetilde{X}$ and the result follows from Proposition 2.1. The proof for $\overline{X}$ is similar. ∎

If $X$ is a metrizable lcHs, then $X$ is sequentially dense in $\widetilde{X} = \widehat{X} = \overline{X}$. It is a consequence of Goldstein's theorem that if $X$ denotes a Banach space $Y$ equipped with its weak topology $\sigma(Y, Y')$, then $X$ is non-metrizable (if $\dim(Y) = \infty$) and $\widetilde{X}$ is precisely $Y''$ equipped with its weak-star topology $\sigma(Y'', Y')$. Non-trivial spectral measures in such spaces $X$ are never equicontinuous [10; Proposition 4], and so Proposition 2.3 is not applicable to $\widetilde{P}$. But, if $Y'$ is norm separable, then $X$ is sequentially dense in $\widetilde{X}$ and so Proposition 2.6 is applicable to any spectral measure in $X$. However, with $Y = \ell^1$ (so $Y'$ is not separable) we see that $X = \widehat{X}$ is sequentially complete and so $X$ is not sequentially dense in $\widetilde{X} = ((\ell^\infty)', \sigma((\ell^\infty)', \ell^\infty))$; this is the point on which Example 2.2 is based. Fortunately, this example is not a paradigm, i.e. the sequential denseness of $X$ in $\widetilde{X}$ is not a necessary condition for $\widetilde{P}$ to be $\sigma$-additive.

To see this, let $Y = \mathbb{C}^{[0,1]}$ be the complete lcHs of all $\mathbb{C}$-valued functions on $\Gamma = [0, 1]$, equipped with the topology of pointwise convergence on $\Gamma$, and $\Sigma$ be the Borel subsets of $\Gamma$. Let $X$ be the space of those $\varphi \in Y$ for which $\{w \in \Gamma : \varphi(w) \neq 0\}$ is a countable set. Then $X$ is sequentially complete. Moreover, $\widetilde{X} = Y$ since each $\varphi \in Y$ is the limit of the *bounded* net $\{\varphi\chi_{F}\}_{F\in\mathcal{F}} \subseteq X$, where $\mathcal{F}$ is the family of finite subsets of $\Gamma$ directed by inclusion. Since $\widehat{X} = X$ is a proper subspace of $Y$ it is clear that $X$ is not sequentially dense in $\widetilde{X} = Y$. However, for the spectral measure $P : \Sigma \to L_s(X)$ given by $P(E)\varphi = \chi_E \varphi$, for $E \in \Sigma$ and $\varphi \in X$, it is clear that $\widetilde{P} : \Sigma \to L_s(\widetilde{X})$ is $\sigma$-additive.

It can be argued that the above example is already a consequence of Proposition 2.3 since $P$ is equicontinuous. To produce an example where neither Proposition 2.3 nor 2.6 apply we proceed as follows; the notation is still from the previous paragraph. Let $\ell^2_\sigma$ denote the (non-separable) Hilbert space $\ell^2(\Gamma)$ equipped with its weak topology. Then the product space $Z = \ell^2_\sigma \times X$ is sequentially complete (as $\ell^2_\sigma$ is quasicomplete and $X$ is sequentially complete) and its quasicompletion is $\widetilde{Z} = \ell^2_\sigma \times Y$ (as $\widetilde{X} = Y$). Since $X$ is not sequentially dense in $Y$ it follows that $Z$ is not sequentially dense in $\widetilde{Z}$ and so Proposition 2.6 is not applicable. Define a spectral measure $R : \Sigma \to L_s(\ell^2_\sigma)$ by $R(E)\varphi = \chi_E \varphi$, for $E \in \Sigma$ and $\varphi \in \ell^2_\sigma$, in which case $R$ is *not* equicontinuous [10; Proposition 4]. Accordingly, the spectral measure $Q : \Sigma \to L_s(Z)$ given by $Q(E)(\varphi, \psi) = (R(E)\varphi, P(E)\psi)$, for $E \in \Sigma$ and $(\varphi, \psi) \in Z$, also fails to be equicontinuous. So, Proposition 2.3 is also not applicable. However, the set function $\widetilde{Q} : \Sigma \to L_s(\widetilde{Z})$, which is given by $\widetilde{Q}(E)(\varphi, \psi) = (R(E)\varphi, \widetilde{P}(E)\psi)$ for $E \in \Sigma$ and $(\varphi, \psi) \in \widetilde{Z}$, is clearly $\sigma$-additive.

In conclusion, we remark that Propositions 2.1, 2.3, 2.5 and 2.6 actually hold for arbitrary operator-valued measures (same proof), not just spectral measures.

**3. The $\mathcal{L}^1$-space of extended measures.** Let $X$ be a lcHs and $P : \Sigma \to L_s(X)$ be a spectral measure defined on a $\sigma$-algebra $\Sigma$ of subsets of a set $\Gamma$. The main aim of this section is to identify the relationship between $\mathcal{L}^1(P)$ and the various spaces of integrable functions $\mathcal{L}^1(\widehat{P})$, $\mathcal{L}^1(\widetilde{P})$ and $\mathcal{L}^1(\overline{P})$, which satisfy the inclusions $\mathcal{L}^1(\overline{P}) \subseteq \mathcal{L}^1(\widetilde{P}) \subseteq \mathcal{L}^1(\widehat{P})$ whenever $\widetilde{P}$ and $\overline{P}$ are $\sigma$-additive (cf. Lemma 1.2).

