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ON THE MOMENT MAP OF A MULTIPLICITY FREE ACTION

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The purpose of this note is to show that the Orbit Conjecture of C. Benson, J. Jenkins, R. L. Lipsman and G. Ratcliff [BJLR1] is true. Another proof of that fact has been given by those authors in [BJLR2]. Their proof is based on their earlier results, announced together with the conjecture in [BJLR1]. We follow another path: using a geometric quantization result of Guillemin–Sternberg [G-S] we reduce the conjecture to a similar statement for a projective space, which is a special case of a characterization of projective smooth spherical varieties due to Brion [B2].

Let V be a finite-dimensional complex representation space for a connected reductive complex group G. Choose a maximal compact subgroup $K \subseteq G$ and a K-invariant positive definite hermitian form $(\cdot\,,\cdot)$ on V. Let

$$\langle u, v\rangle = \operatorname{Im}(u, v) \qquad (u, v \in V) \tag{1}$$

be the associated symplectic form. Recall the unnormalized moment map

$$\tau_{\mathfrak{k}} : V \to \mathfrak{k}^*, \quad \tau_{\mathfrak{k}}(v)(X) = \langle X(v), v\rangle \qquad (v \in V), \tag{2}$$

and the normalized moment map

$$\mu_{\mathfrak{k}} : \mathbf{P}(V) \to \mathfrak{k}^*, \quad \mu_{\mathfrak{k}}(\widetilde{v})(X) = \frac{\langle X(v), v\rangle}{(v, v)} \qquad (v \in V), \tag{3}$$

where $\mathbf{P}(V)$ is the projective space of lines in V and $\widetilde{v}$ is the line passing through v. It is easy to see that these maps are K-equivariant.

Let $\mathbb{C}[V]$ be the space of polynomial functions on V. Clearly the group K acts on $\mathbb{C}[V]$. Recall from [BJLR1] that the action of K on V is called *multiplicity-free* if the action of K on $\mathbb{C}[V]$ has no multiplicities, i.e. the multiplicities of the irreducible representations of K in $\mathbb{C}[V]$ are at most one.

Here is the Orbit Conjecture (see [BJLR1]), stated as a theorem.

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THEOREM. *The map $\tau_{\mathfrak{k}}$ is one-to-one on K-orbits* (*i.e. distinct orbits are mapped onto distinct orbits*) *if and only if the action of K on V is multiplicity-free.*

Before we give the proof of the theorem, we will recall a result of Brion on moment maps of smooth projective G-varieties.

An algebraic variety X with an action of a complex reductive group G is called *spherical* if some (or equivalently each) Borel subgroup B of G has a dense orbit in X. It is well known (see [Se]) that an affine G-variety X is spherical if and only if it is multiplicity-free, i.e. its ring $\mathbb{C}[X]$ of polynomial functions has no multiplicities as a G-module. For a good introduction to the theory of spherical varieties the reader may consult [B1].

Assume that the variety X is contained in the projective space $\mathbf{P}(V)$ for some complex representation space V of G, and that the action of G on X is induced by that on V. Let $\mu_X : X \to \mathfrak{k}^*$ be the normalized moment map of X, i.e. the composite $X \hookrightarrow \mathbf{P}(V) \to \mathfrak{k}^*$ of the normalized moment map (3) and inclusion. Assume that X is smooth and projective (closed in $\mathbf{P}(V)$). Then the theorem of Brion (see [B2, 5.1], [B1, Theorem 3.2]) says that

(4) X is spherical if and only if μ_X is one-to-one on K-orbits.

P r o o f o f t h e t h e o r e m. We notice first that

(5) if $\tau_{\mathfrak{k}}$ is one-to-one on K-orbits, then so is the normalized moment map $\mu_{\mathfrak{k}}$.

Indeed, we can view this normalized map as the restriction of $\tau_{\mathfrak{k}}$ to the unit sphere S in V composed with the canonical map $S \to \mathbf{P}(V)$.

Let U be the full isometry group of the hermitian form $(\cdot\,,\cdot)$. We have $K \subseteq U$. Let Z denote the center of U. Let $\mathcal{P}_d(V) \subseteq \mathbb{C}[V]$ be the subspace of homogeneous polynomials of degree d. Then the spaces $\mathcal{P}_d(V)$ are the eigenspaces for the action of Z on $\mathbb{C}[V]$, corresponding to distinct eigenvalues (weights). Notice that

(6) if $Z \subseteq K$ and if the map $\mu_{\mathfrak{k}}$ is one-to-one on K-orbits, then so is the unnormalized map $\tau_{\mathfrak{k}}$.

