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# CONCERNING TOPOLOGIZATION OF REAL OR COMPLEX ALGEBRAS

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A *topological algebra* is a real or complex (associative) algebra $A$ provided with a Hausdorff vector space topology making the mutiplication a jointly continuous bilinear map. That means that for each neighbourhood $U$ of the origin in $A$ there is another such neighbourhood $V$ satisfying

$$V^2 \subset U. \tag{1}$$

We say that an algebra $A$ is *topologizable* if there is a topology making it a topological algebra. An example of a non-topologizable algebra was given in [2] (see also [1], [5] and [6]). If we drop the assumption that the considered topology is Hausdorff then every algebra $A$ can be given an algebra topology, i.e. a vector space topology satisfying (1). This relation is satisfied by the anti-discrete topology (following Wilansky [4] we say that a vector space topology is *anti-discrete* if the whole space is the only neighbourhood of the origin). It follows that each algebra $A$ has the strongest (maximal) algebra topology $\tau^{\mathrm{a}}_{\max}$, and $A$ is topologizable if and only if the topology $\tau^{\mathrm{a}}_{\max}$ is Hausdorff. The purpose of this paper is to provide the reader with an example of an algebra which is not only non-topologizable but also for which the topology $\tau^{\mathrm{a}}_{\max}$ is anti-discrete (this is the worse possible situation).

PROPOSITION. *Let $X$ be an infinite-dimensional real or complex vector space. Put $A = L_{\mathrm{FD}}(X)$, the algebra of all finite-dimensional endomorphisms of $X$. Then the topology $\tau^{\mathrm{a}}_{\max}$ on $A$ is anti-discrete.*

Proof. First we show that $A$ is non-topologizable. To this end fix a non-zero linear functional $f_0 \in X'$ and a non-zero element $\xi_0 \in X$ and consider the one-dimensional operator $f_0 \otimes \xi_0$ in $A$ given by $(f_0 \otimes \xi_0)\eta = f_0(\eta)\xi_0$. If $A$ has a Hausdorff algebra topology, then there are a neighbourhood $U$ of the origin with

$$f_0 \otimes \xi_0 \notin U, \tag{2}$$

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and a neighbourhood $V$ such that (1) is satisfied (without loss of generality we can assume that all considered neighbourhoods of the origin are balanced, i.e. satisfy $\lambda U \subset U$ for all scalars $\lambda$ with $|\lambda| \le 1$; for all concepts concerning topological vector spaces we refer the reader to [3] and [4]). Define

$$\widetilde{V} = \{\xi \in X : f_0 \otimes \xi \in V\}; \tag{3}$$

it is clearly a balanced absorbing subset of $X$.

Let $\xi \in \widetilde{V}$ and consider an arbitrary linear functional $f$ in $X'$. Since the set $V$ is absorbing, there is a positive $M_f$ such that

$$M_f^{-1} f \otimes \xi_0 \in V. \tag{4}$$

By (3) we have $f_0 \otimes \xi \in V$. Since multiplication in $L_{\mathrm{FD}}(X)$ is the composition of operators, by (4), (1) and an easy calculation we obtain

$$M_f^{-1} f(\xi) f_0 \otimes \xi_0 = (M_f^{-1} f \otimes \xi_0)(f_0 \otimes \xi) \in V^2 \subset U.$$

Thus by (2), $M_f^{-1}|f(\xi)| < 1$, or

$$|f(\xi)| < M_f \tag{5}$$

for all $\xi$ in $\widetilde{V}$, and so for all $\xi \in K = \mathrm{conv}(\widetilde{V})$. Clearly $K$ is an absorbing balanced subset of $X$ so that its Minkowski functional $|\cdot|_K$ is a seminorm on $X$. Relation (5) now implies

$$|f(\xi)| < M_f |\xi|_K \tag{6}$$

for all $\xi$ in $X$, which holds true also for all $f$ in $X'$. Since for each non-zero $\xi$ in $X$ we can find a functional $f \in X'$ with $f(\xi) \ne 0$, relation (6) implies that $|\cdot|_K$ is a norm on $X$. Thus $(X, |\cdot|_K)$ is a normed space, and by (6) all its linear functionals are continuous. But this can only happen when the dimension of $X$ is finite. The contradiction proves that the topology $\tau_{\max}^{\mathrm{a}}$ is non-Hausdorff on $A$, which implies that the intersection $I = \bigcap U$ of all neighbourhoods of the origin in $A$ in this topology is a non-zero (closed) subspace.

We shall show that it is a two-sided ideal in $A$. In fact, let $x \in I$ and $y \in A$. Choose $\tau_{\max}^{\mathrm{a}}$-neigbourhoods $U$ and $V$ of the origin in $A$ so that (1) is satisfied, and choose a positive scalar $\lambda$ such that $\lambda y \in V$. Since $I$ is a vector subspace of $A$ we have $\lambda^{-1} x \in V$ and so $xy = \lambda^{-1} x \lambda y \in V^2 \subset U$ and similarly $yx \in U$. Since $U$ was chosen arbitrarily we have $xy, yx \in I$ so that it is a non-zero two-sided ideal of $A$. It can be easily verified that the algebra $A$ is simple, i.e. the only two-sided ideals of $A$ are (0) and $A$. Thus $I = A$ and the only neighbourhood of the origin in the topology $\tau_{\max}^{\mathrm{a}}$ is $U = A$. The conclusion follows.

This gives a much simpler proof of the following fact:

COROLLARY [2]. *Let $X$ be a real or complex vector space. The algebra $L(X)$ of all endomorphisms of $X$ is topologizable if and only if the dimension of $X$ is finite* (see also [5]).

Note that by a result of [6] every at most countably generated algebra is topologizable as a complete locally convex algebra.

## REFERENCES

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