Let $X$ be a lcHs and $T \in L(X)$. Recall that $\widehat{T} X_\alpha \subseteq X_\alpha$ for every ordinal number $\alpha \in [0, \Omega_1)$; see the proof of Lemma 1.6. Let $T^{(\alpha)} \in L(X_\alpha)$ denote the restriction of $\widehat{T}$ to $X_\alpha$, for each $\alpha \in [0, \Omega_1)$. Given a spectral measure $P : \Sigma \to L_s(X)$ and $\alpha \in [0, \Omega_1)$, let $P^{(\alpha)} : \Sigma \to L(X_\alpha)$ be defined by $P^{(\alpha)}(E) = P(E)^{(\alpha)}$, for each $E \in \Sigma$. Since $X_\alpha$ is $\widehat{P}$-invariant it follows from Lemma 1.4 and Proposition 2.1 that $P^{(\alpha)}$ is a spectral measure. Moreover, Remark 1.5(iii) and the density of $X$ in both $X_\alpha$ and $\widehat{X}$ imply that $\mathcal{L}^\infty(P) = \mathcal{L}^\infty(\widehat{P}) = \mathcal{L}^\infty(P^{(\alpha)})$, for every $\alpha \in [0, \Omega_1)$.

LEMMA 3.1. *If $P$ is a spectral measure in $X$, then $\mathcal{L}^1(P) \cap \mathcal{L}^\infty(P) \subseteq \mathcal{L}^1(P^{(1)})$.*

Proof. Fix $x \in X_1$. Let $f \in \mathcal{L}^1(P) \cap \mathcal{L}^\infty(P)$. We show that $f$ is $P^{(1)}x$-integrable. Choose vectors $x_n \in X$, for $n \in \mathbb{N}$, converging to $x$ in $X_1$. Since $f \in \mathcal{L}^\infty(\widehat{P})$ it is clear that $f \in \mathcal{L}^\infty(\widehat{P}x)$, and hence $f \in \mathcal{L}^1(\widehat{P}x)$ by sequential completeness of $\widehat{X}$ [6; II, Lemma 3.1]. Choose functions $s_k \in \mathrm{sim}(\Sigma)$ which satisfy $|s_k| \leq |f|$, for $k \in \mathbb{N}$, and converge uniformly to $f$ on $\Gamma$. By the dominated convergence theorem applied to $\widehat{P}x$ in the sequentially complete space $\widehat{X}$ [6; II, Theorem 4.2], it follows that $\int_E s_k \, d\widehat{P}x \to \int_E f \, d\widehat{P}x$ in $\widehat{X}$, as $k \to \infty$, for each $E \in \Sigma$. Since $\widehat{P}(F)x = \lim_{n\to\infty} P(F)x_n$, for $F \in \Sigma$, it is clear that $\int_E s_k \, d\widehat{P}x \in X_1$, for $k \in \mathbb{N}$, and that $\widehat{P}x$ takes its values in $X_1$ and coincides with $P^{(1)}x$. Accordingly, $\int_E f \, d\widehat{P}x \in X_2$ for each $E \in \Sigma$. It follows by the remark after Lemma 1.1 that $f$ is actually $P^{(2)}x$-integrable and $\int_E f \, d\widehat{P}x = \int_E f \, dP^{(2)}x$, for $E \in \Sigma$. But the measures $Px_n$ for $n \in \mathbb{N}$ (considered as being $X_2$-valued) converge setwise to $P^{(2)}x$ as $n \to \infty$. It follows from Lemma 1.1, applied in $X_2$, that

$$\lim_{n\to\infty} \int_E f \, dPx_n = \int_E f \, dP^{(2)}x, \qquad E \in \Sigma. \tag{2.2}$$

But $\{\int_E f \, dPx_n\}_{n=1}^\infty \subseteq X$ (as $f \in \mathcal{L}^1(P)$) and we see from (2.2) that actually $\int_E f \, dP^{(2)}x \in X_1$, for $E \in \Sigma$. Since $P^{(2)}x = P^{(1)}x$ (as $x \in X_1$) it follows again by the remark after Lemma 1.1 that $f \in \mathcal{L}^1(P^{(1)}x)$. Hence, the right-hand side of (2.2) is equal to $\int_E f \, dP^{(1)}x$, for $E \in \Sigma$. Since $P(f)^{(1)}$ is continuous and $\int_\Gamma f \, dPx_n = P(f)x_n$, for $n \in \mathbb{N}$, it follows that the left-hand side of (2.2) equals $P(f)^{(1)}x$. Since $x \in X_1$ is arbitrary Lemma 1.2 implies that $f \in \mathcal{L}^1(P^{(1)})$. ■

It may be interesting to note that, in general, $\mathcal{L}^\infty(P)$ need not be contained in $\mathcal{L}^1(P)$; consider the spectral measure $P_{Y_1}$ acting in $Y_1$ of Remark 1.5(ii).