Indeed, the restriction of $\tau_{\mathfrak{k}}$ to any sphere in V is one-to-one on K-orbits and the composition of $\tau_{\mathfrak{k}}$ with the restriction map $\mathfrak{k}^* \to \mathfrak{z}^*$ distinguishes the spheres.

Clearly

(7) if $Z \subseteq K$, then $\mathbf{P}(V)$ is spherical if and only if V is spherical.

This is obvious because under the assumption (7), $\mathbb{C}^\times\cdot$ identity is contained in every Borel subgroup of G.

By combining (4), for $X = \mathbf{P}(V)$, with (5)–(7) we see that the theorem holds if $Z \subseteq K$.

Assume from now on that Z is not contained in K.

Suppose $\tau_{\mathfrak{k}}$ is one-to-one on K-orbits. Then by (4) and (5), $\mathbf{P}(V)$ is G-spherical. Hence V is $(\mathbb{C}^\times \cdot G)$-spherical. Hence the group $Z \cdot K$ acts on $\mathbb{C}[V]$ without multiplicities. Therefore K acts on each $\mathcal{P}_d(V)$ without multiplicities.

Recall that each $\mathcal{P}_d(V)$ is irreducible for the action of U. Let $\mathcal{O}_d \subseteq \mathfrak{u}^*$ denote the corresponding orbit, as in [G-S, Theorem 3.7]. This is the coadjoint orbit passing through a highest weight of this representation, divided by $2\pi i$. Then it is easy to see that $\mathcal{O}_d \subseteq \tau_{\mathfrak{u}}(V)$, where $\tau_{\mathfrak{u}} : V \to \mathfrak{u}^*$ is defined as in (3). This map is one-to-one on U-orbits. In fact, $V_d = \tau_{\mathfrak{u}}^{-1}(\mathcal{O}_d)$ is a sphere of radius $d \cdot \mathrm{const}$, where the const does not depend on d.

Let $q : \mathfrak{u}^* \to \mathfrak{k}^*$ be the restriction map. Then $\tau_{\mathfrak{k}} = q \circ \tau_{\mathfrak{u}}$. Suppose $\pi \in \widehat{K}$ occurs in $\mathbb{C}[V]$ at least twice. Then it occurs in $\mathcal{P}_d(V)$ and in $\mathcal{P}_{d'}(V)$ for some $d \neq d'$. Let $\mathcal{O}_\pi \subseteq \mathfrak{k}^*$ be the corresponding orbit (as in [G-S]). Then by [G-S, Theorem 6.3],

$$\mathcal{O}_\pi \subseteq q(\mathcal{O}_d) = \tau_{\mathfrak{k}}(V_d) \quad \text{and} \quad \mathcal{O}_\pi \subseteq q(\mathcal{O}_{d'}) = \tau_{\mathfrak{k}}(V_{d'}). \tag{9}$$

But V_d and $V_{d'}$ are spheres of distinct radii. Hence (9) contradicts the assumption that $\tau_{\mathfrak{k}}$ was one-to-one on K-orbits.

Conversely, suppose K acts on $\mathbb{C}[V]$ without multiplicities. Then $\mathbf{P}(V)$ is spherical. Hence $\mu_{\mathfrak{k}}$ is one-to-one on K-orbits. Therefore the map $V/(Z \cdot K) \to \mathfrak{k}^*/K$ induced by $\tau_{\mathfrak{k}}$ is one-to-one. Thus it will suffice to show that each $(Z \cdot K)$-orbit in V is a K-orbit.

It is well known (see [O-V, p. 138]) that functions in the algebra $\mathbb{C}[V_{\mathbb{R}}]^K$ separate K-orbits. As a U-module, $\mathbb{C}[V_{\mathbb{R}}] = \mathbb{C}[V] \otimes \mathbb{C}[V]^{\mathrm{c}}$, where the superscript c indicates the contragredient. Let $\mathbb{C}[V] = \sum \pi$ be the decomposition into irreducible K-modules. Then, by Schur's lemma, $\mathbb{C}[V_{\mathbb{R}}]^K = \sum (\pi \otimes \pi^{\mathrm{c}})^K$. Hence $\mathbb{C}[V_{\mathbb{R}}]^K$ consists of Z-invariant functions, and we are done. ∎

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