We come to one of the main results of this section.

PROPOSITION 3.2. *If $P : \Sigma \to L_s(X)$ is a spectral measure, then $\mathcal{L}^1(P) \subseteq \mathcal{L}^1(\widehat{P})$ and*

$$\mathcal{L}^1(P) = \{f \in \mathcal{L}^1(\widehat{P}) : \widehat{P}(f)X \subseteq X\}. \tag{2.3}$$

*Moreover, if $f \in \mathcal{L}^1(P)$, then*

$$\widehat{P}(f\chi_E) = (P(f\chi_E))\widehat{\ }, \qquad E \in \Sigma. \tag{2.4}$$

Proof. To establish the inclusion $\mathcal{L}^1(P) \subseteq \mathcal{L}^1(\widehat{P})$, let $f \in \mathcal{L}^1(P)$. Suppose that $\alpha \in (0, \Omega_1)$ is an ordinal number such that $f \in \mathcal{L}^1(P^{(\beta)})$ and $P^{(\beta)}(f\chi_E) = P(f\chi_E)^{(\beta)}$ for every $E \in \Sigma$, whenever $0 \leq \beta < \alpha$. Let $Y$ be the $\widehat{P}$-invariant subspace $\bigcup_{0\leq\beta<\alpha} X_\beta$ of $\widehat{X}$. The restriction of $\widehat{P}$ to $Y$

is denoted by $\widehat{P}_Y$. Then the spectral measure $\widehat{P}_Y : \Sigma \to L_s(Y)$ satisfies $(\widehat{P}_Y)^{(1)} = P^{(\alpha)}$ because $Y_1 = [Y] = X_\alpha$.

The claim is that $f \in \mathcal{L}^1(\widehat{P}_Y)$. In fact, let $T \in L(Y)$ denote the unique extension of $P(f)$ to $Y$. Fix $y \in Y$ and choose $\beta \in [0, \alpha)$ such that $y \in X_\beta$. Since $\widehat{P}_Y y = P^{(\beta)} y$ (as $Y$-valued measures) and $f \in \mathcal{L}^1(P^{(\beta)})$ by the inductive hypothesis, it follows that $f$ is $\widehat{P}_Y y$-integrable. Moreover, since $P^{(\beta)}(f)y = Ty$ as elements of $Y$ we have $Ty = P^{(\beta)}(f)y = \int_\Gamma f\, dP^{(\beta)}y = \int_\Gamma f\, d\widehat{P}_Y y$. Lemma 1.2 implies that $f \in \mathcal{L}^1(\widehat{P}_Y)$.

For each $n \in \mathbb{N}$, let $A(n) = \{\gamma \in \Gamma : |f(\gamma)| \le n\}$. Since $f \in \mathcal{L}^1(\widehat{P}_Y)$ it follows that $f_n = f\chi_{A(n)}$ belongs to $\mathcal{L}^1(\widehat{P}_Y) \cap \mathcal{L}^\infty(\widehat{P}_Y)$, for $n \in \mathbb{N}$. Lemma 3.1 ensures that $f_n$ is $(\widehat{P}_Y)^{(1)}$-integrable, i.e. $f_n \in \mathcal{L}^1(P^{(\alpha)})$, for $n \in \mathbb{N}$. Fix $x \in X_\alpha$. We claim that

$$\lim_{n\to\infty} \int_\Gamma f_n\, dP^{(\alpha)}x = P(f)^{(\alpha)}x. \tag{2.5}$$

Let $y_k \in Y$, for $k \in \mathbb{N}$, be a sequence converging to $x$ in $X_\alpha = [Y]$. If $J : Y \to X_\alpha$ is the natural injection, then the sequence of $X_\alpha$-valued measures $J \circ \widehat{P}_Y y_k$, for $k \in \mathbb{N}$, is setwise convergent to $P^{(\alpha)}x$. Since $P^{(\alpha)}y_k = J \circ \widehat{P}_Y y_k$ as $X_\alpha$-valued measures, for $k \in \mathbb{N}$, it follows from $f_n \in \mathcal{L}^1(P^{(\alpha)})$ that $f_n$ is $J \circ \widehat{P}_Y y_k$-integrable and $P^{(\alpha)}x$-integrable, for $k \in \mathbb{N}$, and hence, by Lemma 1.1,

$$\lim_{k\to\infty} \int_\Gamma f_n\, d(J \circ \widehat{P}_Y y_k) = \int_\Gamma f_n\, dP^{(\alpha)}x, \qquad n \in \mathbb{N}. \tag{2.6}$$

Since $f \in \mathcal{L}^1(\widehat{P}_Y)$ we have $f \in \mathcal{L}^1(J \circ \widehat{P}_Y y_k)$, for each $k \in \mathbb{N}$, and so the dominated convergence theorem applied to $J \circ \widehat{P}_Y y_k$, considered as taking its values in the sequentially complete space $\widehat{X}$, implies that

$$\lim_{n\to\infty} \int_\Gamma f_n\, d(J \circ \widehat{P}_Y y_k) = \int_\Gamma f\, d(J \circ \widehat{P}_Y y_k) = J \circ \widehat{P}_Y(f)y_k, \qquad k \in \mathbb{N}. \tag{2.7}$$

Of course, the value of the limit (2.7) lies in the subspace $Y$ of $\widehat{X}$. Consequently,

$$\lim_{k\to\infty} \lim_{n\to\infty} \int_\Gamma f_n\, d(J \circ \widehat{P}_Y y_k) = \lim_{k\to\infty} J \circ \widehat{P}_Y(f)y_k = P(f)^{(\alpha)}x. \tag{2.8}$$

Once we show that the limit in (2.7) is uniform with respect to $k \in \mathbb{N}$, we can exchange the order of limits in (2.8), by applying [4; I, Lemma 7.6] in the completion of the normed space $X/p^{-1}(\{0\})$, for each $p \in \mathcal{P}(X)$, so that (2.5) will follow from (2.6). But the sequence of indefinite integrals $f(J \circ \widehat{P}_Y y_k) : \Sigma \to X_\alpha$, for $k \in \mathbb{N}$, is setwise convergent, and hence they are uniformly $\sigma$-additive with respect to $k \in \mathbb{N}$ by the Vitali–Hahn–Saks theorem. Accordingly,

$$\lim_{n\to\infty}\Big(\int_\Gamma f_n\,d(J\circ \widehat{P}_Y y_k)-\int_\Gamma f\,d(J\circ \widehat{P}_Y y_k)\Big)=\lim_{n\to\infty}\int_{\Gamma\setminus A(n)} f\,d(J\circ \widehat{P}_Y)y_k=0$$

uniformly in $k\in\mathbb{N}$, which establishes (2.5).

Since $f_n\to f$ pointwise as $n\to\infty$, and (2.5) holds, Lemma 1.2 implies that $f\in\mathcal{L}^1(P^{(\alpha)})$. So, transfinite induction establishes that $f\in\mathcal{L}^1(P^{(\alpha)})$ for every $\alpha\in[0,\Omega_1)$. Since $\widehat{X}=\bigcup_{0\le\alpha<\Omega_1}X_\alpha$ it follows by an argument similar to that where we established $f\in\mathcal{L}^1(\widehat{P}_Y)$ that $f\in\mathcal{L}^1(\widehat{P})$.

Finally, (2.3) and (2.4) follow from Lemma 1.4 as $P$ is the restriction of $\widehat{P}$ to $X$. ■

COROLLARY 3.3. *Let $X$ be a normed space and $P:\Sigma\to L_s(X)$ be a spectral measure. Then $\mathcal{L}^1(P)\subseteq\mathcal{L}^\infty(P)$.*

Proof. The sequential completion $\widehat{X}$ of $X$ is a Banach space, and hence $\mathcal{L}^1(\widehat{P})=\mathcal{L}^\infty(\widehat{P})$; see [5; XVIII, Theorem 2.11(c)] or [18; (1), p. 436], for example. Since $\mathcal{L}^\infty(P)=\mathcal{L}^\infty(\widehat{P})$ the conclusion follows from Proposition 3.2. ■

Whether or not the inclusion $\mathcal{L}^1(P)\subseteq\mathcal{L}^1(\widehat{P})$ is strict depends on $P$. Indeed, for the spectral measure $P_{Y_1}$ in Remark 1.5(ii) we see that $\widehat{P}_{Y_1}$ is the spectral measure $P$ given there, and hence $\mathcal{L}^1(P_{Y_1})\subseteq\mathcal{L}^1(\widehat{P}_{Y_1})$ is a strict inclusion. However, for the spectral measure $P_{Y_2}$ in Remark 1.5(ii) we see that $\widehat{P}_{Y_2}$ is also the spectral measure $P$ given there (note that $\widehat{Y}_1=\widehat{Y}_2$), and hence $\mathcal{L}^1(P_{Y_2})=\mathcal{L}^1(\widehat{P}_{Y_2})$.

LEMMA 3.4. *Let $P:\Sigma\to L_s(X)$ be an equicontinuous spectral measure. Then, for $f\in\mathcal{L}^1(P)$, the set $\{P(g):|g|\le|f|,\ g\in\mathcal{L}^1(P)\}$ is an equicontinuous part of $L(X)$.*

Proof. Let $r\in\mathcal{P}(X)$. By the equicontinuity of $P$ there is $q\in\mathcal{P}(X)$ such that $r(P(E)x)\le q(x)$, for $x\in X$ and $E\in\Sigma$. Since $P(f)\in L(X)$ there is $p\in\mathcal{P}(X)$ such that $q(P(f)x)\le p(x)$, for $x\in X$. It follows, for $x\in X$ and $E\in\Sigma$, that

$$r(P(E)P(f)x)\le q(P(f)x)\le p(x). \tag{2.9}$$

Let $g\in\mathcal{L}^1(P)$ satisfy $|g|\le|f|$. Then

$$r(P(g)x)=r\Big(\int_\Gamma g\,d(Px)\Big)\le\sup_{x'\in U_r^0}\int_\Gamma|g|\,d|\langle x',Px\rangle|, \tag{2.10}$$

where $U_r^0$ is the polar of the closed $r$-unit ball [6; II, Lemmas 1.2 & 2.2].

But the right-hand side of (2.10) does not exceed

$$\sup_{x'\in U_r^0} \int_\Gamma |f|\, d|\langle x', Px\rangle| \le 4 \sup_{E\in\Sigma} r\Big(\int_E f\, d(Px)\Big) = 4\sup_{E\in\Sigma} r(P(E)P(f)x); \tag{2.11}$$

see [6; II, Lemmas 1.2 & 2.2] for the inequality in (2.11). Combining (2.9), (2.10) and (2.11) gives $r(P(g)x) \le 4p(x)$, for $x \in X$. Since $g \in \mathcal{L}^1(P)$ satisfying $|g| \le |f|$ is arbitrary the result follows. ■

The next result is concerned with the quasicompletion and completion; in view of Propositions 2.3 and 2.5 it has some generality.

PROPOSITION 3.5. *Let $P : \Sigma \to L_s(X)$ be an equicontinuous spectral measure. Then the equicontinuous spectral measure $\widetilde{P} : \Sigma \to L_s(\widetilde{X})$ satisfies $\mathcal{L}^1(P) \subseteq \mathcal{L}^1(\widetilde{P})$ and*

$$\mathcal{L}^1(P) = \{f \in \mathcal{L}^1(\widetilde{P}) : \widetilde{P}(f)X \subseteq X\}. \tag{2.12}$$

*A similar result holds for $\overline{P} : \Sigma \to L_s(\overline{X})$.*

Proof. To establish $\mathcal{L}^1(P) \subseteq \mathcal{L}^1(\widetilde{P})$, let $f \in \mathcal{L}^1(P)$. By Lemma 1.2, it is possible to choose functions $s_n \in \mathrm{sim}(\Sigma)$, for $n \in \mathbb{N}$, converging pointwise to $f$, such that $\{P(s_n)\}_{n=1}^\infty$ strongly converges to $P(f)$. Moreover, it is possible to choose $s_n$ such that $|s_n| \le |f|$, for $n \in \mathbb{N}$ [11; Proposition 1.2]. By Lemma 3.4 the set $H = \{P(s_n) : n \in \mathbb{N}\} \cup \{P(f)\}$ is an equicontinuous part of $L(X)$. Then $\widetilde{H} = \{\widetilde{T} : T \in H\}$ is an equicontinuous subset of $L(\widetilde{X})$ by Lemma 1.8. Since $X$ is dense in $\widetilde{X}$, the sequence $\{\widetilde{P}(s_n)\}_{n=1}^\infty$ strongly converges to $(P(f))^\sim$ in $L_s(\widetilde{X})$ because $\widetilde{P}(s_n)$ is the continuous extension of $P(s_n)$ to $\widetilde{X}$, for $n \in \mathbb{N}$, and because on the equicontinuous set $\widetilde{H}$ the pointwise convergence topologies over $X$ and $\widetilde{X}$ coincide. Lemma 1.2 implies that $f \in \mathcal{L}^1(\widetilde{P})$. The identity (2.12) is a consequence of Lemma 1.3 as $P$ is the restriction of $\widetilde{P}$.

A similar proof applies to $\overline{P} : \Sigma \to L_s(\overline{X})$. ■

PROPOSITION 3.6. *Let $X$ be a lcHs such that $X$ is sequentially dense in $\widetilde{X}$. Then, for every spectral measure $P$ in $X$, we have $\mathcal{L}^1(P) \subseteq \mathcal{L}^1(\widetilde{P})$ and (2.12) holds. A similar result holds for $\overline{P}$ if $X$ is sequentially dense in $\overline{X}$.*

Proof. As in the proof of Proposition 2.6 it follows that $\widetilde{X} = \widehat{X}$ and so Proposition 3.2 implies the conclusion. ■

We now present a result related to Proposition 3.5 (cf. Proposition 3.8) without the equicontinuity requirement. First we need an alternative description of the quasicompletion. Given a lcHs $X$ let $X_{[0]} = X$ and $X_{[1]}$

denote the linear space of all elements in $\overline{X}$ which are the limit of some *bounded net* of elements from $X$. Suppose that $\alpha > 0$ is an ordinal number and $X_{[\beta]}$ has been defined for all ordinals $\beta$ satisfying $0 \le \beta < \alpha$. Define $X_{[\alpha]} = (\bigcup_{0\le\beta<\alpha} X_{[\beta]})_{[1]}$. By considering the cardinality of $X$ it follows that there must exist an ordinal number $\Omega$ such that $X_{[\Omega]} = \bigcup_{0\le\beta<\Omega} X_{[\beta]}$. If $\Omega(X)$ is the smallest such ordinal number, then $\widetilde{X} = \bigcup_{0\le\alpha<\Omega(X)} X_{[\alpha]}$. It turns out that $\widetilde{T} X_{[\alpha]} \subseteq X_{[\alpha]}$ for every ordinal number $\alpha \in [0, \Omega(X))$. Let $T^{[\alpha]} \in L(X_{[\alpha]})$ be the restriction of $\widetilde{T}$ to $X_{[\alpha]}$, for each $\alpha \in [0, \Omega(X))$. Given a spectral measure $P : \Sigma \to L_{\mathrm{s}}(X)$ and $\alpha \in [0, \Omega(X))$, let $P^{[\alpha]} : \Sigma \to L(X_{[\alpha]})$ be defined by $P^{[\alpha]}(E) = P(E)^{[\alpha]}$, for $E \in \Sigma$. Since $X_{[\alpha]}$ is $\widetilde{P}$-invariant, Lemma 1.4 implies that *if* $\widetilde{P} : \Sigma \to L_{\mathrm{s}}(\widetilde{X})$ happens to be $\sigma$-additive, then $P^{[\alpha]}$ is a spectral measure. Moreover, Remark 1.5(iii) and the density of $X$ in both $X_{[\alpha]}$ and $\widetilde{X}$ imply that $\mathcal{L}^\infty(P) = \mathcal{L}^\infty(\widetilde{P}) = \mathcal{L}^\infty(P^{[\alpha]})$, for every $\alpha \in [0, \Omega(X))$.

LEMMA 3.7. *Let $P : \Sigma \to L_{\mathrm{s}}(X)$ be a spectral measure. If $\widetilde{P} : \Sigma \to L_{\mathrm{s}}(\widetilde{X})$ is also a spectral measure* (*i.e.* $\widetilde{P}$ *is $\sigma$-additive*), *then $\mathcal{L}^1(P) \cap \mathcal{L}^\infty(P) \subseteq \mathcal{L}^1(P^{[1]})$.*

Proof. Fix $x \in X_{[1]}$. Let $f \in \mathcal{L}^1(P) \cap \mathcal{L}^\infty(P)$. Choose a bounded Cauchy net $\{x_\alpha\}_{\alpha\in A} \subseteq X$ converging to $x$ in $X_{[1]}$. The measures $Px_\alpha$ for $\alpha \in A$ (considered as being $X_{[2]}$-valued) converge setwise to $P^{[2]}x$. Since $B = \{x_\alpha\}_{\alpha\in A}$ is a bounded subset of $X \subseteq X_{[2]}$ it follows from Lemma 1.3 that $\sup\{p(P(E)x_\alpha) : E \in \Sigma,\ \alpha \in A\} < \infty$, for each $p \in \mathcal{P}(X_{[2]})$. An argument as in the proof of Lemma 3.1 (via Lemma 1.1) shows that $\lim_\alpha \int_E f\,dPx_\alpha = \int_E f\,dP^{[2]}x$, for $E \in \Sigma$. By noting that $\int_E f\,dPx_\alpha = P(E)P(f)x_\alpha$, $\alpha \in A$, is a bounded net in $X$ it follows that actually $\int_E f\,dP^{[2]}x \in X_{[1]}$, for $E \in \Sigma$, and the proof can be completed as for Lemma 3.1. ∎

PROPOSITION 3.8. *Let $P : \Sigma \to L_{\mathrm{s}}(X)$ be a spectral measure such that $\widetilde{P} : \Sigma \to L_{\mathrm{s}}(\widetilde{X})$ is also a spectral measure. Then $\mathcal{L}^1(P) \cap \mathcal{L}^\infty(P) \subseteq \mathcal{L}^1(\widetilde{P})$.*

Proof. Let $f \in \mathcal{L}^1(P) \cap \mathcal{L}^\infty(P)$. Suppose that $\alpha \in (0, \Omega(X))$ is an ordinal number such that $f \in \mathcal{L}^1(P^{[\beta]})$ whenever $0 \le \beta < \alpha$. An argument as in the proof of Proposition 3.2 (first 2 paragraphs) shows that $f$ is $\widetilde{P}_Y$-integrable in $L_{\mathrm{s}}(Y)$, where $Y$ is the $\widetilde{P}$-invariant subspace $\bigcup_{0\le\beta<\alpha} X_{[\beta]}$ of $\widetilde{X}$. Since $Y_{[1]} = X_{[\alpha]}$ and $(\widetilde{P}_Y)^{[1]} = P^{[\alpha]}$ with $f \in \mathcal{L}^\infty(\widetilde{P}_Y) \cap \mathcal{L}^1(\widetilde{P}_Y)$, it follows from Lemma 3.7 applied to $\widetilde{P}_Y$ that $f$ is $P^{[\alpha]}$-integrable. Accordingly, it follows by transfinite induction that $f \in \mathcal{L}^1(\widetilde{P})$. ∎

We give a non-trivial application of Proposition 3.8.

EXAMPLE 3.9. Let $W$ be a normed space and $X = W_\sigma$ denote $W$ equipped with its weak topology. Since $W$ has its Mackey topology [7; §21.5], it follows that $L(W) = L(X)$ as vector spaces [7; (6) in §21.4]. Moreover, $L_{\mathrm{s}}(W)$ and $L_{\mathrm{s}}(X)$ have the same continuous dual space. Hence, a multiplicative set function $P : \Sigma \to L_{\mathrm{s}}(X)$ is a spectral measure iff $P_{\|\cdot\|} : \Sigma \to L_{\mathrm{s}}(W)$ is a spectral measure, where $P_{\|\cdot\|}$ denotes $P$ interpreted as taking its values in $L_{\mathrm{s}}(W)$. Moreover, Lemma 1.2 implies that $\mathcal{L}^1(P) = \mathcal{L}^1(P_{\|\cdot\|})$ as vector spaces. By Corollary 3.3 we have $\mathcal{L}^1(P_{\|\cdot\|}) \subseteq \mathcal{L}^\infty(P_{\|\cdot\|}) = \mathcal{L}^\infty(P)$, showing that $\mathcal{L}^1(P) \subseteq \mathcal{L}^\infty(P)$. Then Proposition 3.8 implies the following

FACT. *Let $W$ be a normed space, $X = W_\sigma$ and $P : \Sigma \to L_{\mathrm{s}}(X)$ be a spectral measure. If $\widetilde{P} : \Sigma \to L_{\mathrm{s}}(\widetilde{X})$ is $\sigma$-additive, then $\mathcal{L}^1(P) \subseteq \mathcal{L}^1(\widetilde{P})$.*

Note that $\widetilde{P}$ may not be $\sigma$-additive in general; see Example 2.2. The point of the above Fact is that $P : \Sigma \to L_{\mathrm{s}}(X)$ is equicontinuous only in trivial cases. Indeed, if $P(\Sigma)$ is an infinite subset of $L_{\mathrm{s}}(X)$, then $P$ is *not* equicontinuous. To see this, note that the extended spectral measure $(P_{\|\cdot\|})^\frown$ acting in the Banach space $\widehat{W}$ also has the property that its range is an infinite subset of $L(\widehat{W})$. So, if $(P_{\|\cdot\|})^\frown_\sigma$ denotes $(P_{\|\cdot\|})^\frown$ considered as taking its values in $L_{\mathrm{s}}((\widehat{W})_\sigma)$, then $(P_{\|\cdot\|})^\frown_\sigma$ fails to be equicontinuous [10; Proposition 4(ii)]. Since $X$ is norm dense in $\widehat{W}$ it is also dense in $(\widehat{W})_\sigma$. So, if $P$ is equicontinuous, then Lemma 1.8 (with $Y = X$, $Z = (\widehat{W})_\sigma$ and $H = P(\Sigma)$) implies that $(P_{\|\cdot\|})^\frown_\sigma$ is equicontinuous, which is not the case. Hence $P$ is not equicontinuous. ∎

It is relevant, perhaps, to make some comments concerning a related point. For the spectral measure $P_{Y_1}$ mentioned in the comments after Corollary 3.3 we note that its $\mathcal{L}^1$-space is precisely $\mathrm{sim}(\Sigma)$, which is rather poor from the point of view of analysis. However, the $\mathcal{L}^1$-space of $\widehat{P}_{Y_1}$ is $\mathcal{L}^\infty([0,1])$, which is significantly larger; the difference is that $\widehat{Y}_1$ is sequentially complete whereas $Y_1$ is not. For a general vector measure $m : \Sigma \to Z$ it was noted that $\mathcal{L}^\infty(m) \subseteq \mathcal{L}^1(m)$ whenever $Z$ is a sequentially complete lcHs; see (1.1). To apply this to spectral measures $P : \Sigma \to L_{\mathrm{s}}(X)$ requires the sequential completeness of the lcHs $L_{\mathrm{s}}(X)$.

So, the question is: How are the completeness properties of a lcHs $X$ reflected in those of $L_{\mathrm{s}}(X)$? This is particularly relevant to this note since the spaces $\widehat{X}$, $\widetilde{X}$ and $\overline{X}$ are sequentially complete, quasicomplete and complete, respectively. It is relatively straightforward to exhibit examples of sequentially complete and quasicomplete spaces $X$ for which $L_{\mathrm{s}}(X)$ fails to be sequentially complete [15]. However, we have been unable to find an explicit example in the literature of a *complete* lcHs $X$ for which $L_{\mathrm{s}}(X)$ is not sequentially complete. We conclude this note with such an example.

EXAMPLE 3.10. Given a lcHs $Y$ let $\tau_{\mathrm{f}}$ be the finest topology on $Y'$ which agrees with the weak-star topology $\sigma(Y', Y)$ on equicontinuous subsets of $Y'$. In general, $\tau_{\mathrm{f}}$ is not a lc-topology.

FACT 1 ([7; §21.9 & §21.10]). *If $Y$ is a Fréchet lcHs, then $\tau_{\mathrm{f}}$ is the topology of uniform convergence on the compact subsets of $Y$. In particular, $Y = (Y'_{\tau_{\mathrm{f}}})'$ and $\tau_{\mathrm{f}}$ is a lcH-topology, where $Y'_{\tau_{\mathrm{f}}}$ denotes $Y'$ equipped with the topology $\tau_{\mathrm{f}}$.*

So, let $Y$ be a Fréchet lcHs. By Krein's theorem the family $\mathcal{S}$ of all compact sets in $Y$ is saturated (i.e. the closed convex hull of a compact set is again compact). So, Fact 1 implies that $\tau_{\mathrm{f}}$ is the topology of uniform convergence on members of $\mathcal{S}$. Let $\varphi : Y \to \mathbb{C}$ be any linear functional which is continuous on each $E \in \mathcal{S}$, where $E$ has the relative (metric) topology from $Y$. Since $Y$ is metrizable, to show that $\varphi \in Y'$ it suffices to show that $\varphi(y_n) \to 0$ in $\mathbb{C}$ whenever $y_n \to 0$ in $Y$. But $E = \{0\} \cup \{y_n\}_{n=1}^{\infty}$ is then a member of $\mathcal{S}$ and so $\varphi(y_n) \to 0$ by the assumption on $\varphi$. Applying Grothendieck's completeness theorem [7; §21.9] establishes the following

FACT 2. *If $Y$ is a Fréchet lcHs, then $Y'_{\tau_{\mathrm{f}}}$ is a complete lcHs.*

Now, let $Y$ be the Banach space $c_0$. By Facts 1 & 2 the lcHs $X = Y'_{\tau_{\mathrm{f}}}$ is complete. Let $e_n$ denote the standard $n$th basis vector in $c_0$, for $n \in \mathbb{N}$. Fix any non-zero vector $\xi \in \ell^1 = X$ and define linear operators $T_n : X \to X$ by $T_n x = \langle e_n, x\rangle \xi$, for $x \in X$. Since $e_n \in X'$, for each $n \in \mathbb{N}$, it follows that $\{T_n\}_{n=1}^{\infty} \subseteq L(X)$. By using the fact that the seminorms generating the topology in $L_{\mathrm{s}}(X)$ have the form $T \mapsto \sup_{y \in B} |\langle y, Tx\rangle|$, for some $x \in X$ and norm compact set $B \subseteq c_0 = Y$, it is routine to verify that $\{T_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $L_{\mathrm{s}}(X)$. For each $x \in X$, we have $T_n x \to Tx$ in $X$, where $T : X \to X$ is the linear operator given by $Tx = \langle \mathbf{1}, x\rangle \xi$, for $x \in X$. Since the constant function $\mathbf{1}$ (on $\mathbb{N}$) belongs to $\ell^{\infty} \backslash c_0$ we see that $T \notin L(X)$, i.e. $\{T_n\}_{n=1}^{\infty}$ has no limit in $L_{\mathrm{s}}(X)$. This establishes that $L_{\mathrm{s}}(X)$ is *not* sequentially complete. ∎

## REFERENCES


[1] J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Math. Surveys 15, Amer. Math. Soc., Providence, R.I., 1977.

[2] P. G. Dodds and and B. de Pagter, *Orthomorphisms and Boolean algebras of projections*, Math. Z. 187 (1984), 361–381.

[3] P. G. Dodds and W. J. Ricker, *Spectral measures and the Bade reflexivity theorem*, J. Funct. Anal. 61 (1985), 136–163.

[4] N. Dunford and J. T. Schwartz, *Linear Operators I: General Theory*, Wiley-Interscience, New York, 1958.

[5] —, —, *Linear Operators III: Spectral Operators*, Wiley-Interscience, New York, 1972.

[6] I. Kluvánek and G. Knowles, *Vector Measures and Control Systems*, North-Holland, Amsterdam, 1975.
[7] G. Köthe, *Topological Vector Spaces I*, Grundlehren Math. Wiss. 159, Springer, Heidelberg, 1969.
[8] —, *Topological Vector Spaces II*, Grundlehren Math. Wiss. 237, Springer, New York, 1979.
[9] D. R. Lewis, *Integration with respect to vector measures*, Pacific J. Math. 33 (1970), 157–165.
[10] S. Okada and W. J. Ricker, *Spectral measures which fail to be equicontinuous*, Period. Math. Hungar. 28 (1994), 55–61.
[11] —, —, *Vector measures and integration in non-complete spaces*, Arch. Math. (Basel) 63 (1994), 344–353.
[12] —, —, *The range of the integration map of a vector measure*, ibid. 64 (1995), 512–522.
[13] E. G. Ostling and A. Wilansky, *Locally convex topologies and the convex compactness property*, Proc. Cambridge Philos. Soc. 75 (1974), 45–50.
[14] W. J. Ricker, *Closed spectral measures in Fréchet spaces*, Internat. J. Math. Math. Sci. 7 (1984), 15–21.
[15] —, *Remarks on completeness in spaces of linear operators*, Bull. Austral. Math. Soc. 34 (1986), 25–35.
[16] —, *Completeness of the* $L^1$*-space of closed vector measures*, Proc. Edinburgh Math. Soc. 33 (1990), 71–78.
[17] —, *Uniformly closed algebras generated by Boolean algebras of projections in locally convex spaces*, Canad. J. Math. 34 (1987), 1123–1146.
[18] W. J. Ricker and H. H. Schaefer, *The uniformly closed algebra generated by a complete Boolean algebra of projections*, Math. Z. 201 (1989), 429–439.
[19] B. Walsh, *Structure of spectral measures on locally convex spaces*, Trans. Amer. Math. Soc. 120 (1965), 295–326.

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